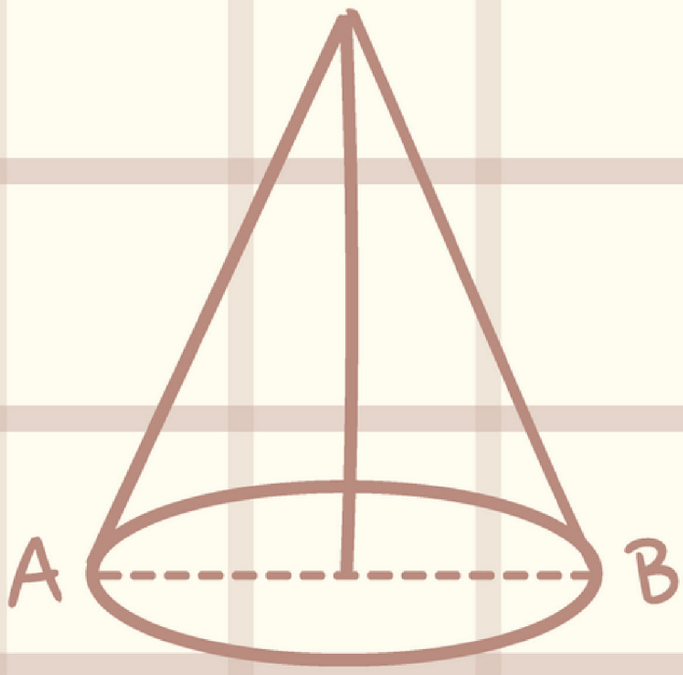

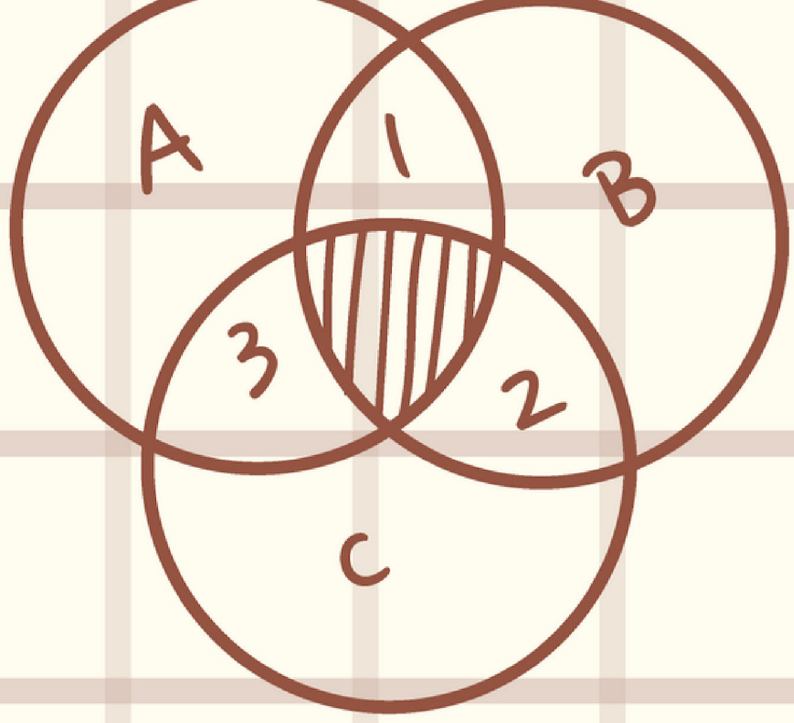

MATHEMATICAL DIGEST (MAGAZINE)

https://archive.org/details/@mks75

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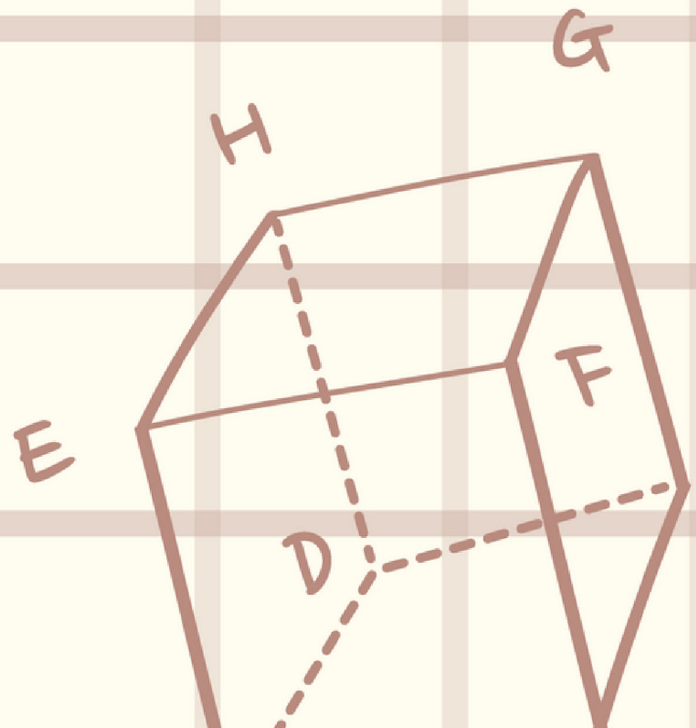
University of Cape Town

$y = x^2$



$a(b \times c)$

mathematical digest

No. 145 October 2006

Baden Powell's Wave Machine

mathematical digest

Number 145 **October 2006**

Just as in the microcosm there are seven 'windows' in the head (two nostrils, two eyes, two ears, and a mouth), so in the macrocosm God has placed two beneficent stars (Jupiter, Venus), two maleficent stars (Mars, Saturn), two luminaries (sun and moon), and one indifferent star (Mercury). Besides, the Jews and other ancient nations as well as modern Europeans, have adopted the division of the week into seven days, and have named them from the seven planets; now if we increase the number of planets, this whole system falls to the ground. Furthermore, since ancient times the alchemists had made each of the seven metals correspond to one of the planets; gold to the sun, silver to the moon, copper to Venus, quicksilver to Mercury, iron to Mars, tin to Jupiter, lead to Saturn.

From these and many other similar phenomena of nature, which it would be tedious to enumerate, we gather that the number of planets is necessarily seven. Moreover, the satellites [of Jupiter] are invisible to the naked eye and therefore can have no influence on the earth, and therefore would be useless, and therefore do not exist.

(Francesco Sizzi, an astrologer, when told of Galileo's discovery of the satellites of Jupiter)

Department of Mathematics and Applied Mathematics
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THE 2006 FIELDS MEDALS

The Fields Medals are the most important international prize in the world of mathematics. Between two and four medals are awarded at the International Congress of Mathematicians (ICM), which takes place every four years. Fields Medals are awarded for outstanding achievement in mathematics by mathematicians under the age of 40 at the beginning of the year in which the ICM is held.

The medals are named after the Canadian mathematician John Fields (1863-1932) and were first awarded at the International Congress of Mathematicians in 1932.

At the 2006 ICM held in August in Madrid, it was announced that Fields Medals had been awarded to Andrei Okounkov, Grigori Perelman, Terence Tao and Wendelin Werner. Though their achievements are in a wide number of branches of mathematics, they share the feature that they make significant connections between apparently very different branches of mathematics.

Andrei Okounkov was born in Russia. His work bridges probability and algebraic geometry, with new insights into problems in physics.

Grigori Perelman, also born in Russia, has achieved worldwide recognition for his solution of a century-old problem, the Poincaré Conjecture (see *Mathematical Digest*, July 2003, pp.4-5).

Terence Tao, born in Australia, has made major advances in an astonishingly varied number of fields, including partial differential equations, combinatorics, harmonic analysis and number theory (see *Mathematical Digest*, July 2004, p.7).

Wendelin Werner, born in France, has achieved breakthroughs in statistical physics, especially in the understanding of Brownian motion (see *Mathematical Digest*, April 2005, p.8).

The medals were presented by King Juan Carlos of Spain during the ICM. However, Grigori Perelman was conspicuously absent, having announced beforehand that he would not accept the award. A recluse, who currently holds no academic appointment, he believes that the mere achievement of solving a difficult mathematical problem ahead of everybody else is sufficient reward. □

BADEN POWELL'S WAVE MACHINE

It is not generally known that the father of the founder of the Boy Scouts, Robert Stephenson Smyth Baden-Powell, was a Professor of Mathematics. Baden Powell (the father) was born in 1796. His first name, Baden, was many years later incorporated by the family to make a double-barrelled surname.

Baden Powell (the father) was an eminent scientist with a wide range of interests. He graduated at the age of 21 from Oriel College, Oxford, with a first class degree in mathematics and was elected a Fellow of the Royal Society at the age of 27. He became Savilian Professor of Geometry at the University of Oxford at the age of 31. He was also a Fellow of the Royal Astronomical Society and the Royal Geographical Society. A strong proponent of the wave theory of light, Baden Powell designed a "wave machine" to demonstrate wave motion in the classroom.

The machine is shown on the front cover. The photograph was supplied by the Whipple Museum of Science in Cambridge and is used here with their kind permission. As the handle is turned, the white balls at the tops of the rods move up and down, giving the effect of a wave moving along. Baden Powell wrote: "For the sake of those readers ... who may be commencing their acquaintance with the undulatory theory, it may not be out of place here to mention a method of imitating the different kinds of vibrations producing a wave by mechanical means. ... A glance at an illustration of this kind enables the mind to grasp the idea at once, even when unaccustomed to the mathematical analysis of it." The balls appear to form a sine wave, but since they are moving back and forth as well as up and down, the exact shape of the curve, as Baden Powell hints, is not quite straightforward.

Baden Powell, an ordained minister in the Church of England, was a noted theologian and was tipped for a bishopric. However, his courageous support of Charles Darwin's theory of evolution, and his rejection on scientific grounds of the creationist theory that the universe was created in 4004 B.C., put paid to his chances. He died in 1860, the year of the famous Oxford debate on evolution, which featured the Bishop of Oxford, "Soapy Sam" Wilberforce and "Darwin's Bulldog", T H Huxley.

When Baden Powell died, his son Robert was only three years old. Robert's mother had hopes that her son would follow his late father into a career of science, especially since her brother, Charles Piazzi Smyth, was Astronomer Royal of Scotland. But it was not to be. The young Robert showed no aptitude or interest in mathematics at school, and when he went to Oxford to be interviewed for a place he was very firmly rejected. His examiner was the Christ Church mathematician, Charles Dodgson, better known as Lewis Carroll. His mother appealed against the decision to the Dean of Christ Church, who confirmed the rejection. (Dean Liddell was the father of the real-life Alice.) Robert joined the army. The rest (Mafeking and the Boy Scouts) is history. □

HYMN-BOARD PROBLEM

A hymn-board in a church has five slots in which the numbers of the hymns to be sung in the service are placed. The numbers are displayed by means of cards, each carrying a single digit, that are slotted into the board.

The hymn-book contains 800 hymns. What is the smallest number of cards, each showing one digit, so that any selection of four different hymns can be displayed?

How many cards are needed if you can turn a 6 over to give a 9? A Prize Voucher worth R50 will be awarded for the first correct answers opened on 1 December, from a reader in grade 8 or lower. □

GIMPS STRIKES AGAIN

The Great Internet Mersenne Prime Search, a distributed computing project, has discovered the largest known prime number: $2^{32\,582\,657} - 1$, a number with 9 808 358 digits. It is the 44th known Mersenne prime number (one of the form $2^p - 1$).

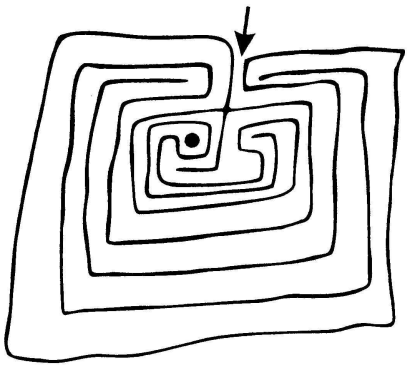
The number was discovered at Central Missouri State University by Curtis Cooper and Steve Boone, who also discovered the previous prime record holder, a number with 9 152 051 digits (see *Mathematical Digest*, April 2006, p.6).

A US$100 000 award will go to the discoverer of a prime with 10 million digits. For details on how to compete for the prize, go to `www.mersenne.org`. □

LABYRINTHS AND MAZES

The history of labyrinths and mazes goes back at least 3500 years. The most famous labyrinth is that of Knossos in Crete, at the centre of which was the half-human, half animal monster, the Minotaur. The daughter of King Minos of Crete, Ariadne, gave Theseus a magic ball of string and instructed him to tie one end to the lintel of the entrance to the maze. The ball of string then unrolled itself, leading Theseus to the centre of the maze, where he found the Minotaur and killed it. He then found his way out by rolling up the ball of string, and married Ariadne.

The design shown below on the right is found on many ancient coins, and is thought to be a map of the labyrinth of Crete. It inspired the design of the logo for the 2006 International Mathematical Olympiad in Slovenia, shown below on the left.

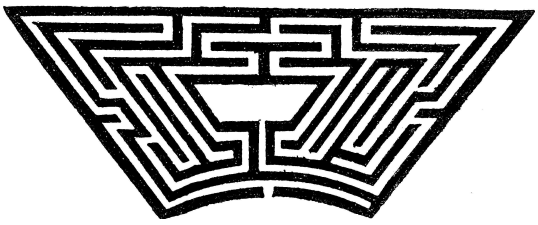


Labyrinths may be found in different cultures all over the world. They were often associated with religious ceremonies, symbolizing the journey of a pilgrim through life. Priests would lead a procession of the faithful along a tortuous path, stopping at each twist in the path to say a prayer or recite an incantation. A shrine of special significance was often found at the centre of the labyrinth.

Labyrinths make interesting tiling patterns for the walls and floors of buildings. One of the most famous is in the cathedral of Chartres, near Paris, and a copy of this labyrinth was recently laid outside St George's Cathedral in Cape Town. Labyrinths are often found cut into the turf of churchyards or village greens in Europe, and in recent years labyrinths have been laid out in many other places in South Africa.

The photograph of the St George's Labyrinth on the next page was taken by Rebecca Malambo, the Editor of *Good Hope*, the newspaper of the Diocese of Cape Town.

Traversing a labyrinth is a ritual, and does not present a problem since it is simply a twisted path, with no forks and no dead ends. Mazes, on the other hand, have forks and dead ends. The most famous maze is the hedge maze of Hampton Court Palace, near London, which is over 300 years old. The aim is to get to the centre of the maze, and then get out again. The maze attracts hundreds of thousands of visitors every year.



The Hampton Court maze is not very challenging, as it can be traversed by the simple rule of consistently following the hedge on one side. This rule works for many mazes, but will fail if at some point the route to the centre forks and the two forks come together again.

The Hampton Court maze is simple for another reason: at no node (fork) in the maze are there more than three paths.

A feature of the English countryside during summer are "Maize Mazes": paths cut through fields of tall maize. Aerial views of some very imaginative mazes can be viewed at `www.maizemaze.com`.

Labyrinths and mazes could make an interesting classroom investigation and Expo project. □

THE GÖDEL CENTENARY

Kurt Gödel was born one hundred years ago on 2 April 1906 in the town of Brünn in the former Austro-Hungarian Empire. Today the town is in of the Czech Republic, and known as Brno.

Gödel was a logician and mathematician, famous for the "consistency theorem" theorem that bears his name. It states that the consistency of any axiomatic system can never be verified within the system itself, provided the system itself is rich enough to develop elementary arithmetic. The theorem established a fundamental distinction between what is *true* about the natural numbers, and what is *provable*.

This astonishing result meant that attempts to establish definitive sets of axioms for geometry or arithmetic were doomed to failure. The theorem, published in 1931, ended nearly 100 years of attempts to provide a rigorous basis for all of mathematics.

Gödel's method was to construct a coding system by means of which every expression of a formal system could be assigned its own natural number, now known as its *Gödel number*. He then showed how to construct a statement $\mathcal{S}$

"The formula with Gödel number n is not provable."

such that the Gödel number of $\mathcal{S}$ itself is n.

He then noted that if statement $\mathcal{S}$ is false, it would be provable, and that is self-contradictory. Thus statement $\mathcal{S}$ is true, but not provable!

That shows that there are statements that are true, but not provable. Next, if you tried to make your axiomatic system complete by adding statement $\mathcal{S}$ as a new axiom, the whole process could be repeated to obtain a new true, but unprovable statement.

Some might find the situation discouraging. But there is an up side. Mathematics is never finished. So there will always be opportunities to create and develop new areas of mathematics.

Within the same decade Gödel established the relative consistency of two important issues: the Axiom of Choice and the Continuum Hypothesis.

The Axiom of Choice seems self-evident. It states that given any family of sets, it is always possible to make up a new set containing one element from each set of the original family. This is intuitively true for finite families, but not quite so obvious for infinite families. Moreover, the Axiom of Choice can be shown to be logically equivalent to other statements that are not nearly so "obvious".

The Continuum Hypothesis was first formalised by Georg Cantor, who had shown that the set of real numbers $\mathbb{R}$ was larger (in a sense that he made quite precise) than the set of natural numbers $\mathbb{N}$. The question then arose as to whether a set of numbers could be found that was larger than $\mathbb{N}$ but not as large as $\mathbb{R}$.

From 1940 Gödel was at the Institute of Advanced Study in Princeton, where his interests turned to philosophy and physics. He demonstrated the existence of paradoxical solutions to Einstein's field equations of general relativity.

Gödel was rated by *Time* as one of the hundred most important personalities of the twentieth century. He died in 1978. □

NEW RUBIK RECORD

"The Rubik's cube is a puzzle you can always solve. I'm a physics major, and we get problems we can't solve all the time. This is easy."

(Caltech student Leyan Lo, who solved a Rubik's cube in 11.13 seconds, setting a new world record. Reported in *The Mercury News*, San Francisco, 15 January 2006.) □

FIND THE FRACTION

Find two three-digit numbers whose quotient (to 7 decimal places) is 0.9323308.

The first correct entry opened on 1 December will win a R50 Prize Voucher. □

TWENTY-FIVE YEARS ON

Twenty-five years ago, in 1981, *Mathematical Digest* was already in its eleventh year of publication. The previous year had seen the beginning of Old Mutual's sponsorship of the magazine, with a marked improvement in the quality of printing. A leisurely browse through the four editions of 1981 (all long out of print, unfortunately) revealed interesting snippets of information, brought back some memories and provoked some insights and afterthoughts.

The subscription for the four issues of *Mathematical Digest* for 1981 was a mere 80c, and a single issue could be ordered for 20c (including postage. Prizes of R1 and R2 were offered for solutions to problems.

In 1981 Rubik's Cube was all the rage, and the October edition contained full instructions on how to sort out a scrambled cube.

There were instructions on how to make a polyhedral lampshade (very much in vogue at the time).

Ronald Reagan had been inaugurated as President of the United States of America, having defeated Jimmy Carter in the elections at the end of 1980. We noted that the statisticians and opinion pollsters had been unable to predict the result, though Reagan had defeated Carter in a landslide.

Among the names of contributors and prizewinners mentioned in the magazine that year were Derek Fish (Rondebosch Boys' High School), now Director of the highly successful Science Centre at the University of Zululand in Richards' Bay, and J H Meyer (Huis Marais, University of Stellenbosch), now Professor of Mathematics at the University of the Free State.

The October 1981 edition reported on the holding of the International Mathematical Olympiad in July 1981 in Washington, the first time it had been held in the United States. A total of 27 countries took part, with 192 participants. This year there were 498 contestants from 90 countries at the IMO in Slovenia. In 1981 the IMO was won by the United States, with West Germany second and the United Kingdom third. The contestants in Washington included Noam Elkies (USA), and Tim Gowers (UK), both of whom are now distinguished mathematicians in Harvard and Cambridge respectively. Elkies, the youngest person ever to be appointed a full professor at Harvard, achieved distinction by solving an old problem of Euler. Gowers was awarded a Fields Medal in 1998.

In the year before (1980) the IMO had not taken place, when the promised host country had failed to fulfil its commitment to stage the event. The participating countries had been forced to acknowledge that their informal arrangements from one year to the next had proved inadequate to ensure the continuity of the IMO, and a permanent "Site Committee" was appointed to ensure that the IMO would take place every year.

The Scientific Counsellor of the South African Embassy in Washington, Mr C G Hide, attended the IMO in Washington as an observer. His report to the Chairman of the Mathematical Association of South Africa concluded: "In view of the shortage of mathematicians in South Africa I think entry to this international competition could serve as a useful and stimulating experience for our youth." South Africa's participation in the IMO had to wait, however, until the fall of apartheid ten years later.

The October 1981 edition (number 45) quoted an extract from C P Snow's essay *Elitism and Excellence* which is well worth repeating.

"There are various tests you can apply to a society's education. But one of them is – does it bring out a number of creative mathematicians who, by world standards, can hold their own? This is, of course, not the only test. But it is a harsh, objective and essential one. If a society's education does not reach it, then there is something wrong with the society or with the education. A society which doesn't choose to encourage excellence, and this particular kind of excellence, won't be a decent society for long."

From then (1981) till today, in pursuit of just that sort of excellence, one hundred editions of *Mathematical Digest* have been published, containing more than 2500 articles spread over 3000 pages. □

CUTTING THE CAKE

A square cake of uniform height has a uniform layer of vanilla icing on the top and a uniform layer of chocolate icing on its four sides. Five children share the cake. They want to cut it into five pieces, using vertical cuts only, so that each piece contains the same amount of cake, the same amount of vanilla icing and the same amount of chocolate icing.

How can the cake be cut to satisfy the children? The best solution, from a reader in grade 10 or lower, will win a Prize Voucher worth R50. □

IMO IN SLOVENIA

The 47th International Mathematical Olympiad was held in Ljubljana, the capital of Slovenia, from 6 to 18 July. The South African team was

Charles Bradshaw (grade 12, Rondebosch Boys' High School)
Dirk B Coetzee (grade 12, Stellenbosch High School)
Ralf Kistner (grade 12, Paul Roos Gymnasium)
Saadiq Moolla (grade 11, Rondebosch Boys' High School)
Henry Thackeray (Grade 9, St Alban's College)
Hendrie Uys (grade 11, Northern Cape High School)

The Team Leader was Professor Dirk Laurie (University of Stellenbosch) and the Deputy Leader was David Hatton (University of Cape Town).

47th INTERNATIONAL MATHEMATICAL OLYMPIAD SLOVENIA 2006

The team met at the University of Cape Town from 30 June to 4 July, and then travelled to Slovenia via Austria. In Austria they collected their Slovnian visas in Vienna, and then enjoyed a stopover in the town of Millstatt where they joined the Austrian IMO team for a joint training programme.

Meanwhile the Team Leaders of the 90 participating countries were already in the small seaside town of Portoroz, setting the papers. The teams arrived on 10 July, and there was an official Opening Ceremony on 11 July. The competition took place on the next two days.

While the papers were being marked, the students were taken on a number of excursions, and on the last day there was a closing ceremony and banquet. The next day provided an opportunity for sightseeing in nearby Venice, after which the team returned home with five Honourable Mentions and an overall place of 62nd.

The excellent website imo2006.dmfa.si gives full details of the question papers, solutions and results.

South Africa's participation in the International Mathematical Olympiad was supported by the South African Agency for Science and Technology Advancement. □

SAASTA

PAMO IN SENEGAL

The 2006 Pan African Mathematics Olympiad took place in Dakar, the capital of Senegal, from 31 July to 6 August. Eleven teams took part: Benin, Burkina Faso, Cameroun, Ivory Coast, Mali, Nigeria, Senegal, South Africa, Swaziland, Tunisia and Zimbabwe. The South African team (all students who had not previously participated in the International Mathematical Olympiad) consisted of

Francois Conradie (De Kuilen High School)
Kylie Fenner (Reddam House)
Melissa Munnik (Stellenbosch High School)
Carl Veller (St John's College)

The Team Leader was David Hatton (University of Cape Town) and the Deputy Leader was Koos van Zyl (University of Stellenbosch).

The papers of the Pan African Mathematics Olympiad have the same format as those of the International Mathematical Olympiad, but are much more accessible.

The South African team came through with flying colours, with Carl Veller winning top place and a Gold Medal, followed by Melissa Munnik (tied second, and the top girl in the event) who also won a Gold Medal. Francois Conradie was joint fourth, winning a Silver Medal, and Kylie Fenner was twelfth, winning a Bronze Medal. South Africa was the top team, followed by Tunisia and Benin.

South Africa's participation in the Pan African Mathematics Olympiad was supported by the South African Agency for Science and Technology Advancement. □

SQUARES WITH SQUARE DIGITS

Have you noticed that 49 is a square number, and both of its digits are square. The same is true for $441(= 21^2)$ and $9409(= 97^2)$. What other numbers can be found with this property?

A Prize Voucher worth R50 will be awarded for the first correct solution opened on 1 December 2006. □

THE EIGHT QUEENS PROBLEM

A queen in chess can attack any other piece on the same horizontal, vertical or diagonal line. In the German chess magazine *Schachzeitung* of 1848 the following problem was posed:

> Place 8 queens on a chess board so that no two queens attack each other.

The problem is harder than it looks, and it took two years for a solution to be found. The solver was none other than Carl Friedrich Gauss, the greatest mathematician of the 19th century. Can you find a solution? There are 92 solutions, but only twelve of them are really different; the others are symmetrical variations (reflections or rotations) of them. A solution will be given in the next issue.

It is obvious that it is not possible to place more than eight queens on a chess board without having two attacking each other. There are only eight rows on a chess board, so if there are nine queens on the board, there must be a row with at least two queens in it and they attack each other.

Turning the problem around, we can ask:

> What is the *smallest* number of queens that can be placed on a chess board so that every square is attacked?

The Eight Queens Problem can be converted into a game. Two players successively place queens on the board so that no queens attack each other. The player who at any stage cannot make a move is the loser. The game can clearly not go beyond eight moves, but it could end earlier.

Is there a winning strategy for either player? Once again, the answer is not obvious. However, the problem is very easy to solve on a 7×7 board, or on any chess board with an odd number of squares: the first player can win by placing a queen in the centre square, and then simply responding to a move of the second player by placing the next queen in the square symmetrical about the centre of the board to the last move.

This "symmetrical response" winning strategy is not available when the chess board contains eight squares, or any even number of squares along a side. However, it can be shown (but not easily) that the first player can always win on an 8×8 board by starting in one of the four central squares. $\square$

DOUBLETS

Given two words with the same number of letters, can one be changed to the other by a sequence of words, all with the same number of letters, such that successive words in the sequence differ by just one letter? Every word in the sequence should be a good dictionary word, and the first and last words should have a meaningful connection. A simple example is getting a SHIP into DOCK:

SHIP-SHOP-SHOT-SOOT-SORT-
PORT-PERT-PERK-PECK-POCK-DOCK

A shorter sequence is possible, with just six steps:

SHIP-SLIP-SLAP-SOAP-SOAK-SOCK-DOCK.

Can the change be made with fewer than six steps? Four steps, of course, is the absolute minimum.

This word puzzle was devised by Lewis Carroll, who named it *Doublets*. Its popularity during the 19th century is said to have been as great then as Sudoku is today.

Making word ladders of five-letter words is harder. Here is a 12-step ladder, making ORDER out of CHAOS:

CHAOS-CHAPS-CHOPS-COOPS-CORPS-CORES-
CODES-CODER-CIDER-EIDER-ELDER-OLDER-
ORDER

Can you change CHAOS into ORDER with fewer steps?

Can you change MONEY into RANDS? Here's one solution:

MONEY-HONEY-HONES-BONES-BONDS-
BANDS-RANDS

Can you change SLIDE into RULES? A R50 Prize Voucher will be awarded for the first correct entry from a reader in grade 10 or lower opened on 1 December 2006. □

SHARP CALCULATOR COMPETITION

The Sharp Calculator Competition is sponsored by Sharp Electronics (S.A.) (Pty) Limited, who have donated three calculators as prizes.

First Prize	:	a Sharp EL–6850 electronic organizer
Second Prize	:	a Sharp EL–6180 scientific programmable calculator
Third Prize	:	a Sharp EL–506 scientific calculator

These valuable prizes (or equivalents, subject to availability) are offered for the three best sets of solutions of the problems which appear below. Solutions may be in English or Afrikaans. The Editor reserves the right to make no awards if no satisfactory entries are received, and his judgement will be final.

Those who are unable to solve all the problems, but obtain interesting partial solutions, are encouraged to submit solutions.

A winner of an nth prize in a previous competition will only be eligible for an mth prize, where $m < n$.

Only high school students in grade 11 or below may compete. Entrants must give their full name and home address, age, school and grade. Solutions *must* be certified by a mathematics teacher at the school as the entrant's own unaided work.

The closing date for receipt of entries is 5 December 2006 and the results will be announced in our April 2007 issue.

SHARP

1. Find all prime numbers p and q such that $p^q + q^p$ is a prime number.
2. Prove that the solutions of the equation

$$x^4 + x^3 + x^2 + ax + b = 0$$

(where a and b are real numbers) cannot all be real.

3. Let P be a point inside equilateral triangle ABC, with $AP = a, BP = b, CP = c$ and $AB = BC = CA = d$. If $s = \frac{1}{2}(a + b + c)$, prove that

$$d^2 = \frac{1}{2}(a^2 + b^2 + c^2) + 2\sqrt{3s(s-a)(s-b)(s-c)}. \quad \square$$

THE GREATEST MATHEMATICAL ERROR

The Mariner I space probe was launched from Cape Canaveral on 28 July 1962 towards Venus. After 13 minutes' flight a booster engine would give acceleration up to 25820 mph; after 44 minutes 9800 solar cells would unfold; after 80 days a computer would calculate the final course corrections and after 100 days the craft would circle the unknown planet, scanning the mysterious cloud in which it is bathed.

However, ... Mariner I plunged into the Atlantic Ocean only four minutes after take off.

Inquiries later revealed that a minus sign had been omitted from the instructions fed into the computer. 'It was human error', a launch spokesman said.

This minus sign cost £ 4 280 000.

(From *The Book of Heroic Failures*, by Stephen Pile) □

HIGHLY COMPOSITE NUMBERS

In the world of natural numbers (positive integers) a *prime* number is a number whose only factors are 1 and itself. For convenience, 1 is not regarded as a prime number. Natural numbers with more than two factors are said to be *composite*.

The number 1 has only one factor: 1. The number 2 has just two factors, 1 and 2. The next number with more factors is 4, which has 3 factors: 1, 2 and 4. The next number with more factors is 6, which has 4 factors: 1, 2, 3 and 6. After that, 12 has 6 factors (1, 2, 3, 4, 6 and 12), 24 has 8 factors, 36 has 9 factors and 48 and 10 factors.

The numbers generated in this way are called *highly composite* numbers, a concept introduced by the Indian mathematical genius Ramanujan. The sequence continues with 60, 120, 180 and 240. Each number in the sequence has more factors than its predecessor, and is the smallest such number. How does the sequence continue?

What can you discover about this sequence? □

SHARP SOLUTIONS AND RESULTS

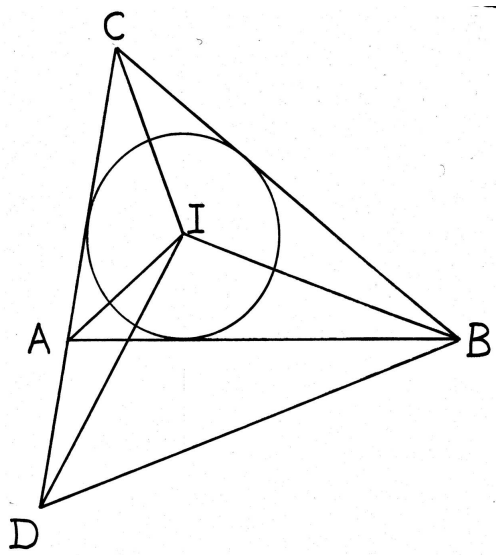
1. *Find all prime numbers p such that p^2+11 has fewer than 11 divisors.*
   If $p = 2$ then $p^2 + 11 = 15$, which has four divisors: 1, 3, 5, and 15.
   If $p = 3$ then $p^2 + 11 = 20$, which has six divisors: 1, 2, 4, 5, 10 and 20.
   If $p = 5$ then $p^2 + 11 = 36$, which has nine divisors: 1, 2, 3, 4, 6, 9, 12, 18 and 36.
   If $p = 7$ then $p^2 + 11 = 60$, which has twelve divisors: 1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30 and 60.

   If p is a prime number greater than 7, then $p \geq 11$ and $p^2 + 11 \geq 132$. Moreover, all prime numbers greater than 3 have the form $6r \pm 1$, so $p^2+11 = 36r^2 \pm 12r+12 = 12a$, where $a > 11$. So the divisors of p^2+11 include at least the numbers 1, 2, 3, 4, 6, 12, a, $2a$, $3a$, $4a$, $6a$ and $12a$, and there are twelve or more divisors (exactly 12 if $a = 12$).

   So the required prime numbers are 2, 3 and 5.
2. *Prove that, in triangle ABC with incentre I,*

$$\angle A = 2\angle B \Leftrightarrow AI + AC = BC.$$

   Produce CA to D such that $AD = AI$ and join DI, IB and DB. Then $AI + AC = CD$.



   Assume that $AI + AC = BC$.
   Then $CD = CA+AD = CA+AI = BC$. So triangle CDB is isosceles, and since I lies on the bisector of $\angle C$, triangle DIB is also isosceles and $\angle ADI = \angle CBI$. So $\angle A = 2\angle CAI = 4\angle ADI = 4\angle CBI = 2\angle B$.

Conversely, assume that $\angle A = 2\angle B$.
Then $\angle ADI = \frac{1}{4}\angle A = \frac{1}{2}\angle B = \angle IBA$. It follows that $AIBD$ is a cyclic quadrilateral and $\angle IDB = \angle IAB = \frac{1}{2}\angle A = \angle B$. So $\angle CDB$
$= \angle ADI + \angle IDB$
$= \angle ABI + \angle IAB$
$= \frac{1}{2}\angle B + \frac{1}{2}\angle A$
$= \frac{3}{2}\angle B$.
But $\angle CBD$
$= \angle B + \angle ABD$
$= \angle B + \angle AID$
$= \angle B + \frac{1}{4}\angle A$
$= \angle B + \frac{1}{2}\angle B$
$= \frac{3}{2}\angle B$.
So $\angle CBD = \angle CDB$, and hence $BC = CD = CA + AI$.

3. *Prove that if a, b and c are positive real numbers such that $abc = 1$, then*

$$\frac{a+1}{ab+a+1} + \frac{b+1}{bc+b+1} + \frac{c+1}{ca+c+1} = 2.$$

$$\begin{aligned}
& \frac{a+1}{ab+a+1} + \frac{b+1}{bc+b+1} + \frac{c+1}{ca+c+1} \\
= & \frac{a+1}{ab+a+abc} + \frac{b+1}{bc+b+1} + \frac{c+1}{ca+c+1} \\
= & \frac{a+1}{a(b+1+bc)} + \frac{b+1}{bc+b+1} + \frac{c+1}{ca+c+1} \\
= & \frac{a+1+a(b+1)}{a(bc+b+1)} + \frac{c+1}{ca+c+1} \\
= & \frac{ab+2a+1}{a(bc+b+abc)} + \frac{c+1}{ca+c+1} \\
= & \frac{ab+2a+1}{ab(c+1+ac)} + \frac{c+1}{ca+c+1} \\
= & \frac{ab+2a+1+ab(c+1)}{ab(c+1+ac)} \\
= & \frac{2ab+2a+2}{1+ab+a} \\
= & 2.
\end{aligned}$$

These problems were set in the final round of the 2005 Slovenian Mathematical Olympiad. No satisfactory solutions were received, so regrettably no prizes can be awarded. $\square$

PRIZEWINNERS AND SOLUTIONS

Chuck Norris Maths (July 2006, p. 3)
"Chuck Norris makes up his own Chuck Norris maths jokes."
Sent in by Rael Chapman (Herzlia Middle School).

"When Chuck Norris proves a theorem in a maths lecture, he always has a second, completely different, proof in reserve – just in case somebody produces a counter-example to his theorem."
Sent in by Chris Nurock.

The Chuck Norris competition will remain open until further notice, so send more jokes to the Editor. The best will be published.

The Inkspot Problem (July 2006, p. 8):
An inkspot of area greater than one square centimeter is made on a piece of 1cm $\times$ 1cm graph paper. Prove that there are two points $A(a_1, a_2)$ and $B(b_1, b_2)$ in the inkspot such that $a_1 - b_1$ and $(a_2 - b_2)$ are both integers.

Cut the inkspot into squares along the grid lines and place them on top of each other, without turning them around. Since the total area of the inkspot is greater than 1 cm^2, there are at least two points in the inkspot with one immediately above the other in the stack. These are the points needed. Henry Thackeray (St Alban's College) sent in a correct solution.

This neat result goes under the name of Blichfeldt's Theorem. It was discovered by a Danish mathematician, Hans Blichfeldt, in 1914. More generally, it says that any bounded planar region with area $A > 1$ placed in any position on the unit lattice can be translated so that the number of lattice points inside the region will be at least $A + 1$. The theorem can be extended to higher dimensions.

Mathematical Clerihews (July 2006, p.10):
The best clerihew came from Daniel Nambassi (Hilton College):

> It is clear that Hipparchus
> Lived a life in sheer darkness.
> Was he too blind to even see
> The pain in trigonometry?

We also received a clerihew about Alan Turing, the famous Bletchley Park codebreaker, logician and computer pioneer, who was elected to a fellowship at King's College, Cambridge, at the age of 22.

Turing
Must have been alluring
To be made a don
So early on.

W. le Riche found this in the recent biography of Turing, *The man who knew too much*, by David Leavitt.

The Sixty-Forty Game (July 2006, p.12)
As explained in the article in which this problem is found, the 60-40 game is not a good game to play, because you lose 97% of your money whenever you play it. However, if the numbers are changed to 64-36, it becomes a good game to play, because then you almost treble your money. However, the 63-37 game gives you a loss of 10%. Henry Thackeray (St Alban's College) sent in the only correct solution.

Calcumaze (July 2006, p.15)
The best pair of paths was sent in by Rofhiwa Mauda (Mbilwi Secondary School).

The Geese Problem (July 2006, p. 27):
Henry Thackeray (St Alban's College) identified the quotation about the "geese that laid the golden eggs – and never cackled". It was the accolade accorded by Winston Churchill to the codebreakers of Bletchley Park, who cracked the German military codes during the Second World War. The "geese" kept the whole project secret throughout the war and even afterwards, thus enabling the British government to crack the diplomatic codes of other countries for another 20 years. You can read about their work in *Mathematical Digest* **126** (January 2002) and **128** (July 2002) and in Simon Singh's *The code book*.

All solutions that did not win prizes went into the Editor's Hat, and three lucky names were drawn: Coetzee Janse van Rensburg, C. Hellman and C. Crouser, all at Hilton College.

Each prizewinner named above wins a 2007 subscription to *Mathematical Digest*.

Competitions are open to all, but only readers who are at school are eligible for prizes. Readers are reminded to submit entries to separate competitions on separate sheets of size A4 paper, each of which must give their name, age, grade, school and home address. The Editor wishes to thank the many adult readers who also send in solutions. □

THE 2006 INTERPROVINCIAL MATHEMATICS OLYMPIAD

The annual Interprovincial Mathematics Olympiad took place this year on the afternoon of Saturday 16 September, and forty teams of ten (21 Junior and 19 Senior) took part. The teams came from ten provinces: Boland, Border, Free State, Gauteng North, Gauteng South, George, KZN Midlands, KZN Piranhas, North-West Province and Western Province, with several provinces entering more than one team in each division.

The first round of the Olympiad was an individual one-hour paper, consisting of 15 multiple-choice problems, with a maximum score was 100. The second paper consisted of ten more difficult problems, to be solved by a team working together. Only answers, worth 100 marks each, were marked. So the maximum total team score for the two papers was 2000.

In a closely-fought contest, the Junior section was won by Western Province, with a score of 1660, followed by Gauteng North (1610), Boland (1605) and Free State (1590). The Senior section was won by Gauteng South, with 1419 points, followed by Free State (1408), Boland (1346) and Western Province (1241).

The provisional date for the 2007 IPMO is Saturday 8 September. For full details, contact

Professor Peter Dankelmann
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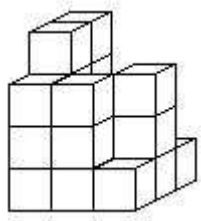
Phone/fax (031) 260 1017
Email: dankelma@nu.ac.za

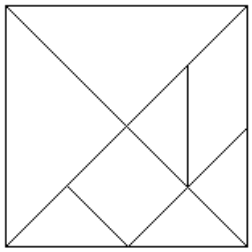
The IPMO is sponsored by the Actuarial Society of South Africa. □

Team Paper: Juniors

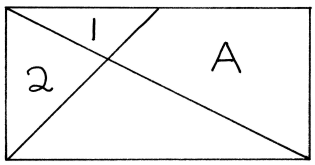
1. Willy is building the following construction by piling a number of identical cubes straight on top of each other on a flat surface. The smallest number of cubes necessary to construct this building is ...



2. In a famous Chinese puzzle (tangram) a square is divided into five isosceles right triangles, one smaller square and a parallelogram (see figure). The ratio of the area of the parallelogram and the area of the large square equals ...



3. A rectangle is divided into four parts by a diagonal and a line segment that connects a vertex and a midpoint of a side (see figure). Two of those parts have area of respectively 1 and 2.
The area of the part denoted with A is ...



4. $$\sqrt{\frac{2+1}{2}\cdot\frac{3+1}{3}\cdot\frac{4+1}{4}\cdot\ldots\cdot\frac{a+1}{a}}=4.$$
So a equals ....

5. If a, b, and c are real numbers such that $a^2+2b=7$, $b^2+4c=-7$, and $c^2+6a=-14$, find $a^2+b^2+c^2$.

6. How many distinct factors does 1 000 000 have? (Do not include 1 and 1 000 000.)

7. Sara's hair is growing 2cm a month. Each four months, 5cm is cut off. After two years, six haircuts later, the length of her hair is doubled. The length of her hair (in cm) at that moment is ...

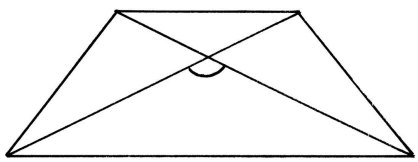
8. $100^2-99^2+98^2-97^2+96^2-95^2+\ldots+2^2-1^2=\ldots$

9. The function f is given by the table

x	1	2	3	4	5
$f(x)$	4	1	3	5	2

If $u_0=4$ and $u_{n+1}=f(u_n)$ for $n\geq 0$, find u_{2006}.

10. The small base of this isosceles trapezoid has the same length as the upright sides while the diagonals have the same length as the large base. The indicated angle formed by the diagonals measures ...



Team Paper: Seniors

1. Given a pair of positive integers a sequence of numbers is formed as follows: the $(k+2)$th term is obtained on subtracting the $(k+1)$th term from the kth term, and the sequence is terminated the moment when one such result of subtraction is negative. The following are two examples of such a sequence:

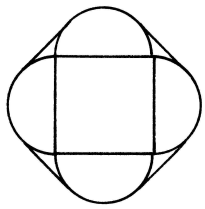
   first example: $25, 16, 9, 7, 2, 5$

   second example: $35, 27, 8, 19$

   The length of the first sequence is 6 while the second sequence has length 4.

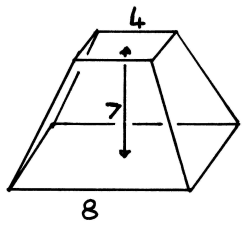
   Given that a sequence starts with 150 find the number(s) x which must come immediately after 150 in the sequence so that the length of the sequence thereby obtained is largest.

2. In the diagram, a square of side length 2 has semi-circles drawn on each side. An elastic band is stretched tightly around the figure. What is the length of the elastic band in this position?



3. A rectangular sheet of paper, $ABCD$, has $AD = 1$ and $AB = r$, where $1 < r < 2$. The paper is folded along a line through A so that the edge AD falls onto the edge AB. Without unfolding, the paper is folded again, along a line through B so that the edge CB also lies on AB. The result is a triangular piece of paper. A region of this triangle is four sheets thick. In terms of r, what is the area of this region?

4. Determine the length of the longest line segment fitting within the regular truncated square pyramid of height 7 centimeters in the figure.



5. Find the sum of all positive integers n for which $n^2 - 19n + 99$ is a perfect square.

6. Three different positive integers are written on three cards. The sum of these numbers equals 13. The smallest number is on the first card, the largest one on the third card. Three members of the Olympiad-jury (who are extremely intelligent and honest) are together around a table.

- Ellen observes the number on the first card and states "Knowing this first number, I cannot deduce the numbers on the other cards."
- Afterwards Eddy sees the number on the third card. He says "I cannot deduce the numbers on the other cards from this."
- Finally, Daniel sees the number on the second card and says "I cannot deduce the numbers on the other cards."

What is the number written on the second card?

7. The factorial of n, written $n!$, is defined by

$$n! = 1 \times 2 \times 3 \times \cdots \times (n-2) \times (n-1) \times n.$$

For how many positive integers $k \leq 50$ is it impossible to find a value of n such that $n!$ ends in exactly k zeros?

8. Given a triangle ABC with $AB = 5 = AC$ and $BC = 6$. P is a point inside the triangle such that $\angle CPA = 90°$, $\angle ABP = \angle BCP$. BP is produced to meet AC at D. Calculate $AD : DC$.
9. The sequence $5, 7, 10, 14, 15, 20, 21, 25, 28, 30, 35, 40, \ldots$ consists of all multiples of 5 and 7 in ascending order. Determine the 100th number in the sequence.
10. The number 18 is **not** the sum of any 2 consecutive integers, but **is** the sum of consecutive positive integers in at least 2 different ways, since $5 + 6 + 7 = 18$ and $3 + 4 + 5 + 6 = 18$.
Determine a positive integer less than 400 that is **not** the sum of 11 consecutive positive integers, but **is** the sum of consecutive positive integers in at least 11 different ways. □

LITERARY QUIZ

A famous American statesman and inventor wrote the following in an essay published in 1735:

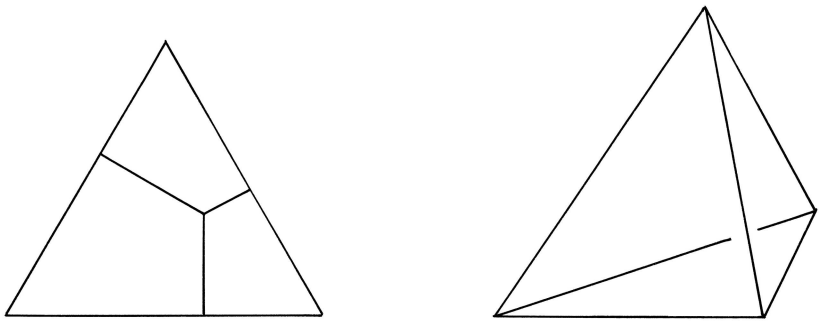
> "What science can there be more noble, more excellent, more useful for men, more admirably high and demonstrative, than this of mathematics?"

Who was he, and for which "spectacular" achievement is he remembered today? Send your answer to the Editor. The first correct answer opened on 1 December will win a Prize Voucher worth R50. □

VIVIANI REVISITED

Viviani's Theorem was discussed in *Mathematical Digest* **142** (January 2006). If, from a point P inside an equilateral triangle three perpendiculars are dropped onto the three sides, then the sum of the lengths of the perpendiculars is the same for all positions of P.

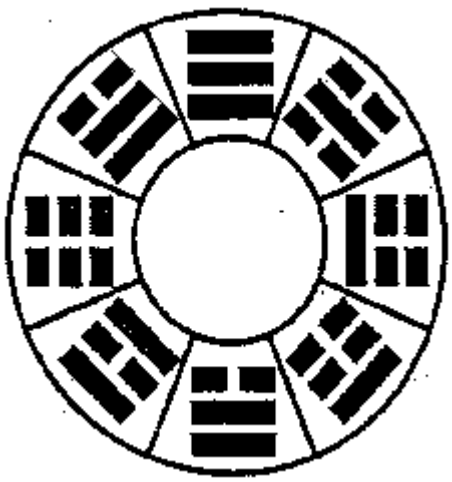
The article then went on to point out that there is a three-dimensional version, obtained by replacing the equilateral triangle by a regular tetrahedron.



Professor Michael de Villiers (University of KwaZulu-Natal) pointed out that the tetrahedron does not have to be regular for the 3-D Viviani to hold. Only the areas of the four faces have to be equal. □

NEW UCT MATHS COMPETITION BOOK

Every year a full account of the annual UCT Mathematics Competition is published. The 2006 edition has just appeared, and is available at R30 (including packing, postage and VAT). The book contains the three question papers set in this year's competition, full solutions, a detailed item analysis of each question, graphs showing overall performances, the names of all the prize winners and a list of the schools that took part. Use the order form at the back of the magazine to order your copy. □



TRANSPORT PROBLEM

Three lorries are to be used to transport k empty barrels, k half-empty barrels and k full barrels. Each lorry must carry the same weight. For which natural numbers k can the problem be solved?
A Prize Voucher will be awarded for the best solution. □

TENNIS PROBLEM

Five girls play tennis, each playing the other four just once. When all the game have been played, it is found that Amy won all her games and Betty lost all hers. Cathy won only one match, and Dolly did better than Emmy.

Who beat whom? The first correct answer opened on 1 December, from a reader in grade 8 or lower, will win a Prize Voucher worth R50. □

FERRY BOAT PROBLEM

Two ferry boats shuttle back and forth across a river at constant speeds. They start at the same time from opposite banks and pass each other 700 metres from one bank. On reaching the opposite shore they turn immediately and pass each other 400 metres from the opposite shore. How wide is the river?

A Prize Voucher worth R50 will be awarded for the first correct answer opened on 1 December. □

ERRATA SEARCH

Keen-eyed readers are invited to look for misprints and errors in this edition of the magazine. We can guarantee at least one mistake. (If there are no errors anywhere else in the magazine, the previous sentence will be incorrect.)
A Prize Voucher worth R50 will be awarded for the best list of errors. □

OLYMPIAD DIARY

University of Cape Town Mathematics Competition
Tuesday 24 April 2007

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Website: `http://www.mth.uct.ac.za/competition`

The UCT Mathematics Competition is for schools in the Western Cape only.

South African Mathematical Olympiad

First round:	20 March
Second round:	15 May
Third round:	6 September

Contact:
Mrs Ellie Olivier, Project Manager
South African Mathematics Olympiad
South African Mathematics Foundation
P O Box 1728
Pretoria 0001

☎ 012 392 9323

Interprovincial Mathematics Olympiad

15 September (provisional)

Contact:
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Interprovincial Mathematics Olympiad
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Durban 4001

☎ /fax: (031) 260 1017 Email: dankelma@nu.ac.za

South African Computer Olympiad

First round:	17 April (in schools)
Second round:	25 May (in schools)
Third round:	15 - 17 September (at the University of Cape Town)

Contact:
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Mowbray 7705

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e-mail: info@olympiad.org.za

The Pan African Mathematics Olympiad will (provisionally) be held in Nigeria in April 2007.

The International Mathematical Olympiad will be held in Vietnam from 19 July to 31 July 2007. □

ERRATA

The following misprints (and worse) have been found in the July 2006 edition of *Mathematical Digest*.

On page 13, ten lines from the bottom of the page, *appetitie* should be *appetite*.

On page 13, three lines from the bottom of the page, the indefinite article should be deleted.

There were two errors in the article *The Body-Mass Index* on page 23. The first could be dismissed as a quibble, but the inequality in the second line of the second paragraph would be better if it read "If $18.5 < \text{BMI} \leq 24.9\ldots$". Further along in the same article was a more serious mistake. In the last but one paragraph the second and third sentences should read:

"Weight is a force, and is a vector, measured in Newtons. A Newton is the force that is required to accelerate a mass of one kilogram by one metre per second per second." □

mathematical digest

No. 154 January 2009

mathematical digest

Number 154 **January 2009**

As we speak of poetical beauty, so ought we to speak of mathematical beauty and medical beauty. But we do not do so; and that reason is that we know well what is the object of mathematics, and that it consists in proofs, and what is the object of medicine, and that it consists in healing. But we do not know in what grace consists, which is the object of poetry.

(Blaise Pascal (1623–1662), French philosopher)

Department of Mathematics and Applied Mathematics
University of Cape Town

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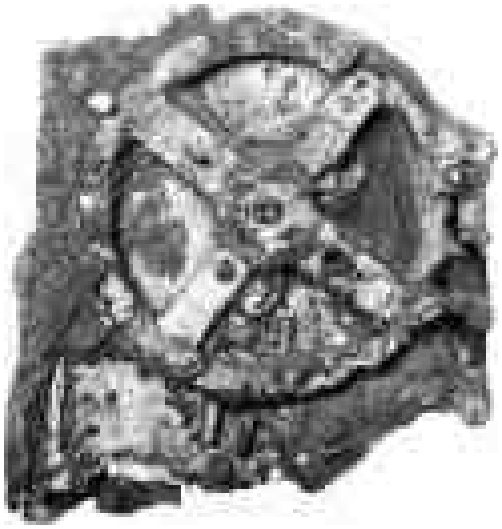
Cover: William Blake's watercolour "The Ancient of Days" portrays God the Architect, constructing the cosmos with geometrical precision.

ARCHIMEDES AND THE ANTIKYTHERA MECHANISM

The *Antikythera mechanism* is a curious chunk of corroded bronze salvaged in 1901 from a ship that was wrecked near the Greek island of Antikythera over two thousand years ago.



Careful examination of the device has shown that it consists of more than 30 bronze gears and pointers, and was probably an astronomical analog computer. Over the years increasingly sophisticated X-ray scanning methods and computerised imaging technology have been used to see through the corrosion of two millennia and decipher the gadget's mysteries.

The dial on the front was used to predict the positions of the sun, moon and planets, while on the back there appears to be a 19-year lunar-solar calendar. Meton of Athens had observed that 19 solar years (6540 days) is very nearly the same as 239 lunar months, an observation of great assistance in predicting eclipses. A smaller dial has been found to give the names of the various Panhellenic games, of which the Olympic games are the most famous.

On another dial the names of the months used in the Greek colony of Syracuse, in Sicily, have been found. This immediately links the mechanism with the greatest mathematician of antiquity, Archimedes, who lived in Syracuse. Archimedes died in 212 B.C., too early to have built the Antikythera mechanism. But he is known to have designed similar devices. □

RECORD NEW MERSENNE PRIME

A Mersenne prime is a prime number of the form $2^n - 1$. They were first investigated by a 17th century French monk, Marin Mersenne, who noted that $2^2 - 1 = 3$, $2^3 - 1 = 7$, $2^5 - 1 = 31$ and $2^7 - 1 = 127$: all prime numbers. Are these the first few cases of a general result: when n is a prime number, $2^n - 1$ is prime?

The conjecture seems to be supported by the observation that, when n is composite (i.e. not prime), then $2^n - 1$ is composite. This result is easy to prove, from the factorization

$$x^n - 1 = (x - 1)(x^{n-1} + x^{n-2} + \cdots + x^2 + x + 1).$$

Thus, if $n = pq$ (where neither p nor q is 1), then $2^n - 1 = 2^{pq} - 1 = (2^p)^q - 1$ is divisible by $2^p - 1$.

However, it is not true that, if p is prime, then $2^p - 1$ is prime, as Mersenne himself was aware. For example, 11 is a prime, but $2^{11} - 1 = 2047 = 23 \times 89$.

In modern times, the largest known prime has almost always been a Mersenne prime. Mersenne primes are discovered by an internet distributed computing project, known as the Great Internet Mersenne Prime Search, or GIMPS for short.

GIMPS is a collaborative project of volunteers who use Prime95 and MPrime computer software that can be downloaded from the Internet for free in order to search for Mersenne prime numbers. The project was founded and the prime testing software was written by George Woltman. Scott Kurowski wrote the PrimeNet server that supports the research to demonstrate Entropia-distributed computing software, a company he founded in 1997. This project has been successful: it has found a total of twelve Mersenne primes, each of which except for the latest one was the largest known prime at the time of discovery.

GIMPS relies on a test devised originally by Édouard Lucas and improved by Derrick Henry Lehmer. It is an algorithm that is both specialized

to testing Mersenne primes and particularly efficient on binary computer architectures.

Before August 2008, only 44 Mersenne primes were known, and a $100 000 prize offered by the Electronic Frontier Foundation nearly a decade earlier for a ten-million digit prime was still unclaimed. Then, in quick succession, two new Mersenne primes were found.

GIMPS reported the finding of a potential 45th Mersenne prime on August 23, 2008, by Edson Smith of the Mathematics Department at the University of California, Los Angeles. It is $2^{43,112,609}-1$, which when written out would have nearly 13 million digits. Just two weeks later Hans-Michael Elvenich, a German electrical engineer and prime number enthusiast, found that $2^{37,156,667}-1$ is prime, more than a million digits shorter. Both were verified independently.

The next challenge is to find a 100 million digit prime, and there is a prize of $150 000 for its discovery, again offered by the Electronic Frontier Foundation. The aim is to develop techniques and technologies to apply the incredibly vast power of cooperative research computing at the Internet's 'electronic frontier'.

For more information, see the website `www.mersenne.org`. □

LINCOLN: CIRCLE-SQUARER

Lincoln had arrived earlier than usual one morning at the office. Spread before him on his desk were sheets of paper covered with figures and equations, plenty of blank paper, a compass, rule, pencils, bottles of ink in different colors. He hardly turned his head as Herndon came in. He covered sheet after sheet with more figures, signs, symbols. As he left for the courthouse later in the day he told Herndon he was trying to square the circle.

He was gone only a short time, came back and spent the rest of the day trying to square the circle, and next day again toiled on the famous problem that has immemorially baffled mathematicians. After two days' struggle, worn down physically and mentally, he gave up trying to square the circle.

(From the biography *Abraham Lincoln*, by Carl Sandburg) □

FAWCETT: CAMBRIDGE WRANGLER

At Cambridge University the Mathematics examination is termed (for reasons lost in the mists of time) the Tripos. Students who graduate in the First Class in the Tripos are termed Wranglers, and in the 19th century the student who came top of the class was dubbed the Senior Wrangler. It was a position of tremendous prestige and a guarantee of a top academic appointment. As long as you were male.

Cambridge University was one of the last bastions of masculine exclusivity, and it was only sixty years ago, in 1948, that women were admitted to Cambridge degrees. Before that, women were allowed to attend lectures and take the examinations, but were never ranked with the men. It was assumed that women were not very good at mathematics.

In 1890 a young woman, Philippa Fawcett, proved them wrong. Philippa had excellent feminist credentials. Her mother Millicent Fawcett was one of the leaders in Britain of the fight for votes for women, and her aunt, Elizabeth Garrett Anderson, was the first English woman doctor. Philippa studied mathematics at Newnham College, Cambridge, which had been co-founded by her mother.

In 1890, she wrote the Mathematical Tripos. It was a gruelling series of twelve three-hour papers. The Senior Moderator of the Mathematical Tripos Examinations of 1890 was the eminent mathematician W.W. Rouse Ball, and it was his duty to announce the results in a special ceremony in the Senate House. Since the participation of women was still regarded as unofficial, Rouse Ball read the men's names and rankings first. Then he began the women's list:

> "Miss Philippa Garrett Fawcett — above the Senior Wrangler."

The Newnham College Diary recorded the event vividly.

> "The great event of the year was Philippa Garrett Fawcett's achievement in the Mathematical Tripos. For the first time a woman has been placed above the Senior Wrangler. The excitement in the Senate House when the lists were read was unparalleled. The deafening cheers of the throng of undergraduates redoubled as Miss Fawcett left the Senate House by the side of the Principal. On her arrival at

> the College she was enthusiastically greeted by a crowd of fellow-students, and carried in triumph into Clough Hall. Flowers, letters, and telegrams poured in upon her throughout the day. The College was profusely decorated with flags. In the evening the whole College dined in Clough Hall. After dinner toasts were proposed: the healths drunk were those of the Principal, Miss Fawcett, her Coach (Mr Hobson) and Senior and Junior Optimes. At 9.30 p.m. the College gardens were illuminated, and a bonfire was lighted on the hockey-ground, round which Miss Fawcett was three times carried amid shouts of triumph and strains of "For she's a jolly good fellow."

A triumphant poem, in the style of Macauley's *Lays of Ancient Rome*, was written in 1890 to celebrate Fawcett's achievement in the Tripos.

> Hail the triumph of the corset!
> Hail the fair Philippa Fawcett!
> Victress in the fray.
> Crown her queen of Hydrostatics
> And the other Mathematics
> Wreathe her brow with bay.
>
> If you entertain objections
> To such things as conic sections
> Put them out of sight.
> Rather sing of the essential
> Beauty of the Differential
> Calculus tonight.
>
> Worthy of our approbation
> She who works out an equation
> By whatever ruse.
> Brighter than the Rose of Sharon
> Are the beauties of the square on
> The hypotenuse.
>
> Curve and angle let her con and
> Parallelopipedon and
> Parallelogram.
> Few can equal, none can beat her
> At eliminating theta
> By the river Cam.

May she increase in knowledge daily
Till the great Professor Cayley
Owns himself surpassed.
Till the great Professor Salmon
Votes his own achievements gammon
And admires aghast.

Coming at the height of the women's suffrage movement, Philippa Fawcett's achievement won worldwide media coverage and highlighted the movement for women's rights. After her triumph in the Mathematical Tripos, Philippa was awarded a scholarship for a further year's mathematical study at Newnham and then became a College Lecturer, a position she held for ten years. During this time she published papers on fluid dynamics.

But Philippa looked for more out of life than a career as a mathematics lecturer. When the South African War broke out in 1899 and the horrors of the concentration camps were revealed by Emily Hobhouse, the British Government appointed an investigating commission to go out to South Africa. It was led by Millicent Fawcett, and Philippa was one of its members. It was to be a significant event in her life.

In South Africa she saw that those in the concentration camps were keen to receive an education. Here she felt was something that she could contribute to, and back in England she offered her services to set up an education system in the Transvaal. Once the permission was granted she resigned her position at Newnham and returned to South Africa in July 1902 as a lecturer in mathematics at the Normal School in Johannesburg and private secretary to the acting Director of Education, with the responsibility of developing a system of farm schools.

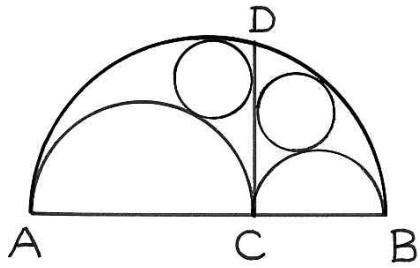
In the aftermath of the bitter Anglo-Boer War there was great tension and distrust between the Boers and the conquering British. It was realised that colonial officials had to demonstrate a true spirit of conciliation. Philippa Fawcett's political ancestry won a measure of confidence from the Boers, whose sympathy had been won by her very distinguished mother.

In 1905 Philippa returned to England as principal assistant to the Director of Education of London County Council. It was a position she held until her retirement in 1934. She died in 1948, just one month after Cambridge finally assented to grant degrees to women. □

ARCHIMEDES AND THE ARBELOS

One of the works attributed to Archimedes is the *Book of Lemmas*, in which a number of interesting minor geometrical results are found. In the book, Archimedes introduces a figure which, because of its shape, has become known as an *arbelos*, the Greek word for a shoemaker's knife.

In the figure, the big semicircle has diameter AB and the two smaller semicircles have diameters AC and CB. The region between the three semicircles is called an *arbelos*.



Archimedes discovered a number of interesting mathematical properties of the arbelos.

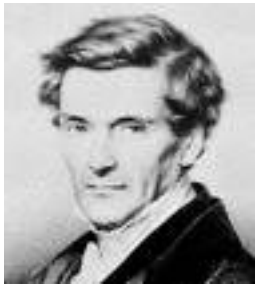
1. The length of arc AC plus the length of arc CB equals the length of arc AB.
2. If a perpendicular is constructed at C to meet the arc AB in D, then $CD = \sqrt{AC.CB}$.
3. The area of the circle with diameter CD is equal to the area of the arbelos.
4. If two circles are drawn tangent to the line CD and the three semicircles as shown, then their radii are equal.

The Arbelos was the title of a mathematical magazine, much on the lines of *Mathematical Digest*. It was founded and edited by the late Sam Greitzer, who pioneered the participation of the United States of America in the International Mathematical Olympiad. So it is appropriate to offer as a prize for solving the arbelos problems above a book that Sam Greitzer would have approved of: *Countdown*: the story of the American team at the 2001 International Mathematical Olympiad.

The Mathematical Association of America has donated a number of copies of *Countdown* to be used as prizes for competitions in *Mathematical Digest*. Send your arbelos solutions to the Editor. The best three submissions will win a copy of *Countdown*. □

GOING DOWN THE DRAIN

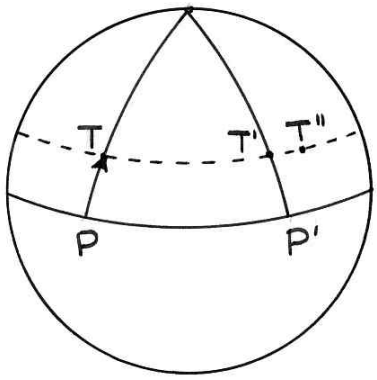
When you pull the plug out of a bath, the water quickly forms a little whirlpool as it runs down the plughole. It is often said that the water will run out clockwise in South Africa (in the southern hemisphere), but anti-clockwise in England (north of the equator). (In America, the water is said to run out in the same direction as in England, but the Americans will describe the motion as "counter-clockwise".)

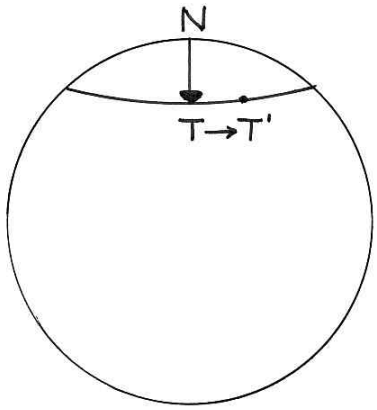


This widespread belief is often explained with a bit of solid science: the direction of rotation of the whirlpool down the plughole is said to be due to the *Coriolis effect*, named after the French physicist Gustave-Gaspard Coriolis (1792–1843).

The Coriolis effect is the apparent deflection of the path of an object in a rotating frame of reference. Little children discover the effect for themselves on playground roundabouts, when they try to throw a ball across to a friend opposite. The spinning roundabout deflects the ball away from the target. And if you stand on the roundabout facing the centre, and try to kick the central pivot, your foot is deflected away from the target.

On a global scale the Coriolis effect is observed in long range gunnery. The Earth rotates from west to east, and points on the equator move eastwards faster than other points on the Earth's surface. A shell fired due north from a point P on the equator towards a target T will land to the east of the target, because when it was fired the shell was moving eastward with the equatorial velocity of the earth (about 1700 km/h), while its target to the north was moving eastward with a lesser velocity. The shell will land at T'', where $PP' = TT''$, but the target will have moved only to T', west of T''.





Similarly, if a shell is fired from the North Pole towards a target due south, the shell (with no initial eastward velocity) will land to the west of its eastward-moving target.

The Coriolis effect in gunnery is not significant over short distances, but is very important in long-range sea battles.

The magnitude of the Coriolis effect is given by the formula $2\nu\omega \sin\phi$, where ν is the velocity of the object, ω is the angular velocity of the Earth, and ϕ is the latitude.

The Coriolis effect is important in understanding weather patterns and ocean currents. In the northern hemisphere, large-scale wind patterns will flow faster near the equator, resulting in hurricanes spiralling anti-clockwise in the northern hemisphere, and clockwise south of the equator. The same effect is observed in ocean currents.

On an even larger scale, the Coriolis effect can be observed in the rotation of sunspots, which are huge storms on the solar surface.

So the Coriolis effect is real, and is of importance in long-range ballistics, meteorology, oceanography and astrophysics. All of these are large-scale phenomena. But on a small scale, in the bathroom, the Coriolis effect on the water running down the plughole is very very small.

Take the diameter of a plughole as 4 cm. The difference of angular velocity of two points on the Earth's surface just four centimetres apart generates a Coriolis effect which is very, very tiny compared with other factors when you pull the plug out. Any movement of the bath water just before you pull out the plug will be the main factor in determining the way the water goes down the plughole. The way you pull out the plug may also affect the outflow. Even a light breeze across the water surface from an open bathroom window will have a much bigger influence than the Coriolis effect.

So the story that the Coriolis effect determines the direction of the bath tub whirlpool is no more than an urban legend. However, in the absence of all other forces on the outflow of the water, the Coriolis effect can be detected. Indeed, the Coriolis effect has been demonstrated in a laboratory bath tub.

In 1962 A.H. Shapiro, of the Mississippi Institute of Technology, published an article in *Nature* with the title "The bath tub vortex". He reported on an experiment that demonstrated that the Coriolis effect can be observed when all other factors have been eliminated.

Shapiro's tub was not a conventional shape, but specially designed. It was two metres across and 15 cm deep, with a very tiny central plughole. Shapiro filled the bath with water, and then allowed the water to settle for several days so that it was free of any movement. Ambient temperature was carefully controlled to prevent convection currents developing in the tub. No air currents were allowed to disturb the water surface. Then the plug was pulled out from below, very carefully, so as not to disturb the water in the tub. Because the outlet was so small, the water started to flow out very slowly. Under such careful experimental conditions, the Coriolis effect was observed: the water in Shapiro's northern hemisphere tub always ran out anti-clockwise.

The challenge was then taken up by a group of Australian scientists who repeated Shapiro's experiment at the University of Sydney. Their conclusions were duly published in *Nature* in 1965: the water in their tub Down Under always ran out clockwise.

In both cases the bathtub experiments were conducted at some distance from the equator. The best places to conduct Shapiro's bath tub experiment would be at the North and South Poles, where the Coriolis effect is greatest. The experiment would probably not have worked at all on the equator, where the Coriolis effect is smallest. (That, by the way, is why hurricanes never start near the equator.)

Designers of space stations and science fiction writers have come up with the idea of building a space station in the form of a large doughnut, and setting it spinning on its axis to create an artificial gravitational effect. But there will be problems. If you are an astronaut sitting in a swivel chair at a control panel in a rotating space station, don't spin around in your chair: the Coriolis effect will make you fall over. □

MARKING MATHS BY WEIGHT

In 1858 the University of Cambridge initiated a national system of school-leaving examinations. The introduction of the system is described in the magazine *History Today*, August 2008, which reported:

"The examiners were paid by the difficulty of the subject and the weight of the scripts they marked. In the 1860s, for every pound weight, those marking arithmetic papers earned 9 shillings and 6 pence, history 12 shillings and classics 18 shillings." □

THE PI CYCLE

A trader and market strategist expounded on a belief he held regarding the cycles of history and the markets. It was based, he said, on the formula for the circumference of a circle—the idea that $2\pi r$ might apply to the financial cycle—and, in combination with various Fibonacci fractal techniques, it had made him a lot of money. You could slice up history into what he called pi cycles, each lasting exactly three thousand one hundred and forty-one days, or 8.6 years. You could subdivide these, by the hours of the clock. say, or the signs of the zodiac, and detect mini-cycles of 8.6 months. He rattled off a series of inflectionary dates and occurrences: the Nikkei high, in late 1989; the ruble collapse, in mid-1998; the historically tight credit spreads in early 2007.

According to pi-cycle theory, after a failed attempt at a rally between now and next spring, the market will not hit bottom until June, 2011, so Obama may be doomed to muddle through a deepening recession and the unpopularity that comes with it, and whoever succeeds him in 2012, should he lose, will then have the chance to ride the recovery—to be, in the public's untrained eye, the transformative F.D.R. or Ronald Reagan that people dearly want Obama to be now. Unless, of course, one adheres to the Venn-diagram view of history, in which case November 4, 2008, could feasibly be Day One in another pi cycle. Where they overlap is anyone's guess.

(Nick Paumgarten, in *The New Yorker*, 17 November 2008). □

RIDDLES

- What did 2 and 3 tell an undisciplined 6?
  You are a product of our times.
- Why was 6 afraid of 7?
  Because 789.
- What did 0 say to 8?
  Nice belt!

(Submitted by Professor Poobhalan Pillay, who found them in *Readers Digest*, August 2007.) □

THE PANCAKE PROBLEM

Suppose you have a pile of pancakes of different sizes. You want to rearrange the pancakes in size, with the biggest at the bottom of the pile and the smallest on the top. You have to do this using only a spatula, which you slip into the stack at some point, and then flip over the pancake pile above the spatula. What is the maximum number of flips that you will need to sort n pancakes into a pile of decreasing size?

The Pancake Problem has been around since 1975. It is hardly of interest to fast food outlets, but for computer scientists it is an important problem, technically known as prefix reversal.

Let P_n (the nth Pancake Number) denote the smallest number of flips needed to put n pancakes in order.

In general, it is not difficult to see that $P_n \leq 2n$. For in whatever order the pancakes start, the largest pancake can be flipped to the top and then to the bottom in just two flips. It takes at most another two flips to get the second-largest pancake on top of the largest, at the bottom of the pile. At most two flips are therefore needed to get each pancake into its correct position. So, for n pancakes, at most $2n$ flips are needed.

But $2n$ flips may never be needed. It is not difficult to work out the values of P_n for small values of n. Clearly $P_2 = 1$ (if a pile of two pancakes is not in order, then one flip will put them in order). For three pancakes you do not need 6 flips: it is not difficult to see that three flips will do the trick. For four pancakes, a little more work is needed to show that $P_4 = 4$.

The table below shows the Pancake Numbers from 1 to 17.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
P_n	0	1	3	4	5	7	8	9	10	11	13	14	15	16	17	18	19

Is there a simple formula for P_n? When the problem was originally proposed, it was proved that, for $n > 2$, $n \leq P_n \leq 2n - 3$. The table above certainly fits into that rule.

An early researcher on the problem was a Harvard undergraduate, William H. Gates, who published a paper with his professor, Christos H. Papadimitriou that proved that

$$\frac{17n}{16} \leq P_n \leq \frac{5(n+1)}{3}.$$

That was Bill Gates' first, and last, publication in computer science. Shortly afterwards he dropped out of Harvard and founded Microsoft.

Last year, a team of students at the University of Texas at Dallas, after working on the problem for two year, and using thousands of hours of computing time, managed to set an improved upper bound: $\frac{18n}{11}$.

Readers are challenged to work out the most efficient way of flipping pancake stacks. Suppose the sizes pancakes are labelled 1 to n, from smallest to largest. How would you flip the pancakes in the stack numbered 52413 to get them in order 12345? And what about a stack numbered 124653?

Send your flipping sequences to the Editor. A Prize Voucher worth R50 will be awarded for the three best entries. □

PROBLEM OF THE YEAR

Using the digits 2, 0, 0 and 9, in that order, and the basic operations of arithmetic, $(+, -, \times, \div, \sqrt{\ })$ and brackets, can you represent the numbers 1, 2, 3, ...?

For example:

$$1 = -2 + 0 + 0 + \sqrt{9}$$
$$2 = 2 + 0 \times 0 \times 9$$
$$3 = 2 \times 0 + 0 + \sqrt{9}$$

Exponential notation (eg $2^0 = 1$), putting two numbers together (eg 20) and recurring decimal notation (eg $.\dot{9} = 1$) are also permitted. You may also use factorial notation: $n! = 1 \times 2 \times 3 \times \ldots \times n$, with the convention that $0! = 1$. These possibilities help to extend the list:

$$4 = 2^0 + 0 + \sqrt{9}$$
$$5 = 20 \div (0! + 3) = 2 + 0 + 0 + 3$$
$$6 = (2 + 0) \times (0 + 3) = (2 + 0 + 0) \div .\dot{3}$$

How far can you extend the list without breaks? Prize Vouchers worth R50 will be awarded for the three best lists, sent in by readers in grade 9 or lower. □

DOUBLING THE CUBE

According to an ancient Greek legend, the city of Athens was afflicted by an outbreak of the plague in 430 B.C. When the citizens asked the Oracle of Apollo at Delios how to stop the plague, the message came back that they had to appease Apollo by doubling the size of his altar.

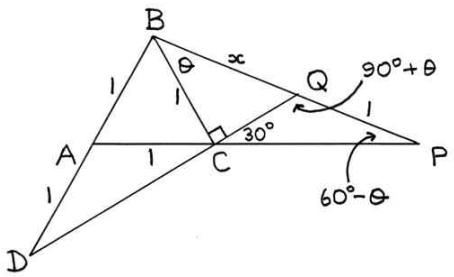
The Athenians immediately replaced the altar, which was a perfect cube, with one with double the dimensions. The plague, to their distress, did not abate. When they sent a further appeal to the Oracle, they were ticked off for not following instructions: they had been commanded to double the volume of the altar, not its dimensions. Doubling the dimensions of the altar had increased its volume by a factor of 8, the Oracle said, and Apollo was not appeased by this extravagance.

Baffled, the worthy leaders of Athens consulted the geometers for their advice. In order to double the volume of the cubical altar, the geometers said, the lengths of the sides had to be increased by a factor of $\sqrt[3]{2}$, the cube root of two. Taking the length of the side of the original altar as one unit, the geometers set about trying to construct a length of side of $\sqrt[3]{2}$ units. The implements that they had at their disposal were simply a straight edge (an unmarked ruler) and compasses. They were unable to solve the problem.

The legend does not relate how the plague ended. But the failure of the geometers of Athens to construct a length of $\sqrt[3]{2}$ units continued to worry them. It took over a hundred years for a solution to be found by Menaechmus. He was the tutor of a young Macedonian prince who, as Alexander the Great, later became one of the greatest conquerors of the ancient world.

The construction devised by Menaechmus is described by the diagram below.

The triangle ABC is equilateral, with sides of unit length. The side BA is produced to D, so that $BA = AD$. The line segments AC and DC are produced.



Now take a ruler, on which two marks, exactly one unit apart, have been marked, and place it at B, so that the first mark lies at P on the line AC produced and the second mark lies at Q on the line DC produced. It turns out that $BQ = \sqrt[3]{2}$.

Before setting out to prove that $BQ = \sqrt[3]{2}$, it must immediately be noted that the construction devised by Menaechmus is not compliant with the fundamental restrictions of Greek geometry, that the ruler must be *unmarked*. Menaechmus needed those two marks, exactly one unit apart, to make his construction work. Moreover, Menaechmus had to use the ruler in quite a complicated way, not simply placing it to touch two given points.

But if the construction is allowed, it can be proved that $BQ = \sqrt[3]{2}$. The proof below uses trigonometry.

Let $BQ = x$ and $\angle PBC = \theta$. Since $\angle B = 60°$ and $BD = 2BC$, it follows that $\angle BCQ = 90°$. So $\cos\theta = 1/x$.
Now $\angle CPQ = 60° - \theta$, $\angle CQP = 90° + \theta$ and $\angle PCQ = 30°$.
By the sine rule in triangle CPQ,

$$\frac{CP}{\sin(90° + \theta)} = \frac{1}{\sin 30°},$$

from which it follows that $CP = 2\sin(90° + \theta) = 2\cos\theta$. (1)
Using the sine rule in triangle BCP we have

$$\frac{CP}{\sin\theta} = \frac{1}{\sin(60° - \theta)},$$

from which it follows that $CP = \dfrac{\sin\theta}{\sin(60° - \theta)}$. (2)
From (1) and (2) we get the trig equation $2\cos\theta\sin(60° - \theta) = \sin\theta$
$\Leftrightarrow 2\cos\theta(\sin 60° \cos\theta - \sin\theta\cos 60°) = \sin\theta$
$\Leftrightarrow \sqrt{3}\cos^2\theta - \sin\theta\cos\theta = \sin\theta$.
Since $\cos\theta = 1/x$, $\quad \sin\theta = \sqrt{x^2 - 1}/x$, and the equation becomes
$\sqrt{3}/x^2 - (\sqrt{x^2 - 1}/x)(1/x) = \sqrt{x^2 - 1}/x$
$\Leftrightarrow \sqrt{3} - \sqrt{x^2 - 1} = x\sqrt{x^2 - 1}$
$\Leftrightarrow (x + 1)\sqrt{x^2 - 1} = \sqrt{3}$
$\Leftrightarrow (x^2 + 2x + 1)(x^2 - 1) = 3$
$\Leftrightarrow x^4 + 2x^3 - 2x - 4 = 0$.
$\Leftrightarrow (x + 2)(x^3 - 2) = 0$ (using the factor theorem)
from which it follows (since $x > 0$) that $x = \sqrt[3]{2}$, as required. $\square$

CUBIC DIGIT SUMS

Start with a whole number, cube it and add the digits of the cube. This is the *cubic digit sum* of the number.

For example: starting with 14, we get $14^3 = 2744$ and $2 + 7 + 4 + 4 = 17$.

Another example: starting with 35, we get $35^3 = 42875$, and $4 + 8 + 2 + 7 + 5 = 26$.

If you start with 23, then $23^3 = 12167$. The cubic digit sum is $1 + 2 + 1 + 6 + 7 = 17$.

Usually this process produces a different number from the starting number, but sometimes you get the original number back.

For example, $26^3 = 17576$ and $1 + 7 + 5 + 7 + 6 = 26$.

The simplest case of a number that is equal to its cubic digit sum is 8: $8^3 = 512$ and $5 + 1 + 2 = 8$. Can you find other examples?

Having found the cubic digit sum of a number, you can repeat the process, and sometime you get back to the original number. For example, the cubic digit sum of 28 is 19 (since $28^3 = 21952$ and $2 + 1 + 5 + 9 + 2 = 19$), and the cubic digit sum of 19 is 28 (since $19^3 = 6859$ and $6 + 8 + 5 + 9 = 28$).

Instead of cubing, we could take fourth powers, getting a *quartic digit sum*. For example, $28^4 = 614656$, and $6 + 1 + 4 + 6 + 5 + 6 = 28$.

Try fifth powers, and look for numbers which are equal to their *quintic digit sums*. One is 36: $36^5 = 60466176$ and $6 + 0 + 4 + 6 + 6 + 1 + 7 + 6 = 36$.

There are lots of opportunities for experimental work (with a calculator, of course) to find patterns of cubic, quartic and quintic digit sums.

Can you find

- a number (other than 8 or 26) that is equal to its cubic digit sum
- a number (other than 28) which is equal to its quartic digit sum
- a number (other than 36) that is equal to its quintic digit sum?

Send your numbers to the Editor. A Prize Voucher worth R50 will be awarded for the best entry. □

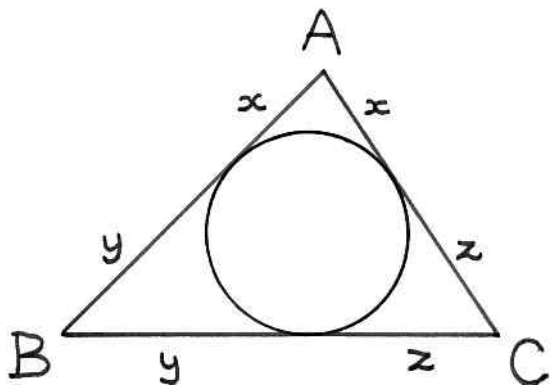
PADOA'S INEQUALITY

Alessandro Padoa (1868–1937) was an Italian mathematician remembered today for his work in mathematical logic. He is also credited with an elementary result in geometry.

If a, b and c are the sides of a triangle, then

$$abc \geq (a+b-c)(b+c-a)(c+a-b).$$

In any inequality involving the sides of a triangle, it is usually a good idea to make a simple substitution, inspired by thinking of the inscribed circle of the triangle, as shown in the diagram below.



The point to note in this diagram is that the tangents to a circle from a point outside the circle are equal. It now follows that

$$a = y+z, \qquad b = z+x \qquad \text{and} \qquad c = x+y.$$

Solving for x, y and z gives

$$x = \frac{1}{2}(b+c-a), \quad y = \frac{1}{2}(c+a-b) \quad \text{and} \quad z = \frac{1}{2}(a+b-c).$$

It is important to note that x, y and z are all positive, since a, b and c are the sides of a triangle.

Making the substitution changes the problem to proving that

$$(y+z)(z+x)(x+y) \geq 8xyz.$$

where x, y and z are positive numbers.

This latter inequality is easily proved. By the Arithmetic Mean–Geometric Mean Inequality, we have

$$x+y \geq 2\sqrt{xy}, \qquad y+z \geq 2\sqrt{yz} \qquad \text{and} \qquad z+x \geq 2\sqrt{zx}.$$

Multiplying these three inequalities gives the required result. $\square$

SHARP CALCULATOR COMPETITION

The Sharp Calculator Competition is sponsored by Sharp Electronics (S.A.) (Pty) Limited, who have donated three calculators as prizes.

These prizes are offered for the three best solutions of the problems below. Solutions may be in English or Afrikaans. Partial solutions will be judged on their merits. The Editor reserves the right to make no award is no satisfactory entries are received, and his judgement will be final. The winner of an nth prize in a previous competition will only be eligible for an mth prize if $m < n$.

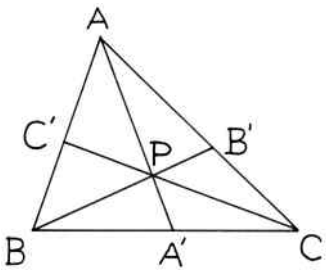
Only high schools students may compete. Entrants must give their full name and home address, age, school and grade. Solutions must be certified by a mathematics teacher at the school as the entrant's own unaided work.

The closing date for receipt of entries is 28 February 2009, and the results will be announced in our April 2009 edition.

SHARP

1. In triangle ABC, points A', B' and C' lie on sides BC, CA and AB respectively, with AA', BB' and CC' meeting in P. Prove that

$$\frac{PA'}{AA'} + \frac{PB'}{BB'} + \frac{PC'}{CC'} = 1.$$



2. Let a, b and c be positive integers. Prove that, if p is a prime number greater than 3 which is a divisor of both $a+b+c$ and $a^3+b^3+c^3$, then p^2 is a divisor of $a^4+b^4+c^4-2a^2b^2-2b^2c^2-2c^2a^2$.

3. A sequence $a_1, a_2, a_3, \ldots, a_n$ is defined by the recurrence relation

$$a_1 = 1,\ a_{n+1} = 3a_n + \sqrt{8(a_n^2 - 1)}.$$

Prove that a_n is an integer for every n. $\square$

HOMEWORK EXCUSES

- I accidentally divided by zero and my paper burst into flames.
- Isaac Newton's birthday.
- I could only get arbitrarily close to my textbook. I couldn't actually reach it.
- I have the proof, but there isn't room to write it in this margin.
- I was watching the World Series and got tied up trying to prove that it converged.
- I have a solar powered calculator and it was cloudy.
- I locked the paper in my trunk but a four-dimensional dog got in and ate it.
- I couldn't figure out whether i am the square of negative one or i is the square root of negative one.
- I took time out to snack on a doughnut and a cup of coffee. I spent the rest of the night trying to figure which one to dunk.
- I could have sworn I put the homework inside a Klein bottle, but this morning I couldn't find it.
- According to the Julian calendar the homework is not due for another thirteen days.

Can you dream up some better reasons for not doing your maths homework? Send your excuses to the Editor. The best ideas will win a Prize Voucher worth R50. □

QUOTE

If Euclid failed to kindle your youthful enthusiasm, then you were not born to be a scientific thinker.

(Albert Einstein, Herbert Spencer Memorial Lecture, Oxford 1933) □

SHARP SOLUTIONS AND RESULTS

Sharp Problems July 2008:

1. *Prove that* $\sin 20^\circ \sin 40^\circ \sin 80^\circ = \frac{1}{8}\sqrt{3}$.

   We will use the trig identity

$$2\sin A \sin B = \cos(A-B) - \cos(A+B),$$

   which is easily established by adding the two identities

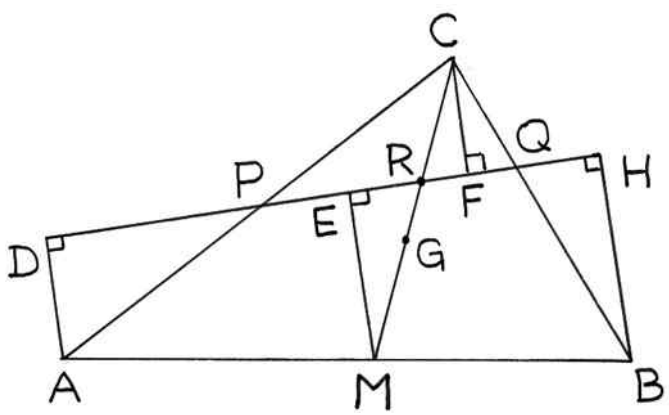
$$\cos(A-B) = \cos A \cos B + \sin A \sin B$$

$$\cos(A+B) = \cos A \cos B - \sin A \sin B.$$

   Applying this identity twice in the reasoning below, we have
   $\sin 20^\circ \sin 40^\circ \sin 80^\circ$
   $= \sin 40^\circ(\sin 80^\circ \sin 20^\circ)$
   $= \frac{1}{2}\sin 40^\circ(\cos 60^\circ - \cos 100^\circ)$
   $= \frac{1}{2}\sin 40^\circ(\frac{1}{2} + \sin 10^\circ)$
   $= \frac{1}{4}\sin 40^\circ + \frac{1}{2}\sin 40^\circ \sin 10^\circ$
   $= \frac{1}{4}\sin 40^\circ + \frac{1}{4}(\cos 30^\circ - \cos 50^\circ)$
   $= \frac{1}{4}\sin 40^\circ + \frac{1}{4}\cos 30^\circ - \frac{1}{4}\sin 40^\circ$
   $= \frac{1}{4}\cos 30^\circ = \frac{1}{8}\sqrt{3}$.

2. *A straight line cuts sides AC and BC of triangle ABC in P and Q respectively. Prove that the centroid G of the triangle lies on PQ if and only if*

$$\frac{AP}{PC} + \frac{BQ}{QC} = 1.$$



Let M be the midpoint of AB. Then CM is a median of the triangle, and the centroid G of the triangle lies on CM, with $CG = 2GM$. Suppose that CM cuts PQ in R. The problem now takes the form:

Prove that $\dfrac{AP}{PC} + \dfrac{BQ}{QC} = 1$ if and only if $R = G$.

Drop perpendiculars AD, ME, CF and BH onto the line PQ. Since AD, ME and BH are parallel, and M is the midpoint of AB, it follows that E is the midpoint of DH and $EM = \frac{1}{2}(AD + BH)$.

Since triangles ADP and CFP are similar, $\dfrac{AP}{PC} = \dfrac{AD}{CF}$.

Since triangles BHQ and CFQ are similar, $\dfrac{BQ}{QC} = \dfrac{BH}{CF}$.

So $$\frac{AP}{PC} + \frac{BQ}{QC} = \frac{AD}{CF} + \frac{BH}{CF} = \frac{AD + BH}{CF} = \frac{2ME}{CF}. \tag{1}$$

Since triangles MER and CFR are similar, $$\frac{MR}{CR} = \frac{ME}{CF}. \tag{2}$$

Now suppose G lies on PQ (i.e. $G = R$).
Since $CG = 2GM$, by (2) $CF = 2ME$, and it follows from (1) that $\dfrac{AP}{PC} + \dfrac{BQ}{QC} = 1$.

Conversely, if $\dfrac{AP}{PC} + \dfrac{BQ}{QC} = 1$, it follows from (1) that $CF = 2ME$. From (2), it follows that $CR = 2MR$. Since CM is a median, R is the centroid of the triangle. So $R = G$, which means that G lies on PQ.

This problem appeared in the German magazine $\sqrt{WURZEL}$, February 2008.

3. *Prove that $\lfloor \sqrt{n} + \sqrt{n+1} \rfloor = \lfloor \sqrt{4n+1} \rfloor$ for every positive integer n, where $\lfloor x \rfloor$ means the greatest integer less than or equal to x (the "whole number part" of x).*

We first prove that $\sqrt{4n+1} \leq \sqrt{n} + \sqrt{n+1}$.
Squaring both sides, this is equivalent to

$$4n + 1 \leq n + 2\sqrt{n(n+1)} + (n+1) \Leftrightarrow 2n \leq 2\sqrt{n(n+1)}$$

which is clearly true.

It is also true that $\sqrt{n} + \sqrt{n+1} \leq \sqrt{4n+2}$, since (squaring) the inequality is equivalent to $n + 2\sqrt{n(n+1)} + (n+1) \leq 4n + 2$
$\Leftrightarrow 2\sqrt{n(n+1)} \leq 2n + 1$
$\Leftrightarrow 4n^2 + 4n \leq 4n^2 + 4n + 1$,
which is clearly true.

We now have two cases to consider:

If there is no integer k between $\sqrt{4n+1}$ and $\sqrt{n}+\sqrt{n+1}$, then the integer parts of $\sqrt{4n+1}$ and $\sqrt{n}+\sqrt{n+1}$ are clearly the same.

On the other hand, suppose that there is an integer k such that $\sqrt{4n+1} \leq k \leq \sqrt{n}+\sqrt{n+1}$. Then (squaring) we have

$4n+1 \leq k^2 \leq n+2\sqrt{n(n+1)}+(n+1) = 2n+1+\sqrt{4n^2+4n}$
$< 2n+1+\sqrt{4n^2+4n+1} = 2n+1+2n+1 = 4n+2.$

So the integer k^2 lies between the successive integers $4n+1$ and $4n+2$. Since $4n+2$ is divisible by 2 but not by 4, it follows that $4n+1 = k^2$. So $\sqrt{4n+1} = k \leq \sqrt{n}+\sqrt{n+1}$, and the stated result follows.

Henry Thackeray (St Alban's College) sent in an excellent set of solutions, but as a former First Prize winner has to be content with an Honourable Mention.

SHARP

Sharp Problems October 2008:

1. Find all positive integers m and n such that $\dfrac{1}{m}+\dfrac{2}{n}=\dfrac{1}{3}$.

   $\dfrac{1}{m}+\dfrac{2}{n}=\dfrac{1}{3}$
   $\Leftrightarrow 3n+6m = mn$
   $\Leftrightarrow mn-6m-3n+18 = 18$
   $\Leftrightarrow (m-3)(n-6) = 18$
   We now consider the different ways of factorising 18:
   $m-3=1$ and $n-6=18$, which gives $m=4$ and $n=24$
   $m-3=2$ and $n-6=9$, which gives $m=5$ and $n=15$
   $m-3=3$ and $n-6=6$, which gives $m=6$ and $n=12$
   $m-3=6$ and $n-6=3$, which gives $m=9$ and $n=9$
   $m-3=9$ and $n-6=2$, which gives $m=12$ and $n=8$
   $m-3=18$ and $n-6=1$, which gives $m=21$ and $n=7$.
   Note that the negative factors of 18 will require $m<3$ and $n<6$, which will give (since m and n are positive): $\dfrac{1}{m}+\dfrac{2}{n}>\dfrac{1}{3}$.

2. Seven points on a circle are joined by 21 straight lines that are either red or blue. Prove that at least four triangles are formed, with their vertices on the circle, having all three sides of the same colour.

We start by noting that there are 35 ($= 7 \times 6 \times 5 \div 6$ triangles in the figure.

Instead of finding the minimum number of monochromatic triangles (triangles with all three sides the same colour), we will work out the maximum number of bichromatic triangles (triangles with one red and two blue sides, or one blue and two red sides).

If a triangle has both red and blue sides, then at exactly two of its vertices there is a red side and a blue side. Now at each vertex there are six lines, some red and some blue. If all the lines are the same colour, there are no bichromatic triangles at that vertex. If one line is red and the other five blue, there are 5 bichromatic triangles at the vertex. If two sides are red and four blue, there are 8 bichromatic triangles at that vertex. The maximum number of bichromatic triangles at the vertex is 9, which occurs when there are three red lines and three blue lines at the vertex.

With at most 9 bichromatic triangles at each of the 7 vertices, there are at most 63 bichromatic triangles in the figure. But since bichromatic triangles are counted twice in this way, there are at most 31 bichromatic triangles in the figure. Since we saw that there are 35 different triangles in the figure, it follows that there are at least four monochromatic triangles.

3. A triangle is inscribed in a circle of diameter 1. Prove that

$$AH = \cos A,$$

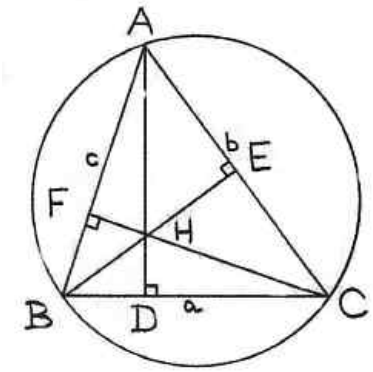
where H is the orthocentre of the triangle (the point of intersection of the altitudes).

By the sine rule, $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 2R$, where R is the radius of the circumcircle.
Since $R = 1$,
$\sin A = BC = BD + DC$
$= HD \tan \angle BHD + HD \tan \angle DHC = HD \tan C + HD \tan B.$

(The last step follows from the observation that $BFHD$ and $ECDH$ are cyclic quadrilaterals. Alternatively, one can reason as follows: $\angle BHD = 180° - \angle DHC - \angle CHE = (90° - \angle DHC) + (90° - \angle CHE) = \angle ECH + \angle HCD = \angle C$; and similarly $\angle DHC = \angle B$.)

So $HD = \dfrac{\sin A}{\tan B + \tan C}$
$= \dfrac{\sin A \cos B \cos C}{\cos B \cos C(\tan B + \tan C)}$
$= \dfrac{\sin A \cos B \cos C}{\sin B \cos C + \cos B \sin C}$
$= \dfrac{\sin A \cos B \cos C}{\sin(B + C)}$
$= \cos B \cos C$

(since $A + B + C = 180°$).
So $AH = AD - HD$
$= AB \sin B - \cos B \cos C$
$= \sin B \sin C - \cos B \cos C$
$= -\cos(B + C)$
$= \cos A$
(since $A + B + C = 180°$).



SHARP

Sharp calculator prizes are awarded to

First Prize: Greg Jackson (Bishops)
Second Prize: Charl du Plessis (Stellenberg High School)
Third Prize: Young Hyun Choi (Pinelands High School).

Liam Baker (Mondale High School), a former First Prize winner, wins an Honourable Mention for his excellent solutions. □

BACK NUMBERS OFFER

New readers of *Mathematical Digest* may obtain all four editions published in 2008 free by sending a stamped (R4.10) self-addressed B5 (250 mm × 176 mm) envelope to

Mathematical Digest 2008 Special Offer
Department of Mathematics and Applied Mathematics
University of Cape Town
7701 RONDEBOSCH

This offer will expire on 31 March 2009. □

PRIZEWINNERS AND SOLUTIONS

The Number of the Beast (April 2008, p. 23):
Young Hyun Choi (Pinelands High School) found the Beastly combinations

$$123 + 456 + 78 + 9 = 666 = 9 + 87 + 6 + 543 + 21,$$

winning a R50 Prize Voucher.

Count Down Competition (October 2008, p.12):
The correct responses were, of course, 5–4–3–2–1. Young Hyun Choi (Pinelands High School), Charl du Plessis (Stellenberg High School), Greg Jackson (Bishops) and Dessi Nikolov (Eunice High School) win copies of Steve Olsen's book, *Count Down*, donated by the Mathematical Association of America.

Carroll's Last Theorem October 2008, p.13):
Just before he died, Lewis Carroll (aka Charles Dodgson) was looking for three Pythagorean triangles with the same area. Two Pythagorean triples giving triangles with the same area (840) are (40, 42, 58) and (24, 70, 74). Dessi Nikolov (Eunice High School) wins a R50 Prize Voucher for finding a third: (15, 112, 113).

From the Press (October 2008, p. 21):
Henry Thackeray (St Alban's College) wins a R50 Prize Voucher for his list of corrections of the list of mathematical errors found in the newspapers.

Alphametric (October 2008, p. 23):
The solution of the Alphametric

$$MATHS \times X = YYYYYY$$

is $95238 \times 7 = 666666$ (a doubly Beastly solution). Liam Baker (Mondale High School), Young Hyun Choi (Pinelands High School) and Charl du Plessis (Stellenberg High School) each win a R50 Prize Voucher for their entries.

Fifty Years On (October 2008, p. 32):
Greg Jackson (Bishops) wins a R50 Prize Voucher for working through pounds, shillings and pence and finding the solution: 240 dozen eggs.

Readers are invited to search through last year's editions of *Mathematical Digest* to find unsolved problems. The prizes offered for these problems are still waiting to be won. □

MATHEMATICS OLYMPIAD RESULTS

The South African Mathematics Olympiad began in 1966. It is held in two divisions: Junior (up to grade 9) and Senior (grades 10–12). In the 2008 Mathematics Olympiad the Junior winner was Ashraf Moolla (Rondebosch Boys' High School). Silver medals were won by the next nine in the rankings (given below in alphabetical order):

Charl du Plessis (grade 9, Hoërskool Stellenberg)
Francois Kassier (grade 9, Hilton College)
Benjamin Levin (grade 9, Crawford College, Sandton)
Vaughan Newton (grade 9, Bridge House School)
Nicola Rule (grade 9, Pietermartizburg Girls High School)
Anne-Mien Schoombie (grade 9, Hoërskool Sentraal)
Rehan Visser (grade 9, Bridge House School)
Sean Wentzel (grade 9, Westerford High School)
Ella Zhu (grade 9, Eunice High School

The winner of the Senior division was Ben-Eben de Klerk (grade 12, Hoërskool Witteberg), followed by nine Silver Medallists (in alphabetical order):

Liam Baker (grade 11, Mondale High School)
Francois Conradie (grade 11, Hoërskool De Kuilen)
Seth Sashen Durais (grade 10, Star College)
Arlton Gilbert, (grade 11, Star College)
Haroon Moolla, (grade 12, Rondebosch Boys High School)
Desislava Nikolov, (grade 10, Eunice High School)
Henry Thackeray, (grade 11, St. Albans College)
Thomas Weighill, (grade 12, Paarl Boys High School)
Kosie van der Merwe, (grade 11, Hoërskool Brackenfell)

The closing date for entries for the 2009 South African Mathematics Olympiad is 9 February 2009. The first round is on 18 March, the second round on 21 May and the final round on 8 September 2009.

Enquiries

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COMPUTER OLYMPIAD RESULTS

The South African Computer Olympiad is an annual competition to identify, encourage and reward programming skills among high school students. The competition attracted nearly 23 000 entries for the first round in 2008, with 2700 winning through into the second round. The top sixteen were invited to take part in the final round, which took place at the University of Cape Town on 27 and 28 September. The winners were announced at a gala function at Kelvin Grove in Cape Town.

Four of the six medal winners in the 2008 Standard Bank/Computer Society of South Africa Computer Olympiad came from the Western Cape. The Gold Medal, the Standard Bank Trophy and R41 000 was won by Francois Conradie, a grade 11 student at De Kuilen High School in Kuilsriver, Western Cape. This is the second time that Francois has reached the Computer Olympiad finals.

Silver Medals were awarded to Kosie van der Merwe (grade 11, Brackenfell High School) and Robert Ketteringham (grade 12, Rondebosch Boys' High School).

Also from Rondebosch Boys' High was Bronze Medal winner Haroon Moolla. Both Ketteringham and Moolla won medals in the 2007 Computer Olympiad. Other Bronze Medals were won by Michiel Baird of Elspark High School in Gauteng, and the youngest medal winner in the competition – Schalk-Willem Krüger, in Grade 11 at Ferdinand Postma High School in North West Province.

Important Dates:
Closing dates for entries: 27 February 2009
First Round: 22 April 2009
Second Round: 22 May 2009
Third Round: 26–28 September 2009

COMPUTER SOCIETY
SOUTH AFRICA

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□

THE UCT MATHEMATICS COMPETITION

The University of Cape Town Mathematics Competition has been an annual event since 1977. This year it will be held on the evening of Tuesday 21 April on the UCT Campus, and is open to all high schools in the Western Cape.

Over 7000 participants are expected this year, and all the big lecture, tutorial and exam venues have been booked to accommodate them. While there are many larger mathematics competitions in other parts of the world, the UCT Mathematics Competition claims to be the largest competition in the world in which all the participants write the papers in the same place and at the same time.

A special feature of the UCT Mathematics Competition is that teams of two are allowed. They write the same paper as the individuals, but are ranked separately.

Question papers for each of the grades 8 to 12 are set, each consisting of 30 multiple-choice problems. The problems range in difficulty, from simple routine calculations to challenging problems. There are several important differences between a competition paper and a school exam.

- There is no "pass" mark, since a competition is designed to rank the participants against each other, not against some official benchmark.
- Unlike stereotyped schools exams, every competition includes problems of a type not met in previous years.
- Full marks in a UCT Mathematics Competition paper is a very rare event.

In recent years it has become progressively easier to score an "A" in the Senior Certificate maths exam, so success in a mathematics competition or Olympiad can be an important addition to a CV for bursary applications or university entrance. Universities are using entrance examinations or placement tests to sort out new students into mainstream or supplementary

courses. These tests use problems seldom seen in standard matric exams, but rather more familiar to those with maths competition experience.

For newcomers to the UCT Mathematics Competition, David Jacobs' *A Mathlete's Training Guide* is recommended, as well as books of papers from previous years. The latter include full solutions, a statistical analysis of all the papers and the names of all the prizewinners of that year. The form at the back of the magazine may be used to order these publications.

Schools or groups of schools in other parts of the country (and even outside South Africa) are welcome to use the questions papers in their own regional competitions. The only restriction is that the papers must be written after 21 April. For details, contact

The Director, UCT Mathematics Competition
Department of Mathematics and Applied Mathematics
University of Cape Town
7701 RONDEBOSCH

The closing date for schools to register for the UCT Mathematics Competition is 20 March 2009. □

ZIMBILLIONS

Rampant inflation in Zimbabwe has necessitated the introduction of new words for large numbers. The Zimbabwe newspaper *The Herald* announced on 27 November 2008 that the commerical bank FBC had been suspended from trading because it had "fallen short of $4.9 hexillion".

"Hexillion" is not a dictionary word, and what was probably meant was "sextillion", or 10^{21}, the next number in the sequence: million (10^6), billion (10^9), trillion (10^{12}), quadrillion (10^{15}), quintillion, (10^{18}).

Recently the Reserve Bank of Zimbabwe knocked ten zeros off the Zimbabwean dollar (in other words, dividing it by ten billion). Not much later it was reported that an ordinary banana cost a million Zimbabwe dollars, and another six zeros were cut off.

The Herald is unreliably reported to be coining new words for large numbers, including *Zimbillion, Bobillion* and *Mugabillion*, to be introduced as needed. The newspaper, the mouthpiece of the government, has denied that it is proposing the term *MDCillion* (for a million decillion). □

BOOK REVIEWS

A Primer for Mathematics Competitions, by Gavin Hitchcock and Alexander Zawaira. Pp. xvii + 344. Oxford University Press, 2008. $100 (hardback).

A Mathematical Olympiad Primer, by Geoff Smith. Pp. x + 178. United Kingdom Mathematics Trust, 2008. £11 (paperback).

Mathematical competitions are recognized worldwide as the most effective means of stimulating interest in mathematics and identifying natural mathematical talent. But once their talent has been uncovered and their interest aroused, the budding young mathematicians need training material as they progress through local, region and national mathematics competitions to the toughest mathematical contest in the world: the International Mathematical Olympiad (IMO).

Gavin Hitchcock, Senior Lecturer in Mathematics at the University of Zimbabwe, has long been involved in mathematics enrichment programmes. One of his students, Alexander Zawaira, went on to win a scholarship to Oxford where he obtained a PhD in Structural Biology. He is currently working in the Medical School of the University of Cape Town.

Their book provides a huge amount of material for developing mathematical skills. It leads the reader from standard school fare through a wide range of topics not included in the standard school curriculum, but accessible to the young mathlete. These include Diophantine equations, mathematical induction, combinatorics and functional equations. Each chapter begins with a "toolchest": basic techniques needed to solve a range of problems that are initially straightforward multiple-choice exercises, ending up with Olympiad mindbenders. On the way, there are cartoons to enjoy and snippets of mathematical history to be savoured. An excellent book, but at a daunting price.

Geoff Smith, Senior Lecturer in Mathematics at the University of Bath, is the current leader of the United Kingdom IMO team. The opening chapter of his book emphasizes the need for a proper understanding of what a real mathematical proof entails —something woefully neglected in school mathematics today. Then there are short chapters on the four topic areas into which IMO problems can be generally classified: Algebra, Combinatorics, Geometry and Number Theory. The real meat of the book is a full discussion of all the problems set in the first round of the British Mathematical

Olympiad, from 1996 to 2007. The problems provide excellent training material for the third round of the South African Mathematics Olympiad. Every solution is preceded by by a thoughtful preliminary discussion, showing the reader how to get started when confronted by a problem that may initially seem totally baffling.

Both books are to be dipped into and explored, with pencil and paper and open mind at the ready. They should be in every school library.

100 Essential Things You Didn't Know You Didn't Know, by John Barrow. Pp. xiv + 284. Bodley Head, 2008. £10 (hardback).

John Barrow is Professor of Mathematics and Director of the Millennium Mathematics Project of Cambridge University. His book is a delighful collection of short essays on off-beat topics, in each of which is a neat little mathematical idea. Sherlock Holmes, rugby, cutting diamonds, roller coasters, snooker, the high jump, voting, bowling averages, secret codes, football logs, wind power, crowd control, choosing a secretary, winning the lottery, mind-reading, Shakespeare, ... This is a book to open at random, settle back comfortably, and enjoy at leisure. □

LITERARY QUIZ

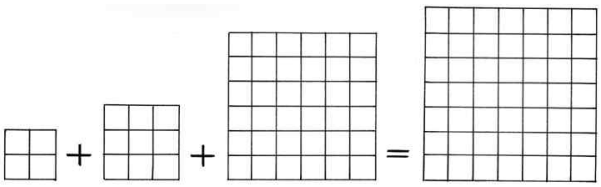
He is a man of good birth and excellent education, endowed by nature with a phenomenal mathematical faculty. At the age of twenty-one he wrote a treatise upon the binomial theorem, which has had a European vogue. On the strength of it he won the mathematical chair at one of our smaller universities, and had, to all appearances, a most brilliant career before him.

But the man had hereditary tendencies of the most diabiolical kind. A criminal strain ran in his blood, which, instead of being modified was increased and rendered infinitely more dangerous by his extraordinary mental powers. Dark rumours gathered round him in the University town, and eventually he was compelled to resign his chair and come down to London.

Who was this mathematical villain, and who wrote the above description? The first correct answer, opened on March 1, will win a Prize Voucher worth R50. □

SQUARE CUTS

Since $7 \times 7 = 2 \times 2 + 3 \times 3 + 6 \times 6$, a 7×7 square can be cut up and rearranged to make three smaller squares.



This can obviously be done by cutting up the 7×7 square into 49 1×1 squares, and then rearranging them to form the three smaller squares.

The dissection can, however, be done with much less cutting.

Show how to cut up the 7×7 square into just FIVE pieces, which can be rearranged to make the three smaller squares. All cuts must be along the grid lines of the big square.

Send your dissection to the Editor. The first three correct solutions opened on 1 March will each win a R50 Prize Voucher. □

BLOCKOUT

Put numbers from 1 to 9 in the empty blocks so that

- each horizontal line of four adds up to the number on the right
- each vertical column of four adds up to the number at the bottom
- the two diagonal lines of four add up to the numbers at the top and bottom of the right hand column.

				23
	4			20
			4	11
		5		30
6				27
23	27	10	28	27

Send your answers to the Editor. The first three correct solutions opened on March 1 will win Prize Vouchers worth R50. □

WHAT IS A LEMMA?

The word *lemma*, when used in mathematics, refers to a minor proposition or theorem which is proved on the way to establishing a more general result. The origin of the word is Greek, and when they have more than one lemma to discuss, very pure mathematicians will use the plural form *lemmata*.

The word is hardly known outside of mathematics, except for its occurrence in the word *dilemma*, which refers to a state of indecision between two unpleasant alternatives. In classical logic, a dilemma is an argument that forces an opponent to make one of two responses, both of which will weaken his case. He is then "impaled on the horns of a dilemma".

Although the prefix *di-* comes from the Greek for "two" or "twice", it is not incorrect to use the word when faced with three or more unpleasant choices. But they must be unpleasant. The term should not be used for a choice between attractive or neutral alternatives.

The plural of *dilemma* is *dilemmas*; the form *dilemmata* would be regarded as very pedantic, even in scholarly works of logic.

Reference:

The Right Word at the Right Time, Reader's Digest, 1985. □

TWENTY-NINE

The year is 2009, or "twenty oh nine", which makes us think of the properties of 29.

For example: 29 is the smallest multidigit prime number whose product of digits of its cube is also a cube:

$$29^3 = 24389 \quad \text{and} \quad 2 \times 4 \times 3 \times 8 \times 9 = 1728 = 12^3$$

Furthermore, 29 is the smallest prime equal to the sum of three consecutive squares:

$$29 = 2^2 + 3^2 + 4^2$$

The 29th power of 2 is the largest power to 2 to have all different digits:

$$2^{29} = 536870912$$

□

MATHS MUDDLE

Five boys are not very good at arithmetic, and not altogether honest either. Each of them is good at just one of Fractions, Decimals and Percentages. Those good at Fractions always tell the truth and those good at Decimals never do — or perhaps it's the other way round. Those good at Percentages make true and false statements alternately.

Each of the boys makes two statements.

James says: "Kevin is good at Decimals. Neville is good at Decimals."

Kevin says: "James is good at Fractions. Martin is good at Fractions."

Lionel says: "James is good at Percentages. I am good at Percentages."

Martin says: "Lionel is good at Decimals. I am good at Fractions."

Neville says: "James is good at Decimals. Kevin is good at Percentages.

Who is good at what, and how honest are they? The first correct answer opened on March 1 will win a Prize Voucher worth R50. □

HOW MUCH IS A YARD?

In the language of currency traders, a *yard* means a billion (10^9). How the term came to be used is not clear. One possible explanation is that it an abbreviation for *milliard*, which is used instead of *billion* in some languages, including Afrikaans and French. □

QUOTE

The scientist in preparing for his work needs three things: mathematics, mathematics and mathematics.

(Wilhelm Conrad Roentgen, the discoverer of X-rays) □

MATHEMATICAL TALENT SEARCH

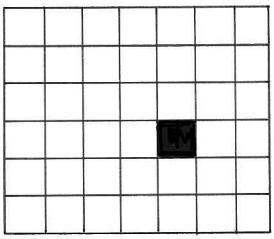
The Mathematical Talent Search is a correspondence course in problem-solving for high schools. Its aim is to identify talented mathematical minds and to develop their skills.

The opening round of the 2009 Mathematical Talent Search is below.

1. Fill in the empty squares so that the sum of the numbers in each row, column and the main diagonals are all the same.

		33
31	28	

2. Write the numbers 1, 2, 3, ..., 16 (1 to 16) in a sequence so that the difference between any two consecutive terms is at least 8.
3. How many squares of all sizes are there on an 8 by 8 grid of unit squares (chessboard)?
4. How many rectangles (of any size) are there below which contain the black square?



5. What is the last digit of 3^{2009}?

Send your answers to:
Mathematical Talent Search, P O Box 201, 7535 BELLVILLE

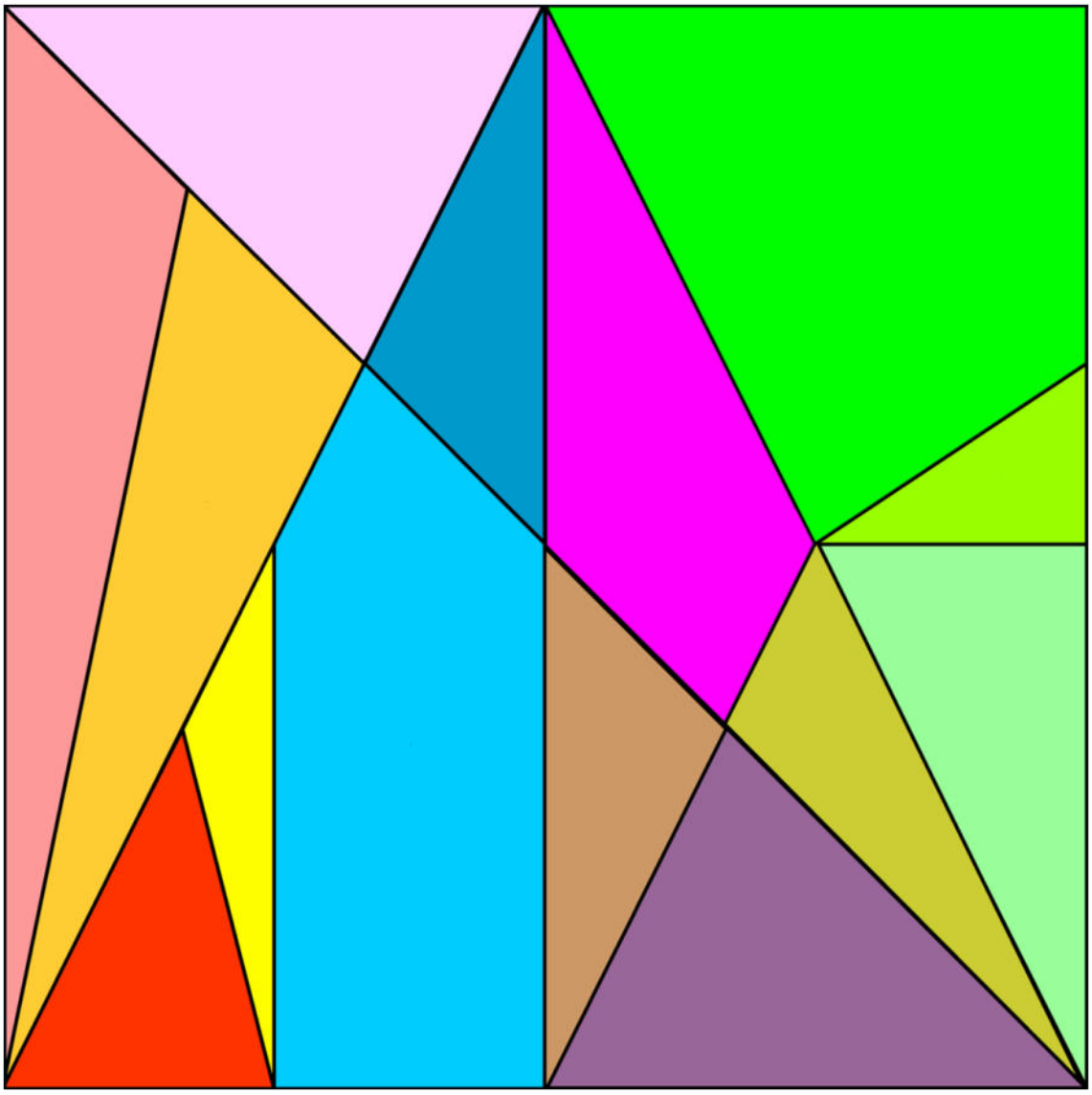
Give your name, grade, school, date of birth, postal address and email address (if available). Include a stamped self-addressed envelope: it will be used to send you the next round of the Talent Search.

Those who make good progress in the Talent Search during the year will be invited to attend a Mathematical Camp at the end of the year in Stellenbosch, after which the teams to represent South Africa at the Pan African Mathematics Olympiad and International Mathematical Olympiad will be chosen.

The Talent Search is a project of the South African Mathematics Foundation. □

mathematical digest

No. 155 April 2009



The Stomachion of Archimedes

mathematical digest

Number 155 **April 2009**

Call Archimedes from his buried tomb
Upon the plain of vanished Syracuse,
And feelingly the Sage shall make report
How insecure, how baseless in itself,
Is the Philosophy, whose sway depends
On mere material instruments; how weak
Those arts, and high inventions, if unpropped
By virtue. He, sighing with pensive grief,
Amid his calm abstractions, would admit
That not the slender privilege is theirs
To save themselves from blank forgetfulness!

(From *The Parsonage*, by William Wordsworth)

Department of Mathematics and Applied Mathematics
University of Cape Town

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Cover: The Stomachion Puzzle – see article on page 2.

QUETELET, PRINCE ALBERT AND SPRING LILACS

Adolphe Quetelet (1796-1874) was born in Ghent, then in Holland but now in Belgium. Quetelet (pronounced Ketter-lay) was a pioneer in the use of statistics in the 19th century.

As a young man, Quetelet coached the princes Ernest and Albert of Saxe Coburg and Gotha in mathematics, and had an extensive correspondence with them afterwards. The letters were later published as *Letters Addressed to H.R.H the Grand Duke of Saxe Coburg and Gotha*. Ernest, the older brother, became Grand Duke, while the younger brother Albert married Queen Victoria.

Albert helped Quetelet make contact with top British scientists of the day, including the computer pioneer Charles Babbage and the economist Thomas Malthus. Quetelet's visit to Britain in 1834 led to the foundation of the Statistical Society of London (subsequently the Royal Statistical Society), and in 1840 Prince Albert became its patron. Albert's address to the fourth meeting of the International Statistical Congress (London, 16 July 1860) was clearly influenced by his former tutor, Quetelet.

Quetelet was a voracious collector of statistics. His early measurement work resulted in the *Quetelet Index of Obesity*, now known as the Body-Mass Index (BMI). To calculate your BMI, divide your weight in kilograms by the square of your height in metres. A BMI of 18.5 to 25 is normal. A BMI lower than 17.5 may indicate anorexia, while a BMI rating above 30 ranks as obesity.

One of Quetelet's letters to the Princes described a rule for predicting when lilacs will bloom in spring. He claimed that they will flower when the sum of the squares of the mean daily temperatures, measured after the last frost, amounts to 4274.

Horticulturalists did not take Quetelet's prediction seriously. It is not on record whether either Albert (or his wife) was amused. □

THE STOMACHION

The *Stomachion* is the name of a puzzle devised by the greatest mathematician of the ancient world, Archimedes (287–212 B.C.).

Though references to the puzzle are known from ancient times, the actual puzzle first came to light just over a century ago.

In 1906 a Danish philologist and Greek scholar, Johan Heiberg, was searching the dusty shelves of a library in Constantinople. There he found an old book of prayers, written some 700 years before. His keen eye noticed that underneath the writing there was another text, very faint, but still readable.

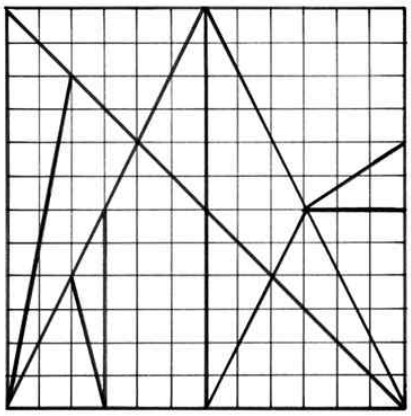
In medieval times books were written on parchment, made from the skin of a calf or sheep. Parchment was very expensive, and was often recycled: the ink was washed off with acid (probably lemon juice) and then scrubbed with pumice stones. The parchment pages were then dried, and the new text was written on them. This type of manuscript is known as a *palimpsest*, from the Greek *palin* (again) and *psestos* (rubbed smooth).

But faint traces of the original text often remain, and can be read. To Heiberg's enormous surprise, he found that underneath the prayers were many important works by Archimedes. He carefully deciphered and translated the text of a very important work, *The Method*, in which Archimedes describes the thought processes that inspired some of his greatest mathematical discoveries.

Another section was on something called a Stomachion. That part of the manuscript was in bad condition and very difficult to read, so Heiberg did not give it much attention. Then the manuscript disappeared in the political turmoil of two World Wars in Europe.

It reappeared again in 1998, when it was auctioned in America for $2.2 million. The buyer, a billionaire philanthropist, presented it to the Walters Art Museum in Baltimore, USA. He also provided funding to commision top scholars to decipher the text. With modern technology (ultra-violet photography and computer-enhanced imaging techniques) the section on the Stomachion could now be read.

The Stomachion is an abstract jigsaw puzzle consisting of a square divided into fourteen pieces, as shown on the cover. On the right, the diagram shows the exact dimensions of the shapes. Copy the figure onto stiff cardboard, cut it up and shuffle the pieces. Now put it back together again.



Finding a solution is the easy part, but that's only the start. Archimedes named his puzzle the Stomachion (which means *bellyache*) because that is what it gives the solver when asked *in how many ways* can the puzzle be put together. Archimedes was therefore the first mathematician to consider what is now known as a *combinatorial* problem.

Counting the number of solutions requires one to consider symmetries of the figure. For example, the solution above can clearly be divided along the central vertical line, and the two pieces then rotated in different ways to give other solutions. Symmetries lead into an important area of mathematics known as *group theory*. Group theory has also proved valuable in understanding the solution of another famous combinatorial problem: Rubik's Cube.

Two husband-and-wife teams of mathematicians, Fan Chung and Ron Graham, and Susan Holmes and Persi Diaconis, joined forces to crack the puzzle and eventually came up with the answer. They found 536 basic solutions, each of which had 32 symmetries. There are therefore $536 \times 32 = 17152$ ways of assembling the square. They were just beaten by a computer program, written to grind out all possible solutions.

The chapter on the Stomachion is just one part of the \$2.2 million manuscript. There was much more, including Archimedes' treatises *On Floating Bodies*, *The Sphere and the Cylinder* and *The Method*: works that established his reputation as one of the greatest mathematicians of all time.

The full story of deciphering the ancient manuscript involves Greek scholars, mathematicians and experts in photography and image enhancement. Read the fascinating story in *The Archimedes Codex*, by Reviel Netz and William Noel (Weidenfeld and Nicholson, 2007). □

NUMBERS ON STRINGS

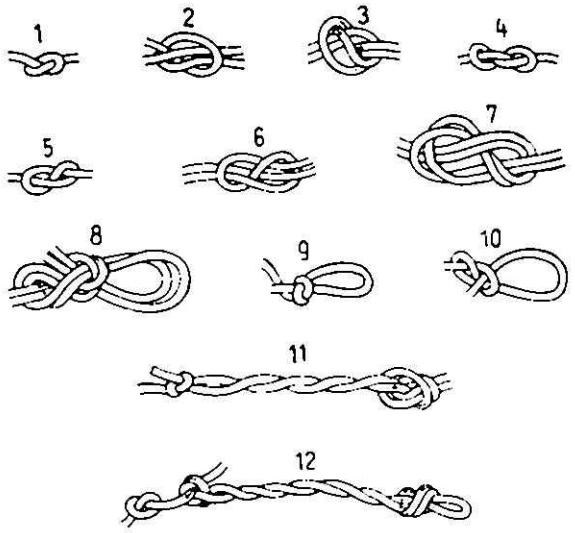
In cultures all over the world, counting is carried out in tens, because we have ten fingers. Counting, and even calculating, can be done on the fingers, but there is then a need to make a permanent record of important numbers, such as the number of goats in your herd or the number of days from one full moon to the next. Numerical records were made long before writing was developed, by cutting notches in a stick or putting pebbles in a pile.

Many cultures use knots to count things. In the most basic way, a knot can be used as a memory prop, such as tying a knot in the corner of a handkerchief to remind you of something important. For thousands of years the major religions of the world have used strings with knots or beads (rosaries) to guide the faithful in their prayers.

The Greek historian Herodotus records how, in the 5th century BC, the Persian King Darius left a squad of his soldiers in defence of a vital bridge. He gave them a leather belt tied with 60 knots, with the instruction to untie one knot every day. "If I have not returned by the time all the knots are undone, take to your boats and return home."

Knots have been used in every continent for recording numbers. Knotted string were used by the Arabs for accounting. The word *aqd* means both "knot" and "contract" in Arabic.

The inhabitants of the island of Okinawa use plaited straw to keep records of the number of days they have worked and the money owed to them.

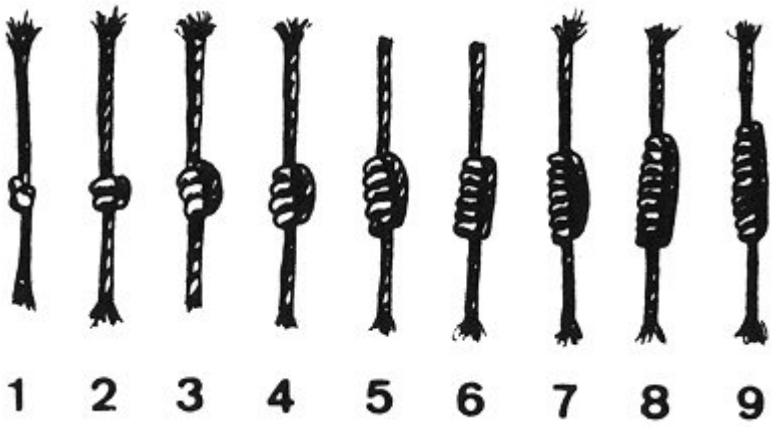


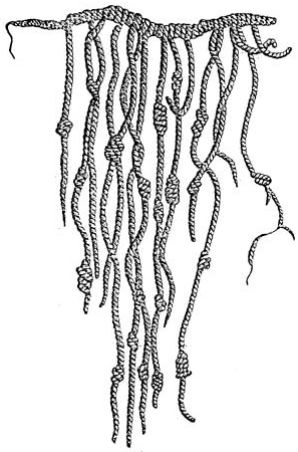
In Germany, up to the end of the 19th century, millers used knots to record the amount and type of flour in their sacks. One unit of flour was indicated by a simple overhand knot, two measures by a simple knot with an extra string passed through it, and a figure eight knot was used for five units. Loops at the end of the string recorded the type of flour in the sack.

An elaborate system of counting and calculating on strings was developed

by the Incas of Peru. The wove their strings, called *quipus*, from the soft wool of alpacas or llamas.

The numbers from 1 to 9 were represented by knots with 1, 2, 3, ... 9 twists. Knots precisely spaced in sequence on a string represented larger numbers.





Officers of the king, called *quipucamayocs* (guardians of the knots) were appointed to conduct censuses of the population, draw up registers of births and deaths and keep records of raw material input and production outputs of every village. Quipus were used in doing arithmetic, assessing taxes and drawing up a calendar. The highly developed and centralized economy of Peru could not have functioned without the meticulous records kept on quipus by the quipucamayocs.

As recently as the mid-nineteenth century, quipus were still in use in Peru. Even today, villagers in Bolivia and Peru use a similar device, the *chimpu*.

Reference

The Universal History of Numbers, by George Ifrah (Wiley, 2000). □

HOW HIGH IS MOUNT EVEREST?

Everybody knows that Mount Everest, on the border between Nepal and Tibet, is the highest mountain in the world. That was not always the name of the peak: the Tibetans knew it as Chomo Lungma, goddess of the earth, and the Nepalese called it Sagarmatha, goddess of the sky. The British surveyors, mapping the subcontinent, simply called it "Peak XV".

In 1829 an observatory was set up in Calcutta, under the direction of the Surveyor-General of India, George Everest, who was engaged in mapping the sub-continent. George Everest had appointed a bright 19-year-old Bengali mathematician, Radhanath Sikdar, to the post of Computor. Sikdar developed into a highly respected surveyor. Under his supervision, in 1852 the height of Peak XV was measured accurately: 29 002 feet, establishing that it was the highest mountain in the world. George Everest had since retired, and Peak XV was renamed Mount Everest in his honour, even though George Everest had never even seen the mountain.

Working out the height of Mount Everest was a major technical achievement. It was measured by theodolites from over 100 miles away. Apart from the possible inaccuracy of the theodolites, a major source of observational error is the refraction of light rays through the atmosphere, caused by changes in the density of the air at different times of the day and year. Then there is the difficulty of determining the height of the observational positions above sea level. And when you pointed your theodolite at the top of the mountain, how did you account for the variation of the thickness of the snow at the peak?

The huge gravitational mass of the Himalayas also introduces errors. The mountains deflect the plumb lines of the theodolites from true vertical, and affect the fluids and bubbles in the levels of the theodolites.

Sikdar worked out how to compensate for all these errors. His measurement of 29002 feet was quickly corroborated and stood for many years. However, because continental drift is pushing the Indian subcontinent northwards, the Himalayas are being pushed up 1.5 inches a year. In 1950 a huge earthquake in Assam was said to have pushed the peak up even more.

The first men to climb Mount Everest were Edmund Hillary and Tensing

Norgay, in May 1953. The next year the new "official" height of Mount Everest was declared to be 29 028 feet (8848 metres), which was confirmed by a 1975 Chinese measurement. In both cases the snow cap, not the rock head, was measured. In May 1999 an American Everest Expedition placed a GPS unit on the peak, which gave the rock head height as 8850 m (29 035 ft), with a snow cover of 1 metre. □

THE YEAR-2038 BUG

In the run-up to the year 2000, the media went into a frenzy of speculation about the Millennium Bug. Would computers that had not recorded the century digits of the date suddenly find their clocks going from 31 December 1999 to 1 January 1900?

In most cases the problem was not serious, but in some it incurred considerable expense. Nine years on, the Millennium Bug has been laid to rest, and it will not occur for another 91 years. But there are several other problems with the way computers handle dates and times.

Most programs work out their dates by counting the seconds from 1 January 1970, at 86 400 seconds per day. On 9 September 2001 the count went from 999 999 999 seconds to 1 000 000 000 seconds. For computers storing time as a nine-digit number of seconds, that could have proved to be a problem. However, not many computers were affected.

The majority of computers today use a 4-byte integer for the second count, storing numbers up to 2^{31} (the remaining bit is used for a plus or minus sign). So when the second count reaches $2^{32}-1 = 2\ 147\ 483\ 647$ it will wrap to $-2\ 147\ 483\ 648$. This momentous event will take place on Tuesday 19 January 2038, at 03:14:07, when computers will suddenly jump to 20:45:52 on Friday 13 December 1901. This is the Year-2038 Bug. The media will probably call it the Friday the Thirteenth Bug.

The predicted time for this event may be off by a couple of seconds. Every few years the astronomers (the world's timekeepers) tell us that because the earth's rotation is slowing down, an extra second has to be added on to the year. At midnight on 31 December of such a year, the new hour is brought in by seven pips instead of six.

We can rest assured that computer boffs are already working on the problem. Follow their progress on the website `www.2038bug.com`. □

THE HARMONIC SERIES

The *arithmetic series*

$$1 + 2 + 3 + \cdots n$$

and the *geometric series*

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots \frac{1}{2^{n-1}}$$

are two important examples of series and are in the school curriculum.

The *harmonic series*

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots \frac{1}{n}$$

looks similar, but has many different features.

The first difference relates to formulas for the sums. While arithmetic and geometric series each have a simple formula for their sum to n terms:

$$1 + 2 + 3 + \cdots n = \frac{1}{2}n(n+1)$$

$$1 + \frac{1}{2} + \frac{1}{4} + \cdots \frac{1}{2^{n-1}} = 2(1 - \frac{1}{2^n})$$

this is not true of the harmonic series: there is no neat formula for its sum.

The second difference relates to the *sum to infinity*. It is known that the sum to infinity of the geometric series $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots$ is 2: the series is said to *converge*. What can one say about the sum to infinity of the harmonic series? Since the terms are getting smaller and smaller, it appears that the sum increases ever more slowly, and it seems plausible that it settles down to some number.

Phrases like *appears that* and *seems plausible* in mathematics are just not good enough. In fact, in this case they are actually deceptive. The fact is that the harmonic series *diverges*: by taking enough terms, the sum can be made as large as you like.

Denote the sum to n terms of the harmonic series by $H(n)$. We first present the standard argument, given in most first-year university maths courses, that $H(n) \to \infty$ as $n \to \infty$.

Start with the just one term: $H(1) = 1$.

For two terms, $H(2) = 1 + \frac{1}{2} = \frac{3}{2}$.

We now look at $H(4) = H(2) + \frac{1}{3} + \frac{1}{4}$. The sum of the two extra terms is greater than $2 \times \frac{1}{4} = \frac{1}{2}$. So $H(4) > \frac{3}{2} + \frac{1}{2} = 2$.

We next look at $H(8) = H(4) + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}$. The sum of the four extra terms is greater than $4 \times \frac{1}{8} = \frac{1}{2}$. So $H(8) > H(4) + \frac{1}{2} > 2 + \frac{1}{2} = \frac{5}{2}$.

Now take the next eight terms: $\frac{1}{9} + \cdots \frac{1}{16} > 8 \times \frac{1}{16} = \frac{1}{2}$, and it follows that $H(16) > H(8) + \frac{1}{2} > \frac{5}{2} + \frac{1}{2} = 3$.

Continuing in this way, we get $H(32) > 3\frac{1}{2}, H(64) > 4, H(128) > 4\frac{1}{2}$, etc.

In general, $H(2^n) \geqslant 1 + \dfrac{n}{2}$.

This argument shows that we can make the sum $1 + \frac{1}{2} + \frac{1}{3} + \cdots \frac{1}{n}$ as large as we like, by taking sufficiently many terms. To make the sum greater than 100, we simply take 2^{200} terms.

The divergence of the harmonic series can be shown in a completely different way. The idea is to focus on *unit fractions*. i.e. fractions with 1 in the numerator.

While any fraction can be expressed as a sum of unit fractions (for example, $\frac{3}{5} = \frac{1}{5} + \frac{1}{5} + \frac{1}{5}$), we can in fact ensure that every fraction can be expressed as a sum of *different* unit fractions. The key idea is to use the identity

$$\frac{1}{n} = \frac{1}{n+1} + \frac{1}{n(n+1)}.$$

For example:

$$\frac{2}{3} = \frac{1}{3} + \frac{1}{3} = \frac{1}{3} + \frac{1}{4} + \frac{1}{12}, \qquad \frac{2}{7} = \frac{1}{7} + \frac{1}{8} + \frac{1}{56} \qquad \frac{5}{9} = \frac{1}{3} + \frac{2}{9} = \frac{1}{3} + \frac{1}{10} + \frac{1}{90}.$$

In this way, any fraction, no matter how large, can be expressed as the sum of different unit fractions, i.e. just a finite number of terms of the harmonic series. So the sum of *all* the terms of the harmonic series must be larger than any fraction you could think of. That proves that the harmonic series diverges.

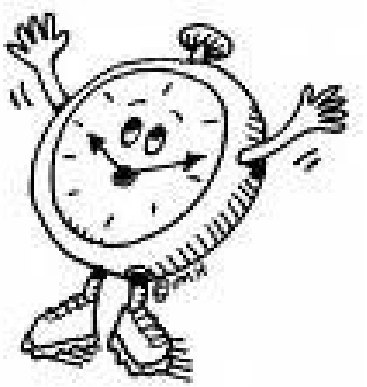
The harmonic series has applications in many different areas of pure and applied mathematics, including traffic flow, weather records and card shuffling. For these and other fascinating features of the harmonic series (including an explanation of why the series is called "harmonic" and a famous unsolved problem), go to the website

`http://plus.maths.org/issue12/features/harmonic/index.html` □

MODULAR ARITHMETIC

The basic principles of *modular arithmetic* are often discussed at primary school level, except that it is likely to be called *clock arithmetic*. The idea is quite simple: 3 hours after 10 o'clock it is 1 o'clock (because $10+3=13$ and we subtract 12). And 17 hours after 9 o'clock it will be 2 o'clock, because $17+9=26$ and we must subtract $2\times 12=24$ to find where the hand is pointing on the clock face. And 9 hours before 4 o'clock is 7 o'clock, since $4-9=-5$, and now we add 12 to get the answer 7, the required number on the clock face.

Clock arithmetic thus deals only with the numbers from 1 to 12, and whenever a calculation takes you outside that range, you add or subtract a multiple of 12 to get back onto the clock face.



You could also do clock arithmetic with a 24-hour clock. For example, 17 hours after 16h00 is 09h00, since $17+16=33$, and 24 is subtracted to get the time. Similarly, 11 hours before 09h00 is 22h00 ($9-11=-2$; add 24 to get 22).

There is nothing special about 12 or 24 in these examples. In general modular arithmetic ("arithmetic mod n") you do arithmetic with the numbers from 1 to n as normal, and if your answer goes outside that range you add or subtract multiples of n to get them back into the range. The number n is called the *modulus*, and the calculations are said to be done *modulo* n or mod n. Instead of the usual equality sign (=), the symbol $\equiv$ is used in modular arithmetic. Here are some examples:

$6+8+1-2\equiv 1 \pmod{12}$

$17-23+14+11-8\equiv 11 \pmod{24}$

Let us now look at general modular arithmetic modulo n, where n is an integer greater than 1. The key idea is that if $a\equiv b \pmod{n}$, then $a-b$ is divisible by n. If a number a leaves a remainder of r after division by n, then $a\equiv r \pmod{n}$.

It is now easy to see that if $a\equiv b \pmod{n}$ and $c\equiv d \pmod{n}$, then

$a+b \equiv c+d \pmod{n}$ and $a-b \equiv c-d \pmod{n}$. For if a and b differ by a multiple of n, and c and d also differ by a multiple of n, then certainly $a+c$ and $b+d$ differ by the sum of those two multiples of n, which is again a multiple of n.

Modular arithmetic also extends to multiplication. If $a \equiv b$ and $c \equiv d \pmod{n}$, then $ab \equiv cd \pmod{n}$. For if $a = b + pn$ and $c = d + qn$ (where p and q are integers), then $ab = (b + pn)(c + qn) = bc + (pc + qb + pqn)n$, showing that ac and bd differ by a multiple of n.

Modular arithmetic obeys the same laws as conventional arithmetic, such as the commutative laws $a + b \equiv b + a$ and $ab \equiv ba$, the associative laws $(a + b) + c \equiv a + (b + c)$ and $(ab)c \equiv a(bc)$ and the distributive laws (laws of brackets) $a(b + c) \equiv ab + ac$ and $(a + b)c \equiv ac + bc$, all of these modulo n. The proofs of these laws are very easy.

Arithmetic modulo n is therefore very simple. Working with integers, you add, subtract and multiply as usual, simply discarding multiples of n as needed.

What about division in modular arithmetic? As in ordinary arithmetic with whole numbers, division may not be possible. This is a good subject for an investigation.

Modular arithmetic is one of those topics not discussed in school mathematics, although there is no reason why it should not be part of the school curriculum. It may not even be mentioned in standard first-year university courses, and when it arises in second-year courses it will probably be dismissed with a wave of a piece of chalk and the comment that "it is really very easy so you can read it up yourself". But if the reader is at all interested in mathematical competitions and Olympiads, modular arithmetic is a valuable tool. □

QUOTE

> Algebra is the gateway to critical thinking, pivotal for success in science, engineering and technology.
>
> (California State Governor Arnold Schwarzenegger) □

SQUARE WORDS

Professor Dirk Laurie
University of Stellenbosch

The logo of the German magazine $\sqrt{WURZEL}$ suggested the following idea.

A *square word* is one in which the letters can be substituted by digits, a different digit for each different letter, such that the resulting number is a perfect square, and the digits follow the same ordering as the letters. For example, if the word contains the letters A and B, the digit for A must be less than the digit for B.

The word SQUARE is not a square word, but the word DIGEST is. Square words longer than six letters, even when repeated letters are allowed, are not easy to find.

Readers are invited to send SQUARE WORDS to the Editor. R50 Prize Vouchers will be awarded for the best entries. □

SORT THE SPORTS

Three boys, Alan, Brad and Carl each play just one sport (cricket, hockey and football) and each belongs to just one of three sports clubs (Northfield, Southgate and Westville).

One of the boys plays cricket for Northfield.
Carl does not belong to Southgate and does not play cricket.
Alan does not belong to Northfield.
Who plays football for Westville?

The first correct answer opened on June 1, from a reader in grade 9 or lower, will win a Prize Voucher worth R50. □

ARCHIMEDES ON A STAMP

The 500 lira Italian stamp shown was issued in 1983 to commemorate the Greek mathematician Archimedes (287–212 BC).

On the left of the stamp one of Archimedes' inventions is shown, the Archimedean screw. When the handle at the top is cranked, the corkscrew inside the cylinder raises the water. Archimedes is said to have invented the screw to pump water out of the bilges of a ship. The Archimedean screw is still widely used in irrigation and water engineering.

However, some historians believe that Archimedes was not the inventor of the screw named after him. There is evidence, they claim, that it was used in Babylon (present-day Iraq) several centuries earlier. King Nebuchadnezar II reigned in Babylon from 605 BC to 562 BC, and built the Hanging Gardens as a gift for his new bride, Queen Amytas. The screw was used to irrigate the Hanging Gardens.

While the true inventor of the screw may have been somebody else, there is no doubt that the bust on the right of the stamp is indeed somebody else. It is the bust of King Archidamos III, King of Sparta in the third century BC, which has been widely used (not just on the Italian stamp) as the likeness of Archimedes. There are many pictures of Archimedes, but none that are authentic. □

QUOTE

James reacted to media reports that 1Time planes were old and unsafe by saying: "We have the second-youngest fleet in South Africa, after SA Airways and Mango."

(*The Times*, 14 January 2009) □

THE GOOGLE ALGORITHM

The key to the success of Google as the number one search engine is a piece of mathematics known as the Page Rank Algorithm, or simply PageRank.

PageRank was developed at Stanford University by Larry Page (hence the name) and (later) Sergey Brin as part of a research project about a new kind of search engine. The project began in 1995 and Google was born in 1998.

PageRank is based on the methodology of citation analysis developed in the 1950s by Eugene Garfield at the University of Pennsylvania. Page and Brin acknowledge Garfield's work in their original paper.

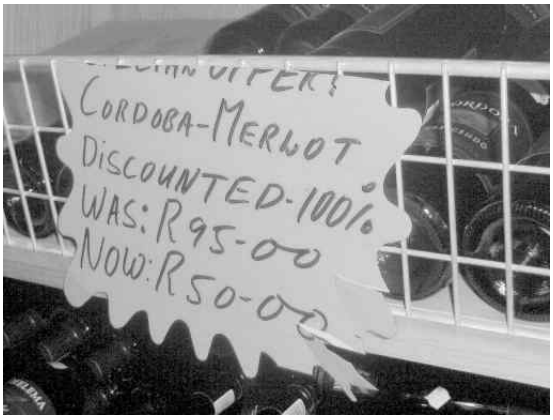
PageRank (the name is copyright and the algorithm itself has been patented) decides which webpages fit your search conditions. Other search engines do this too, but PageRank does more: it decides which of the web pages are most important, and puts them at the top of its results.

To quote Google: "PageRank relies on the uniquely democratic nature of the web by using its vast link structure as an indicator of an individual page's value. In essence, Google interprets a link from page A to page B as a vote, by page A, for page B. But, Google looks at more than the sheer volume of votes, or links a page receives; it also analyzes the page that casts the vote. Votes cast by pages that are themselves 'important' weigh more heavily and help to make other pages 'important'. "

Google assigns a weighting from 0-10 for every webpage on the Internet. The scale is logarithmic, like the Richter scale for measuring earthquakes.

PageRank has been widely acclaimed as one of the most influential and important pieces of mathematics to have been produced in the last ten years. It is also the most lucrative. Google reported revenue of \$5.54 billion in the quarter ended 30 September 2008, an increase of 31% from \$4.23 billion for the previous year. □

DISCOUNT



(Snapped by Mike Rolfe at *Mooiberge*, a discount store near Stellenbosch)

Readers are invited to send pictures of similar mathematical howlers to the Editor. Any picture published will win the sender a Prize Voucher worth R50. □

ISAAC NEWTON'S OXEN PROBLEM

The following problem was devised by Isaac Newton and appears in his book *Arithmetica Universalis* (1707).

> Twelve oxen were put to graze in a field of $3\frac{1}{3}$ acres. After four weeks they had eaten all the grass in the field, including the grass which grew during the four weeks. At the same time 21 oxen were put out to grass in a field of 10 acres; they grazed it bare in 9 weeks. How many oxen would a field of 24 acres support for 18 weeks, if they were put in at the same time?

The best solution to this problem from a reader in grade 10 or lower will win a Prize Voucher worth R50. □

SHARP CALCULATOR COMPETITION

The Sharp Calculator Competition is sponsored by Sharp Electronics (S.A.) (Pty) Limited, who have donated three calculators as prizes.

These prizes are offered for the three best solutions of the problems below. Solutions may be in English or Afrikaans. Partial solutions will be judged on their merits. The Editor reserves the right to make no award if no satisfactory entries are received, and his judgement will be final. The winner of an nth prize in a previous competition will only be eligible for an mth prize if $m < n$.

Only high schools students may compete. Entrants must give their full name and home address, age, school and grade. Solutions must be certified by a mathematics teacher at the school as the entrant's own unaided work.

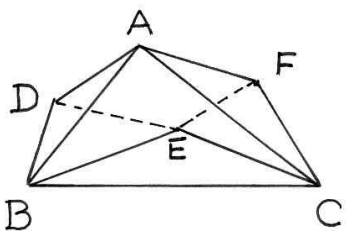
The closing date for receipt of entries is 31 May 2009, and the results will be announced in our July 2009 edition.

SHARP

1. Prove that, if
$$\frac{ac-b^2}{a-2b+c}=\frac{bd-c^2}{b-2c+d}$$
then both sides are equal to
$$\frac{ad-bc}{a-b-c+d}.$$

2. Find all positive integers m and n satisfying the equation
$$m^2-4mn+6n^2-2m-20n=29.$$

3. Three similar isosceles triangles ABD, BCE and CAF are constructed on the sides of triangle ABC as shown. Prove that $ADEF$ is a parallelogram.



□

EINSTEIN ON EUCLID

If Euclid failed to kindle your youthful enthusiasm, then you were not born to be a scientific thinker.

(Albert Einstein, speaking in Oxford in 1933) □

DOMYHOMEWORK.COM

The *Cape Times* of 9 March 2009 carried a report under the headline "French maths website sells solutions to tricky homework". According to the report:

> On `faismesdevoirs.com (domyhomework.com)`, children will be able to buy answers to simple maths problems for €5 (R67).

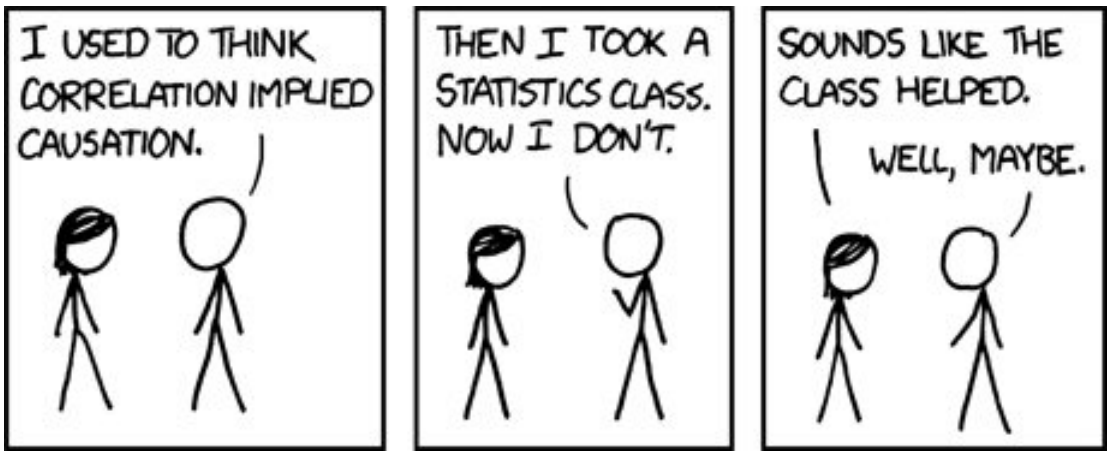
The site went live at the beginning of March, said the report. Teachers had (predictably) reacted with "shock" and "outrage", while the founder, Stephane Boukris, stoutly defended his enterprise.

It was all a storm in a teacup. Logging onto the site on the same day as the newspaper report produced nothing but a notice, signed by Stephane Boukris and dated 6 March, announcing the closure of the site, and offering to repay any payments made to it. No further explanations were given.

Surfing to `www.domyhomework.com` generated a different outcome. A webpage duly popped up with a number of interesting options and offers, such as online tutoring. However, not one of the links worked. A Google search brought up a page offering to sell the domain name `domyhomework.com` for $27 500.

It just goes to show that you can make money out of offering to do maths homework without actually having to do any maths at all. □

CORRELATION AND CAUSATION

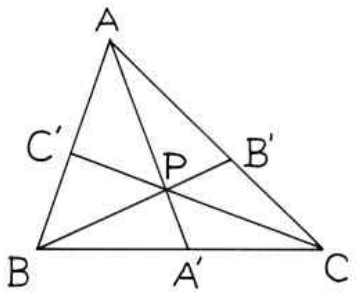


(With acknowledgement to `xkcd webcomics`) □

SHARP SOLUTIONS AND RESULTS

1. *In triangle ABC, points A', B' and C' lie on sides BC, CA and AB respectively, with AA', BB' and CC' meeting in P. Prove that*

$$\frac{PA'}{AA'} + \frac{PB'}{BB'} + \frac{PC}{CC'} = 1.$$



Denote the areas of triangles ABC, BCP, CAP and ABP by T, T_1, T_2 and T_3 respectively. Then

$$\frac{PA'}{AA'} = \frac{T_1}{T}, \quad \frac{PB'}{BB'} = \frac{T_2}{T} \quad \text{and} \quad \frac{PC'}{CC'} = \frac{T_3}{T}.$$

Adding: $\dfrac{PA'}{AA'} + \dfrac{PB'}{BB'} + \dfrac{PC}{CC'} = \dfrac{T_1}{T} + \dfrac{T_2}{T} + \dfrac{T_3}{T} = \dfrac{T_1 + T_2 + T_3}{T} = 1.$

This problem appeared in the German magazine $\sqrt{WURZEL}$ (April 2008).

2. *Let a, b and c be positive integers. Prove that, if p is a prime number greater than 3 which is a divisor of both $a+b+c$ and $a^3+b^3+c^3$, then p^2 is a divisor of $a^4+b^4+c^4-2a^2b^2-2b^2c^2-2c^2a^2$.*

We start with the factorization

$$a^3 + b^3 + c^3 - 3abc = (a+b+c)(a^2+b^2+c^2-ab-bc-ca)$$

(which should be in the toolkit of every Olympiad contender).

Since p is a divisor of both $a+b+c$ and $a^3+b^3+c^3$, it follows that p is a divisor of $3abc$. Since p is a prime number greater than 3, it follows that p is a divisor of at least one of a or b or c. By symmetry, we may assume that p is a divisor of a. Since p divides $a+b+c$, it follows that p divides $b+c$.

Now $a^4+b^4+c^4-2a^2b^2-2b^2c^2-2c^2a^2$
$= a^2(a^2-2b^2-2c^2) + (b^2-c^2)^2$
$= a^2(a^2-2b^2-2c^2) + (b+c)^2(b-c)^2$
which is divisible by p^2 since p is divisor of a and of $b+c$.

This problem appeared in the Romanian magazine *Revista de Matematică din Timişoara*, April 2008.

3. *A sequence $a_1, a_2, a_3, \ldots, a_n$ is defined by the recurrence relation*

$$a_1 = 1, \quad a_{n+1} = 3a_n + \sqrt{8(a_n^2 - 1)} \qquad (n \geq 1).$$

Prove that a_n is an integer for every n.

The recurrence relation is $a_{n+1} = 3a_n + \sqrt{8(a_n^2 - 1)}$. (1)
Starting with $a_1 = 1$, it follows easily that $a_2 = 3$, $a_3 = 17$ and $a_4 = 99$. These numbers are all integers—a promising start. We have to prove that *all* the terms in the sequence are integers.
From (1) we obtain $a_{n+1} - 3a_n = \sqrt{8(a_n^2 - 1)}$. Squaring gives

$$a_{n+1}^2 - 6a_{n+1}a_n + 9a_n^2 = 8(a_n^2 - 1).$$

Adding $8(a_{n+1}^2 - a_n^2)$ to each side, we obtain

$$9a_{n+1}^2 - 6a_{n+1}a_n + a_n^2 = 8(a_{n+1}^2 - 1).$$

Noting that the left hand side is a perfect square, we take square roots:

$$3a_{n+1} - a_n = \sqrt{8(a_{n+1}^2 - 1)}.$$

(Here we observe that, since the sequence is clearly increasing, we need only consider the positive square roots.)
Replacing $n + 1$ by n we obtain $3a_n = a_{n-1} + \sqrt{8(a_n^2 - 1)}$. (2)
Subtract equation (2) from equation (1) and we have:

$$a_{n+1} = 6a_n - a_{n-1}.$$

So it follows that if two successive terms a_{n-1} and a_n of the sequence are integers, so is the next term a_{n+1}. Since the first two terms in the sequence are integers ($a_1 = 1$ and $a_2 = 3$), so is the third term, and the fourth, and the fifth, and so on: by the Principle of Mathematical Induction, all the terms of the sequence are integers.

This problem was inspired by a similar problem in the German magazine $\sqrt{WURZEL}$, April 2008.

First Prize (a top Sharp calculator) is awarded to Arlton Gilbert (Star College, Westville) for his excellent solutions.

Excellent solutions were also received from a number of former First Prize Winners, who were therefore not eligible for a prize: Liam Baker (Mondale High School), Greg Jackson (Bishops), Desislava Nikolov (Eunice High School) and Henry Thackeray (St Alban's College). □

PRIZEWINNERS AND SOLUTIONS

Boat Problem (January 2008, p.19):
The surprisingly tricky problem leads to a cubic equation. It did not daunt Henry Thackeray (St Alban's College), who found that the speed of the boat was 24 km/h and won a Prize Voucher worth R50.

Moebius Problems (January 2008, p.24):
Henry Thackeray) solved the three problems and wins a Prize Voucher worth R50.

Arbelos Problems (January 2009, p.7):
Henry Thackeray solved the problems and wins a copy of *Count Down* by Steve Olsen (donated by the Mathematical Association of America). We offered three copies of the book as prizes, so further solutions of the Arbelos Problems can still be sent in.

Problem of the Year (January 2009, p.13):
Keegan Welken (Elkanah House) wins a copy of Steve Olsen's book *Count Down* for his solutions. Two further copies of *Count Down*, donated by the Mathematical Association of America, are still available for solutions, from readers in grade 9 or lower.

Pancake Problem (January 2009, p.12:
The pancake-flipping sequences are:

$$52413 \to 31425 \to 41325 \to 23415 \to 32145 \to 12345$$

$$\text{and} \quad 1214653 \to 356421 \to 465321 \to 564321 \to 654321 \to 123456.$$

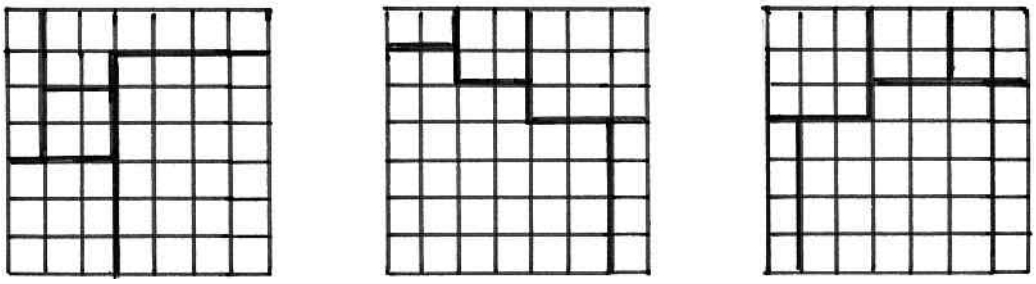
Henry Thackeray wins R50 for his solutions.

Cubic Digit Sums (January 2009, p.16):
For which numbers is it true that the sum of the digits of their ith powers is equal to the number itself? Both Greg Jackson and Henry Thackeray called in the help of a computer to solve the problem. Henry began by proving (for $i \leq 8$) that if the sum of the ith powers of the digits of n is n, then $n \leq 27i$. This inequality gave his computer an upper bound of numbers to test. Henry and Greg's computers then obtained the following results:
2nd power: 1 and 9
3rd power: 1, 8, 17, 18, 26, 27
4th power: 1, 7, 22, 25, 28, 36
5th power: 1, 28, 35, 36, 46

6th power: 1, 18, 45, 54, 64
7th power: 1, 18, 27, 31, 34, 43, 53, 58, 68
8th power: 1, 46, 54, 63.
Henry and Greg each win Prize Vouchers worth R50.

Literary Quiz (January 2009, p.31):
The mathematical villain is Professor James Moriarty, the dastardly enemy of Sherlock Holmes in *The Final Problem* by Arthur Conan Doyle. Greg Jackson (Bishops) wins a R50 Prize Voucher for his answer.

Square Cuts (January 2009, p.32):
Greg Jackson (Bishops) wins a R50 Prize Voucher for showing, in three different ways, how to dissect a 7×7 square an rearrange the pieces to form three squares (2×2, 3×3 and 6×6). His dissections are shown below. We leave the rearrangements as an (easy) problem for the reader.



Blockout (January 2009, p.32):
Younghyun Choi (Pinelands High School) wins a R50 Prize Voucher for his solution.

				23
8	4	1	7	20
1	5	1	4	11
8	9	5	8	30
6	9	3	9	27
23	27	10	28	27

Maths Muddle (January 2009, p.34):
Daniel Untiedt (Constantia Waldorf School) untangled the maths abilities and truthfulness of the five boys and wins a R50 Prize Voucher:
Lionel and James are good at decimals, Martin is good at fractions and Kevin and Neville are good at percentages. Martin and Neville were telling the truth, James and Lionel were lying and Kevin lied about James but was truthful about Martin. □

PENDLEBURY PROBLEMS

Charles Pendlebury (1854 – 1941) was a teacher at St Paul's School in London and the author of a very influential textbook, *New School Arithmetic.* First published in 1886, it ran to many editions.

Ten problems from the 1904 edition are given below. They show the high level of mathematical skills demanded at schools a century ago.

1. Two clocks point to 2 o'clock at the same instant on the afternoon of Christmas Day: one loses 7 sec. and te other gains 8 sec. in 24 hours. When will one be half an hour before the other, and what time will each clock then show?

2. A monkey, climbing up a greased pole, ascends 3 feet and slips down 2 feet in alternate seconds until he reaches the top of the pole. If the pole be 60 feet high, how long will it take him to reach the top?

3. One vessel contains 24 gallons of water; another contains 24 gallons of wine. One gallon is taken from each, and is then poured into the other. This is done 3 times. How much wine and how much water will the vessels then respectively contain?

4. A messenger sets out at the rate of 30 miles a day, but falls off in his speed 4 miles daily. Four days afterwards another sets off from the same place on the same route, travelling 50 miles the first day, but falling off like the first 4 miles daily. After what time will one overtake the other?

5. A and B run a race of $\frac{2}{3}$ mile on a course $\frac{1}{4}$ of a mile round: they run in opposite directions and A wins by 40 yards; where was B when A passed the post the first time?

6. A milkman adulterates his milk as follows:– He takes a pint out of each gallon and replaces it with water. He then takes a pint out of the mixture and replaces that with water; and he repeats the operation a third time. If he then sells his adulterated milk at the price per gallon that pure milk cost him, what is his gain per cent?

7. A train travelling at the rate of $18\frac{1}{2}$ miles an hour, started at 6 o'clock on a journey of 148 miles. A second train started from the same station; its speed was to that of the former as 8:5, and it arrived 15 minutes after it. At what time did the second train start?

8. An up train 88 yds long, travelling at the rate of 35 miles an hour, meets a down train 88 yds long at 12 o'clock, and passes it in 6 seconds. At 15 min 6 sec past 12 o'clock the up train meets a second down train 132 yds long, and passes it also in 6 seconds. At what time will the second down train run into the first?

9. A prisoner of war, in attempting to escape to a friendly ship moored within sight of land, swims out at the rate of 3 miles per hour, and is assisted by the tide, which ebbs at the rate of 200 yards per hour. After he had been swimming for $\frac{3}{4}$ of an hour, the tide turns and impedes him as much as it had previously assisted him. He also slackens his speed from 3 to $2\frac{1}{2}$ miles per hour, and succeeds in reaching the vessel in another hour. How far was the vessel from the shore?

10. Four men and 10 boys together do one-half of a piece of work in 6 days. Two more men and 2 more boys having joined them, another one-third of the work is done in the next 3 days: how many more boys must then be put on if the rest of the work is to be finished in one day more?

Answer are on page 31.

CROSSING THE RIVER

Explorers Andy, Ben, Carl and Dave find themselves in the middle of a jungle at midnight, and must cross a dangerous rope bridge to get to safety. Andy is very fit, and can cross the bridge in 5 minutes. Ben is only moderately fit, and can get across the bridge in 10 minutes. Both Carl and Dave are very tired, and will take 15 minutes and 20 minutes respectively to cross the bridge.

It is pitch dark and anybody crossing the bridge needs a torch. Unfortunately they have only one working torch. The bridge is not strong enough to support more than two of them.

How should they arrange to cross the bridge in as short a time as possible?

The best answer, from a reader in grade 10 or lower, will win a Prize Voucher worth R50. □

TRIGONOMETRIC TRIANGLE

From the basic trigonometric identities

$$\begin{aligned}\cos(A+B) &= \cos A\cos B - \sin A\sin B\\ \sin(A+B) &= \sin A\cos B + \cos A\sin B\end{aligned}$$

we may obtain expressions for $\cos 2\theta$ and $\sin 2\theta$ terms of $\cos\theta$ and $\sin\theta$:

$$\begin{aligned}&\cos 2\theta = \cos\theta\cos\theta - \sin\theta\sin\theta\\ &= \cos^2\theta - \sin^2\theta\\ &= \cos^2\theta - (1-\cos^2\theta)\\ &= 2\cos^2\theta - 1.\\ &\sin 2\theta = \sin\theta\cos\theta + \cos\theta\sin\theta = 2\sin\theta\cos\theta.\end{aligned}$$

In the same way we can express $\cos 3\theta$, $\cos 4\theta$, $\cos 5\theta$ etc in terms of $\cos\theta$:

$$\begin{aligned}&\cos 3\theta = 4\cos^3\theta - 3\cos\theta\\ &\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1\\ &\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta\\ &\cos 6\theta = 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1\end{aligned}$$

A pattern is now becoming clear: the signs alternate and the powers of $\cos\theta$ start at n and decrease by 2. The main problem is to work out how the coefficients are formed.

The pattern of the coefficients is shown in figure 1. The rule used is $c = 2a + b$, where c is related to a and b as shown in figure 2.

						1
					1	0
				2	0	1
			4	0	3	0
		8	0	8	0	1
	16	0	20	0	5	0
32	0	48	0	18	0	1

Figure 1

	b		
		a	
	c		

Figure 2

The key identity in explaining why the trig triangle works is the identity

$$\cos n\theta = 2\cos(n-1)\theta\cos\theta - \cos(n-2)\theta,$$

which is proved as follows:

$$\begin{aligned}
&\cos n\theta + \cos(n-2)\theta \\
&= \cos\big((n-1)\theta + \theta\big) + \cos\big((n-1)\theta - \theta\big) \\
&= \cos(n-1)\theta\cos\theta - \sin(n-1)\theta\sin\theta + \cos(n-1)\theta\cos\theta + \sin(n-1)\theta\sin\theta \\
&= 2\cos(n-1)\theta\cos\theta.
\end{aligned}$$

What about similar formulas for $\sin n\theta$? We leave this to the reader as a suggestion for an interesting investigation.

Reference

Brian Newbould: Trigonometrical triangles, in *The Mathematical Gazette*, LVI(398), December 1972, pp. 316-318.

FERMAT PRIMES

Pierre de Fermat (1601 – 1665) is famous for Fermat's Last Theorem, which took mathematicians over 300 years to crack.

In 1640 Fermat stated that every number of the form $2^{2^n} + 1$ is prime. These number are now called Fermat numbers, and denoted by F_n:

$$\begin{array}{lcccccccccc}
F_1 & = & 2^{2^1} & + & 1 & = & 2^2 & + & 1 & = & 5 \\
F_2 & = & 2^{2^2} & + & 1 & = & 2^4 & + & 1 & = & 17 \\
F_3 & = & 2^{2^3} & + & 1 & = & 2^8 & + & 1 & = & 257 \\
F_4 & = & 2^{2^4} & + & 1 & = & 2^{16} & + & 1 & = & 65537
\end{array}$$

All these numbers are prime, but $F_5 = 2^{32} + 1 = 4294967297$ turned out not to be prime. Fermat was wrong.

A factor of F_5 can be found by hand:

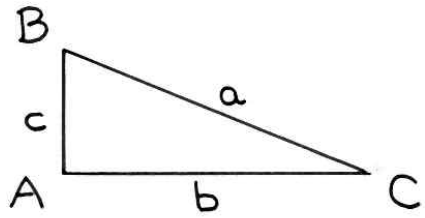
$$\begin{aligned}
F_5 &= 2^{32} + 1 \\
&= 15 \times 2^{28} + 2^{28} + 1 \\
&= 15 \times (2^7)^4 + 2^7 \times (2^7)^3 + 1 \\
&= 15 \times (2^7)^4 + (5^3 + 3) \times (2^7)^3 + 1 \\
&= 3 \times (2^7)^3(5 \times 2^7 + 1) + (5 \times 2^7)^3 + 1
\end{aligned}$$

Since in general $a + b$ is a factor of $a^3 + b^3$, we know that $5 \times 2^7 + 1$ is a factor of $(5 \times 2^7)^3 + 1$. So $5 \times 2^7 + 1 = 641$ is a factor of F_5. $\square$

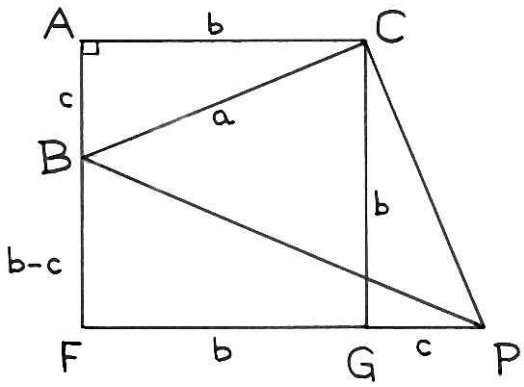
PYTHAGORAS PROVED AGAIN

It is said that there are hundreds of proofs of the Theorem of Pythagoras. Here is one contributed by Benoit Baudelet and Michel Sebille to the first issue of a new journal *Losanges*, published by the *Société Belge des Professeurs de Mathématiques*.

Start with triangle ABC, right-angled at A, and draw the square $ACGF$ as shown. Produce FG to P so that $GP = AB$ and join CP AND . Then triangle CGP is congruent to triangle ABC, and $\angle BCP$ is a right angle.



$$a^2 = b^2 + c^2$$



Then $b^2 = \text{Area}(ACGF)$
$= \text{Area}(BAC) + \text{Area}(BCGF)$
$= \text{Area}(CGP) + \text{Area}(BCGF)$
$= \text{Area}(BCPF)$
$= \text{Area}(BCPF)$
$= \text{Area}(CPB) + \text{Area}(PBF)$
$= \frac{1}{2}a^2 + \frac{1}{2}(b-c)(b+c)$
$= \frac{1}{2}(a^2 + b^2 - c^2)$.

So $2b^2 = a^2 + b^2 - c^2$, from which it follows that $a^2 = b^2 + c^2$. □

PERCENTAGE ERROR

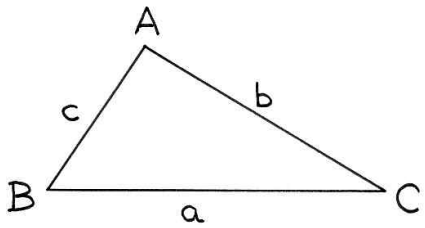
"Commercial radio stations could end up paying 250 percent more in annual licence fees. ... This is based on Icasa's draft licence fee regulations, which would have commercial broadcasters pay 2.5 percent of their annual turnover. At the moment, radio stations pay 1 percent."

(*The Times*, 16 January 2009)

Readers are invited to send similar percentage errors found in the press to the Editor. A Prize Voucher worth R50 will be won for each entry. □

HERON'S FORMULA

How can you find the area of a triangle if you know its side lengths, but not its altitude? Heron of Alexandria (c. 50 A.D.) found a neat formula for the area.

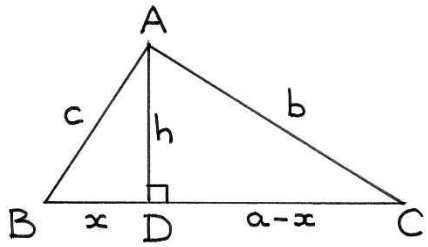


If the sides of the triangle ABC are a, b and c, then the area is

$$\sqrt{s(s-a)(s-b)(s-c)}$$

where $s = \frac{1}{2}(a+b+c)$, the "semiperimeter" of the triangle.

Let Δ denote the area of the triangle. Drop a perpendicular AD from A onto the side BC, meeting BC at D. Let $AD = h$ and $BD = x$. Then $DC = a - x$.



By the usual formula for the area of a triangle (half base times height) we have $\Delta = \frac{1}{2}ah$. The problem is to express h in terms of a, b and c

We start by applying the Theorem of Pythagoras to triangles ABD and ACD), obtaining $h^2 = c^2 - x^2$ and $h^2 = b^2 - (a-x)^2$. Subtracting the two equations gives $ax = \frac{1}{2}(a^2 + c^2 - b^2)$. (1)

It is convenient to work with the square of the area:

$\Delta^2 = (\frac{1}{2}ah)^2 = \frac{1}{4}a^2h^2 = \frac{1}{4}a^2(c^2 - x^2)$.

Multiplying through by 4 gives (using (1))
$4\Delta^2 = a^2c^2 - a^2x^2 = a^2c^2 - \frac{1}{4}(a^2 + c^2 - b^2)^2$.

Multiplying through by 4 again gives
$16\Delta^2 = 4a^2c^2 - (a^2 + c^2 - b^2)^2$
$= (2ac)^2 - (a^2 + c^2 - b^2)^2$
$= \big(2ac - (a^2 + c^2 - b^2)\big)\big(2ac + (a^2 + c^2 - b^2)\big)$ (difference of squares)
$= \big(b^2 - (a-c)^2\big)\big((a+c)^2 - b^2\big)$
$= \big(b - (a-c)\big)\big(b + (a-c)\big)\big((a+c) - b\big)\big((a+c) + b\big)$ (difference of squares)
$= (2s - 2a)(2s - 2c)(2s - 2b)(2s)$ (since $a + b + c = 2s$)
$= 16s(s-a)(s-b)(s-c)$.

Cancelling the common factor of 16 and taking square roots gives Heron's Formula:

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)} \quad \square$$

HASH FUNCTIONS REVISITED

Professor Dirk Laurie
University of Stellenbosch

> "If several documents can have the same MD5 hash value, then MD5 is useless for authentication."
>
> (*Mathematical Digest* (October 2008, page 9))

What Stevens, Lenstra and de Weger showed is that it is possible, by a computer program which searches through about 2^{50} possibilities, to find suffixes S1 and S2 for two given documents P1 and P2 such that P1+S1 and P2+S2 have equal length and give the same MD5 checksum, in other words, are indistinguishable by currently used MD5 checking practice.

If the document is a computer program, in both cases the suffixes are transparent to the user, since if P1 and P2 are correct, they will not attempt to access the part of memory occupied by S1 and S2. If the document is a graphics file, on the other hand, the user will instantly notice that S1 and S2 are pure garbage.

They are careful to point out that this does not imply that they can forge a document which produces a given MD5 checksum: they must be able to modify both documents. However, it does imply that a malicious person on a team responsible for creating a computer program could prepare two versions of the program, give one of them to the other team members to OK and eventually release with their stamp of approval, and use the other one clandestinely for his own nefarious purposes.

For documents like scanned-in cheques, contracts, etc., it has not yet been demonstrated that MD5 is vulnerable. But it is now regarded as suspect.

Reference

`<http://www.win.tue.nl/hashclash/SoftIntCodeSign>` □

QUOTE

Eternity is very long, especially towards the end.

(Woody Allen) □

STAMP PROBLEM

Six stamps form a 2 ×3 rectangle as shown.

In how many ways can two or more stamps be separated from the six so that the shape they form is connected, but is not a rectangle?

This problem was posed by Brian Stokes in *The New Zealand Mathematics Magazine*, November 2008.

If you label the stamps A, B, C, D, E, F as shown and count the arrangements systematically, you get 22 possibilities.

A	B	C
D	E	F

Three stamps: ABD, ABE, BCE, BCF, DEA, DEB, EFB, EFC
Four stamps: ABCD, ABCE, ABCF, DEFA, DEFB, DEFC, ABEF, BCDE
Five stamps: ABCDE, ABCDF, ABCEF, ABDEF, ACDEF, BCDEF

What happens if you start with a 2 × 4 rectangle of stamps? The first correct answer opened on 1 June 2009, from a reader in grade 9 or lower, will win a Prize Voucher worth R50. □

This problem can be extended to larger rectangles of stamps, and could form the basis of an interesting investigation or project. □

THE GEOMETRY OF GOD

If God exists and if He really did create the world, then, as we all know, He created it according to the geometry of Euclid.

(Ivan, in *The Brothers Karamazov*, by Fyodor Dostoyevsky) □

SUMS AND POWERS

If the eight numbers from 0 to 7 are partitioned into two sets: 0, 3, 5, 6 and 0, 1, 2, 4, 7, we have the following results:

$$0^1 + 3^1 + 5^1 + 6^1 = 1^1 + 2^1 + 4^1 + 7^1$$
$$0^2 + 3^2 + 5^2 + 6^2 = 1^2 + 2^2 + 4^2 + 7^2$$

Similarly, if the numbers from 0 to 16 are partitioned into two sets 0, 3, 5, 6, 9, 12, 15, and 1, 2, 4, 7, 8, 11, 13, 14 we have

$$0^1+3^1+5^1+6^1+9^1+10^1+12^1+15^1 = 1^1+2^1+4^1+7^1+8^1+11^1+13^1+14^1$$
$$0^2+3^2+5^2+6^2+9^2+10^2+12^2+15^2 = 1^2+2^2+4^2+7^2+8^2+11^2+13^2+14^2$$
$$0^3+3^3+5^3+6^3+9^3+10^3+12^3+15^3 = 1^3+2^3+4^3+7^3+8^3+11^3+13^3+14^3$$

The pattern continues. The numbers from 0 to 31 can be partitioned into two sets, which have equal sums, equal sums of squares, equal sums of cubes and equal sums of fourth powers.

What are the two sets? Can you work out a simple rule for dividing the numbers into the two sets? The best answer will win a Prize Voucher worth R50. □

CICERO ON MATHEMATICS

Among them [the Greeks] geometry was held in highest honour; nothing was more glorious than mathematics. But we [the Romans] have limited the usefulness of this art to measuring and calculating.

(*Tusculan Disputations*, Book I) □

VERBAL ADDITION

Here is a verbal proof that $2 + 11 - 1 = 12$.

Write down	T	W	O						
Add ELEVEN	T	W	O	E	L	E	V	E	N
Subtract O N E	T	W		E	L		V	E	

□

LITERARY QUIZ

Joe bought a roll, and reduced his purse to the condition (with a difference) of that celebrated purse of Fortunatus, which, whatever were its favoured owner's necessities, had one unvarying amount in it. In these real times, when all the Fairies are dead and buried, there are still a great many purses which possess that quality. The sum–total they contain is expressed in arithmetic by a circle, and whether it be added to or multiplied by its own amount, the result of the problem is more easily stated than any known in figures.

In which novel by Charles Dickens does this paragraph appear? The first correct answer, from a reader in grade 10 or lower, will win a Prize Voucher worth R50. □

ANSWERS TO PENDLEBURY PROBLEMS

1. In 120 days; i.e. on April 24th or April 23rd; 1.46 p.m.; 2.16 p.m.
2. 1 min. 55 sec.
3. The one, $21\frac{35}{144}$ gall. of water and $2\frac{109}{144}$ gall. of wine; the other $2\frac{109}{144}$ gall. of water and $21\frac{35}{144}$ gall of wine.
4. $2\frac{2}{3}$ days after 2nd starts; at the end of 124 miles.
5. 15 yds. from it.
6. $49\frac{93}{343}\%$
7. At 9.15.
8. 15 min. 18 sec. past 1.
9. 4 miles 1270 yds.
10. 12 more. □

mathematical digest

No. 156 July 2009

Schickard's Clock

mathematical digest

Number 156 **July 2009**

A man is like a fraction whose numerator is what he is and whose denominator is what he thinks of himself. The larger the denominator the smaller the fraction.

(Leo Tolstoy)

Department of Mathematics and Applied Mathematics
University of Cape Town

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Cover: Schickard's Clock was the first mechanical calculator. See article on page 10.

ALEXANDRIAN ASTRONOMY

Case Rijsdijk

This is a re-discovery of some aspects of early Greek astronomy as practised by astronomers who worked in the Great Library of Alexandria from about 300 BCE to about 150 CE. Aristarchus used a lunar eclipse to estimate the size of the Moon and later Hipparchus improved on this. The same geometry was again used much later by Copernicus in *De Revolutionibus*.

If the radius of the Earth is calculated using Eratosthenes' method and an eclipse of the Moon is observed, then

- the distance to the Moon and,
- the size of the Moon,

can be found using some straight forward geometry.

Geometry of Lunar Eclipses

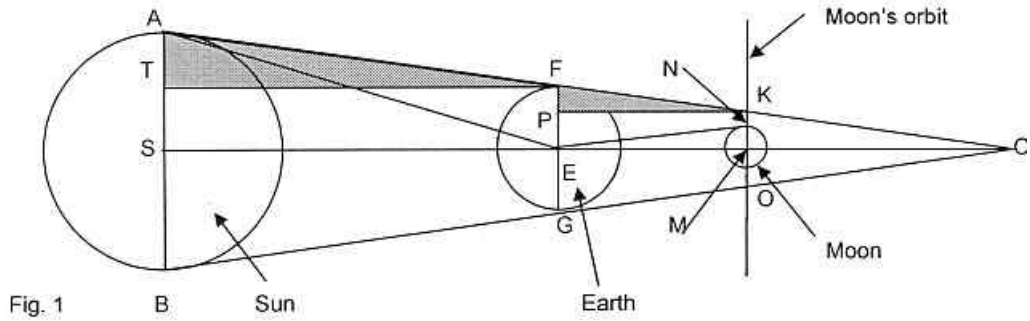


Fig. 1

Earth - Sun distance	ES	$= D$	Earth's radius $EF = r$
Earth - Moon distance	EM	$= d$	Moon's radius $MN = p$
Sun's radius	AS	$= R$	Radius of Earth's shadow $MK = s$

The triangle FGC represents the Earth's shadow and line KMO the Moon's diameter. Assume that the Sun is n times further from the Earth than the Moon is, i.e.

$$SE = n.EM \quad \text{i.e.} \quad D = n.d \quad \text{alternatively} \quad n = \frac{SE}{EM} = \frac{D}{d} \qquad (1)$$

As can be seen during an eclipse of the Sun, the Sun and Moon appear to be the same size, i.e. subtend more or less same angle in the sky and so $\angle AES = \angle MEN$; therefore triangles ASE and MNE are similar. (They don't look similar because the diagram is not drawn to scale.) Then using (1) above:

$$MS = n.EM \quad \text{i.e.} \quad R = n.p \quad \text{alternatively} \quad n = \frac{AS}{MN} = \frac{R}{p} \tag{2}$$

Triangles ATF and FPK are also similar and so:

$$\frac{TF}{PK} = \frac{AT}{FP} \quad \text{i.e.} \quad \frac{SE}{EM} = \frac{AS - TS}{FE - PE} \quad \text{so} \quad \frac{SE}{EM} = \frac{AS - FE}{FE - MK} \tag{3}$$

or using the assigned symbols and substituting (1) and (2) into (3) we get:

$$\frac{D}{d} = \frac{nd}{d} = \frac{R - r}{r - s} \quad \text{i.e.} \quad n = \frac{np - r}{r - s} \tag{4}$$

Equation (4) is now re-arranged to get: $$p(1 + \frac{s}{p}) = r(1 + \frac{1}{n}) \tag{5}$$

To find the distance to the Moon, d, values for n, r and p in terms of d and the ratio s/p are needed.

The ratio s/p can be found during an eclipse of the Moon and using some simple geometry.

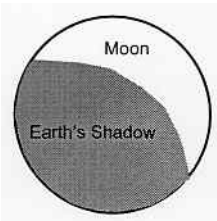


Fig. 2

Fig. 2 shows an image of the Moon passing into the Earth's shadow or Umbra. This can then be used to find the ratio s/p.

A and B are the points of intersection of the circles. The perpendicular bisector of the chord of a circle will pass through the centre of that circle. Referring to Figure 3 alongside, N is the mid-point of AD and N' is the mid-point of DB. The perpendiculars from these points meet at P. DP would then be the radius of the Moon. Similarly OC is the radius of the Earth's shadow at the distance of the Moon's orbit and then the ratio:

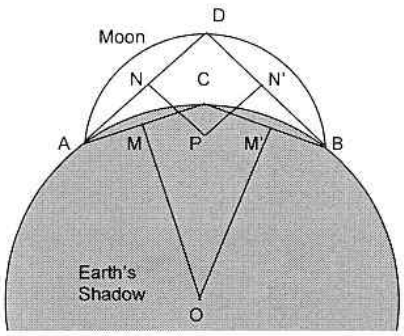


Fig. 3

$$\frac{OC}{DP} = \frac{s}{p} \tag{6}$$

Careful measurement gave them the value $s/p = 8/3$.

The angular diameter of the Moon was measured using a small glass bead, and found that angle subtended was 1/110 for the Moon, so the radius would subtend an angle of 1/220, so $p = d/220$.

In modern day units, Eratosthenes found the Earth's radius to be about 6 500 km = r

So substituting the above into equation (5) $p(1 + \frac{s}{p}) = r(1 + \frac{1}{n})$ we get:

$$d = 390\,000(1 + \frac{1}{n})$$

This leaves n. Now Aristarchus realized that the Sun was further away from the Earth than the Moon was, but he did not know how much further. He tried to work it out using geometry but the value he got, 20, was far too small. However if it is assumed that n is very large, (it is $\tilde{4}00$) then the ratio $1/n$ is very small and can be neglected. This gives a maximum value for the distance to the Moon as 390 000 km and then using the equation $p = d/220$, the minimum radius of the Moon is 1 773 km.

Comparing Aristarchus' values with today's

	Today	Aristarchus
Diameter	3 476 km	3 546 km
Mean Distance	384 404 km	390 000 km
	(range being 358 020 to 405 936 km)	

Not too bad without proper instruments! □

LITERARY QUIZ

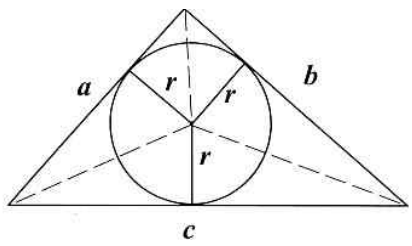
> "While the individual man is an insoluble puzzle, in the aggregate he becomes a mathematical certainty. You can, for example, never foretell what any one man will do, but you can say with precision what an average number will be up to. Individuals vary, but percentages remain constant. So says the statistician."

The above was said by a famous fictional detective. Who was he, in what novel does the quotation occur, and who wrote it? The first three readers to send in correct answers will win free 2009 subscriptions to *Mathematical Digest*. □

THE IN-RADIUS OF A RIGHT-ANGLED TRIANGLE

Marcus Bizony
Bishops

The incentre of a triangle is at the same distance r from each of its sides.
The accompanying diagram shows a triangle and its incircle, and also three dotted lines demonstrating how the area of the triangle can be seen as the sum of the areas of three triangles.



Thus the area of the whole triangle is

$$\frac{1}{2}ra + \frac{1}{2}rb + \frac{1}{2}rc = \frac{1}{2}r(a+b+c),$$

and we deduce that for any triangle the length of its inradius is given by the simple formula $r = \dfrac{2 \times \text{area}}{\text{perimeter}}$.

In the case of a right-angled triangle with legs of length a and b, this becomes

$$r = \frac{ab}{a+b+\sqrt{a^2+b^2}} \qquad (1)$$

Now if the sides of the triangle are all integers, so that a, b, $\sqrt{a^2+b^2}$ form a Pythagorean triple, it is obvious that r will be rational, since the numerator and denominator in (1) will be integers.

In fact the expression simplifies further:

$$\begin{aligned} r &= \frac{ab}{a+b+\sqrt{a^2+b^2}} = \frac{ab}{a+b+\sqrt{a^2+b^2}} \cdot \frac{a+b-\sqrt{a^2+b^2}}{a+b-\sqrt{a^2+b^2}} \\ &= \frac{ab(a+b-\sqrt{a^2+b^2})}{(a+b)^2-(a^2+b^2)} = \frac{ab(a+b-\sqrt{a^2+b^2})}{2ab} = \frac{1}{2}\left(a+b-\sqrt{a^2+b^2}\right) \end{aligned}$$

Now we see that r is not only rational, but has the form $\frac{R}{2}$, where R is an integer.

Looking a bit more closely, let us assume that a and b are co-prime, so either they are both odd or one of them (say b) is odd and the other even. Then in $a+b-\sqrt{a^2+b^2}$ we either have even + odd $+\sqrt{\text{odd}}$ or we have odd + odd $+\sqrt{\text{even}}$. But we know that for a perfect square (such as a^2+b^2 is

in these cases), the parity of its square root is always the same as its own – and so it is clear that $a + b - \sqrt{a^2 + b^2}$ must be even, and therefore that r is not just rational, it is actually guaranteed to be an integer.

Now if a and b are not co-prime, i.e. if they have some common factor h, h will also be a factor of the hypotenuse, and the triangle will be an enlarged version of some more primitive triangle. Since the radius of that smaller triangle will be an integer, so will the radius of the larger one.

This completes a proof that the inradius of a right-angled triangle all of whose sides are integral is itself integral.

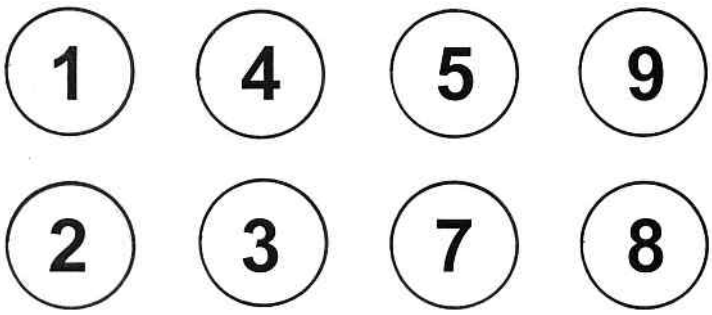
A much quicker proof, and one giving better insight into the actual value of r, is available if we use the well-known result that every Pythagorean triple has the form $m^2 - n^2$, $2mn$, $m^2 + n^2$ for some natural numbers m, n. In this formulation, we get

$$\begin{aligned} r &= \frac{2 \times \text{area}}{\text{perimeter}} = \frac{(m^2 - n^2)mn}{m^2 - n^2 - 2mn + m^2 + n^2} = \frac{mn(m-n)(m+n)}{2m(m+n)} \\ &= n(m-n) \end{aligned}$$

□

COUNTER PROBLEM

Eight numbered counters have been arranged in two rows, as shown.



In this arrangement, the sums of the numbers in the two rows are different. Can you switch a counter in the first row with a counter in the second row, so that the sums in the two rows are the same?

The first three correct solutions will win each win a free 2010 subscription to *Mathematical Digest*. □

TWO GEOMETRY COMMENTS

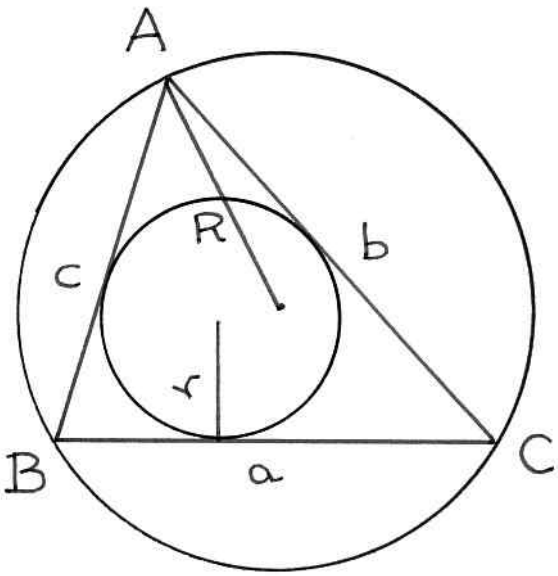
Geoff Smith
University of Bath

Padoa's Inequality

Here is another proof of Padoa's inequality (*Mathematical Digest*, January 2009, page 17) which states:

If a, b and c are the sides of a triangle, then

$$abc \geq (a+b-c)(b+c-a)(c+a-b).$$



If r, R and s denote, respectively, the inradius, circumradius and semiperimeter of triangle ABC, and $[ABC]$ denotes the area of the triangle, then $\dfrac{abc}{4R} = [ABC] = rs$ and hence $\dfrac{abc}{a+b+c} = 2Rr$.

It therefore suffices to show that, by Heron's Formula (see *Mathematical Digest* **155** (April 2009) page 27):

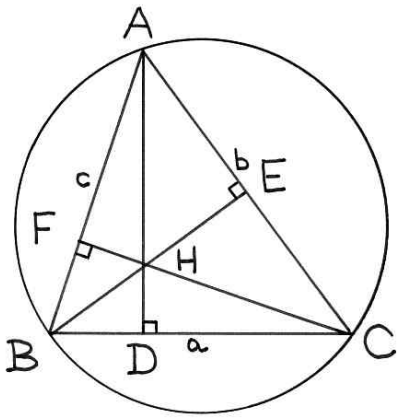
$$2Rr(a+b+c)^2 \geq 16[ABC]^2$$

But $r(a+b+c) = 2[ABC]$, so it suffices to show that $R(a+b+c) \geq 4[ABC]$.

Now $R \geq 2r$ by Euler's Inequality (see *Mathematical Digest* 128 (July 2002)), so $R(a+b+c) \geq 2r(a+b+c) = 4rs = 4[ABC]$.

A Sharp Problem Revisited

The solution to Sharp problem 3 in *Mathematical Digest* **154** (January 2009) on pp 23-24 can be made simpler. The problem stated was:



A triangle is inscribed in a circle of diameter 1. Prove that $AH = \cos A$, where H is the orthocentre of the triangle (the point of intersection of the altitudes).

$AH \sin C = AB \cos A$ ($\angle AHE = \angle C$) $= \sin C \cos A$ ($2R = 1$ and sine rule). So $AH = \cos A$. □

COUNT DOWN COMPETITION

The Mathematical Association of America has donated copies of Steve Olsen's book *Count Down*, to be used as prizes for competitions. Here is a "count down" problem.

Can you put the standard arithmetical symbols between the numbers

$$10 \qquad 9 \qquad 8 \qquad 7 \qquad 6 \qquad 5 \qquad 4 \qquad 3 \qquad 2 \qquad 1$$

so that the calculation gives exactly 2009?

For example, using the factorization $2009 = 49 \times 41$ we have:

$$(-10 + 9 + 8) \times 7 \times (6 \times 5 + 4 \times 3 - 2 + 1) = 2009$$

$$\big(10 + (9 - 8)^7 + 6 \times 5)\big) \times \big((4 + 3)^2 \times 1\big) = 2009.$$

Noting that $2002 = 2 \times 7 \times 11 \times 13$, we have:

$$(10 + \sqrt{9}) \times (87 + 65 + \sqrt{4}) + 3 \times 2 + 1 = 2009.$$

and a quite devious solution is

$$\sqrt{10 \times \sqrt{(9 + 8 - 7)^6}} \times 5 \times 4 + 3^2 \times 1 = 2009.$$

What other solutions can be found? Send your solutions to the Editor. The three best entries will each win a copy of *Count Down*. □

THE HAPPY ENDING PROBLEM

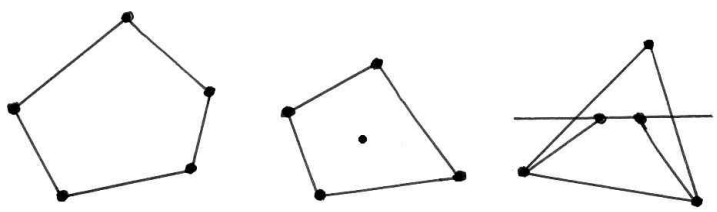
The Happy Ending Problem was devised in Hungary in 1933 by Esther Klein, who posed it to her friends Paul Erdös and George Szekeres.

> Any set of five points in the plane, with no three lying on the same straight line, has a subset of four points that form the vertices of a convex quadrilateral.

(A "convex" figure is one that has no indentations.)

The problem is easily solved, by considering the *convex hull* of the set of five points. This is the smallest convex set that contains all five points.

There are three possibilities for the convex hull.



If the convex hull is a pentagon, then any four points form a convex quadrilateral.

If the convex hull is a quadrilateral, that's it.

If the convex hull is a triangle, then two points lie inside the triangle. Those two points lie on a line that cuts the triangle in two. That line cannot pass through a vertex, since it was stated at the beginning that no three points lie on the same line. So two points of the triangle lie on one side of the line. These two points, and the two points inside the triangle, form a convex quadrilateral.

What about similar problems of the same type? Szekeres and Erdös proved that, if there are nine points in the plane (no three collinear), then it is always possible to select five that will form a convex pentagon.

How many points in the plane do you need to guarantee that among them there are six that form a convex hexagon? Szekeres and Erdös suggested in 1935 that 17 points would be needed, but could not prove their conjecture, nor to this day has anybody else.

Erdös called it the "Happy Ending Problem", because George Szekeres and Ester Klein got married and lived Happily Ever After. □

ON CONVERSES

In Euclidean geometry, the theorem and its converse are often linked. For example, the converse of

> "Congruent triangles are similar"

(which is known to be true) is

> "Similar triangles are congruent"

which is not true.

Consider the famous theorem, known as the *Pons Asinorum*, or "Bridge of Asses":

> "The angles at the base of an isosceles triangle are equal."

This result, proved by Euclid as Proposition 5 of Book I of *The Elements* is well-known to be true. Its converse is

> "If the angles at the base of a triangle are equal, the triangle is isosceles."

Is the converse true? The usual answer is YES, followed by a simple congruency argument along the lines "Suppose we have a triangle ABC, with $AB = AC$. Drop a perpendicular from A to meet BC in D. The triangles BAD and CAD are then quickly seen to be congruent (two angles and a side) and it follows that $AB = AC$."

There is an opening here for the Class Smartypants to object, with the following example:

> Consider the triangle ABC, with $AB = 8$, $AC = 9$ and $BC = 17$. This is clearly not isosceles, but its base angles are equal.

It won't take long for somebody to protest that the example is not genuine, but is a "degenerate" triangle. Smartypants can then counter with the observation that the original theorem is true for degenerate triangles; even better, in all degenerate triangles two of the angles are equal.

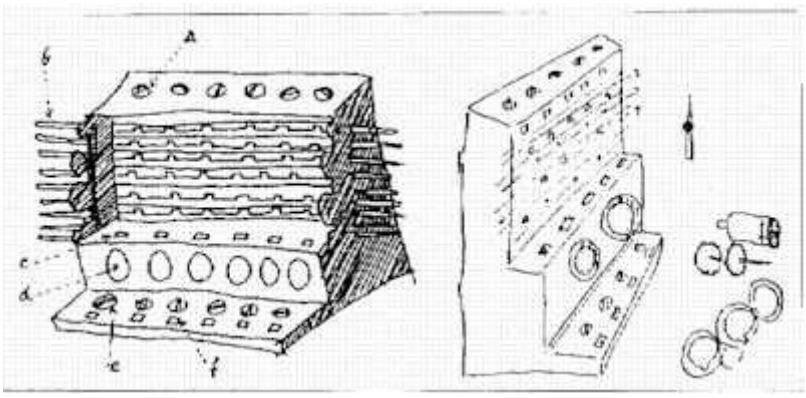
So why does the proof of the converse, as outlined above, break down for degenerate triangles? We leave this to the reader to puzzle out. $\square$

SCHICKARD'S CLOCK

Wilhelm Schickard (1592-1635) was a professor of Hebrew at the University of Tübingen. After meeting the astronomer Johannes Kepler he became fascinated with astronomy, and in fact became Professor of Astronomy in 1631.

Kepler's increasingly accurate observations of the movements of the planets, which eventually led him to deduce that the planets moved in elliptical orbits, required tedious calculations. Schickard, a gifted artist, designer and inventor, came up with the idea of helping his friend by building a machine that could do the four basic calculations of arithmetic.

It was round about that time that logarithms were invented by the Scottish mathematician John Napier, enabling multiplication to be turned into addition. But this process required calculating tables of logarithms. Schickard decided to use another brainwave of Napier, his "calculating rods", which enabled one to multiply large numbers by single digits without memorizing tables. To extend the method to a multiplier with more than one digit, Schickard designed a machine which mimics the school method of "long multiplication": single-digit multiplication, followed by addition of the partial products.



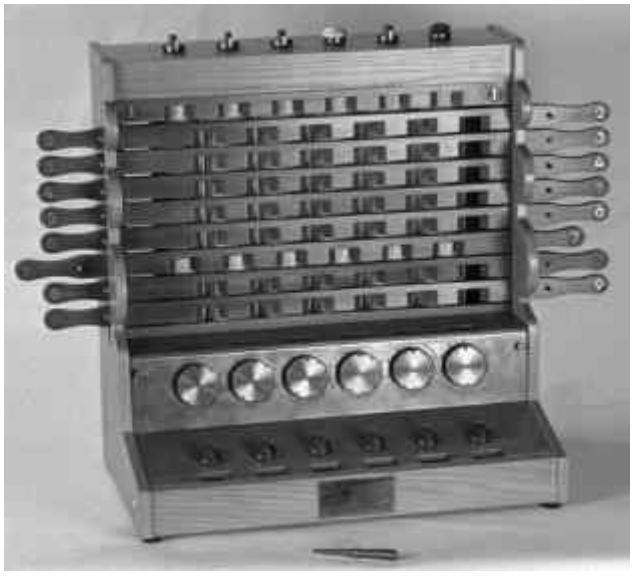
In 1623 Schickard designed his calculator, which he called a Clock, and sent the above sketches to Kepler. The Clock, he explained in a letter to Kepler, consisted of three sections. The main central section performed

addition and subtraction by means of a system of cogs d and dials c. The cogs included a "carry" mechanism: when one wheel completes a full turn, a single tooth moves the wheel to its left one-tenth of a turn. Subtraction was done by turning the wheels backwards.

Multiplication and division were carried out in the upper section, through a system of cylindrical versions of Napier's bones. They were set with knobs a on the top and slides b on the left. At the bottom was a counting mechanism, with knobs e and dials f. The Clock could handle numbers up to six digits long, and included a bell which rang to warn of an overflow.

Schickard built two of his machines. One was to have been sent to Kepler, but was destroyed in a fire. Nobody knows what happened to the other. Schickard died of the plague in 1635 and the plans of his Clock disappeared in the turbulence of the Thirty Years War, which ravaged Central Europe from 1618 to 1648.

Three hundred years later, Schickard's plans finally reappeared, only to be lost again in the turmoil of another war. The plans fortunately survived again, and were rediscovered in 1956. A working model of Schickard's Clock was built in 1960 by Bruno Von Freytag-Löringhoff . It can be seen today be seen in the University of Tübingen.



For many years the French mathematician and philosopher Blaise Pascal (1623-1662) had been regarded as the inventor of the first mechanical calculator, which could handle five-digit calculations. Pascal did not know that Wilhelm Schickard had built a calculator in the very year that he had been born. After the rediscovery of his designs, Schickard's Clock is regarded today as the first example of a mechanical computer. □

PAMO IN PRETORIA

The annual Pan African Mathematics Olympiad took place in Pretoria in April. Thirteen countries entered: Algeria, Benin, Burkino Faso, Ivory Coast, Kenya, Mali, Mozambique, Nigeria, Senegal, South Africa, Swaziland, Uganda and Zimbabwe.

As in the International Mathematical Olympiad, the students write two papers, each $4\frac{1}{2}$ hours in length, on successive days, with each paper consisting of just three questions.

The papers are reproduced below.

While the papers were being set, the teams were taken on an excursion to the Cradle of Humankind and after the papers were written they enjoyed another excursion to Gold Reef City.

The results were announced at a Gala Prize Giving and dinner.

In first place was Isaac Konan (Ivory Coast), with South Africans Arlton Gilbert (Star College, Durban) second and Greg Jackson (Bishops) third. Fourth was Ibrahim Ouattara (Ivory Coast).

These top four contestants won Gold Medals. Desi Nikolov (Eunice High School, Bloemfontein), tied with Mahdi Finnegan (Zimbabwe) for fifth place and a Silver Medal. Hlanganani Shibambo (Wordsworth High School) won a Bronze Medal.

In the team rankings, Ivory Coast came first with 67 points, followed by South Africa (65). Nigeria was third.

Speaking at the awards ceremony, the Director-General of the Department of Science and Technology, Dr Phil Mjwara, said

> "Innovative thinking and problem solving will allow Africa to be competitive in research and development and beyond, and mathematics skills are fundamental to all successful scientific and technological endeavours, whether we're talking about pharmaceuticals or sending a satellite into space."

First Day

1. Do there exist numbers $x_1, x_2, \ldots, x_{2009}$ from the set $\{-1, 1\}$ such that

$$x_1x_2 + x_2x_3 + x_3x_4 + \ldots + x_{2008}x_{2009} + x_{2009}x_1 = 2009?$$

2. Point P lies inside a triangle ABC. Let D, E and F be reflections of the point P in the lines BC, CA and AB respectively. Prove that if the triangle DEF is equilateral, then the lines AD, BE and CF intersect in a common point.

3. Let x be a real number with the following property: for each positive integer q, there exists and integer p such that

$$\left|x - \frac{p}{q}\right| < \frac{1}{3q}.$$

Prove that x is an integer.

Second day

4. Consider n children in a playground, where $n \geq 2$. Every child has a coloured hat, and every pair of children is joined by a coloured ribbon. For every child, the colour of each ribbon held is different, and also different from the colour of that child's hat. What is the minimum number of colours that must be used?

5. Find all functions $f : \mathbb{N}_0 \to \mathbb{N}_0$ for which $f(0) = 0$ and

$$f(x^2 - y^2) = f(x)f(y) \quad \text{for all } x, y \text{ with } \ x > y,$$

where $\mathbb{N}_0$ is the set $\{0, 1, 2, \ldots\}$.

6. Points C, E, D and F lie on a circle with centre O. Two chords CD and EF intersect at a point N. The tangents at C and D intersect at A, and the tangents at E and F intersect at B. Prove that $ON \perp AB$. $\square$

QUOTE

The difference between topological and continuous spacial relationships is captured by Groucho Marx's sweet nothing 'If I held you any closer I'd be on the other side of you.'

(Stephen Pinker, in *The stuff of thought*) $\square$

FINAL FIBONACCI DIGITS

The Fibonacci numbers F_n are defined by the rule

$$F_1 = F_2 = 1, \quad \text{and} \quad F_{n+1} = F_n + F_{n-1} \quad \text{for all} \quad n > 1.$$

So the first ten terms of the sequence are 1, 1, 2, 3, 5, 8, 13, 21, 34, 55.

What is the last digit of F_{2009}?

Since calculating all the Fibonacci numbers up to the 2009th is impractical, some preliminary thinking is needed. There are only ten possibilities for a final Fibonacci digit, and it is clear that as soon as two successive final digits in the Fibonacci sequence repeat, then the subsequent final digits repeat.

The first pattern that can be observed in the final Fibonacci digits

$$1, 1, 2, 3, 5, 8, 3, 1, 4, 5, 9, 4, 3, 7, 0$$

is that the numbers follow the pattern odd, odd, even, odd, odd, even, ... , and it is not difficult to see that this pattern will continue.

While calculating terms of the sequence it may also be noted that every 5th Fibonacci number is divisible by 5 ($F_5 = 5, F_{10} = 55, F_{15} = 610$). Three cases are not, however, enough from which to draw a general conclusion. A general proof is needed.

Such a proof is achieved by showing that $F_{n+5} - F_n = 5F_{n+1}$, as follows:
$F_{n+5} = F_{n+4} + F_{n+3} = 2F_{n+3} + F_{n+2} = 3F_{n+2} + 2F_{n+1} = 5F_{n+1} + F_n$.
It follows that, if F_n is divisible by 5, so is F_{n+5}.
But since $F_5 = 5$ is divisible by 5, so is every 5th Fibonacci number after that.

With every third Fibonacci number even and every fifth divisible by 5, it follows that every 15th Fibonacci number ends in a zero.

Since there are just 100 possibilities for a pair of final digits, we deduce that the sequence must repeat before the 100th term. But the cycle of repetitions must be a multiple of 15.

Moreover, the final digit before a zero must be odd, and not 5, so there are only four possibilities: 1, 3, 7 and 9. So the repetition must occur after the 30th or 60th terms.

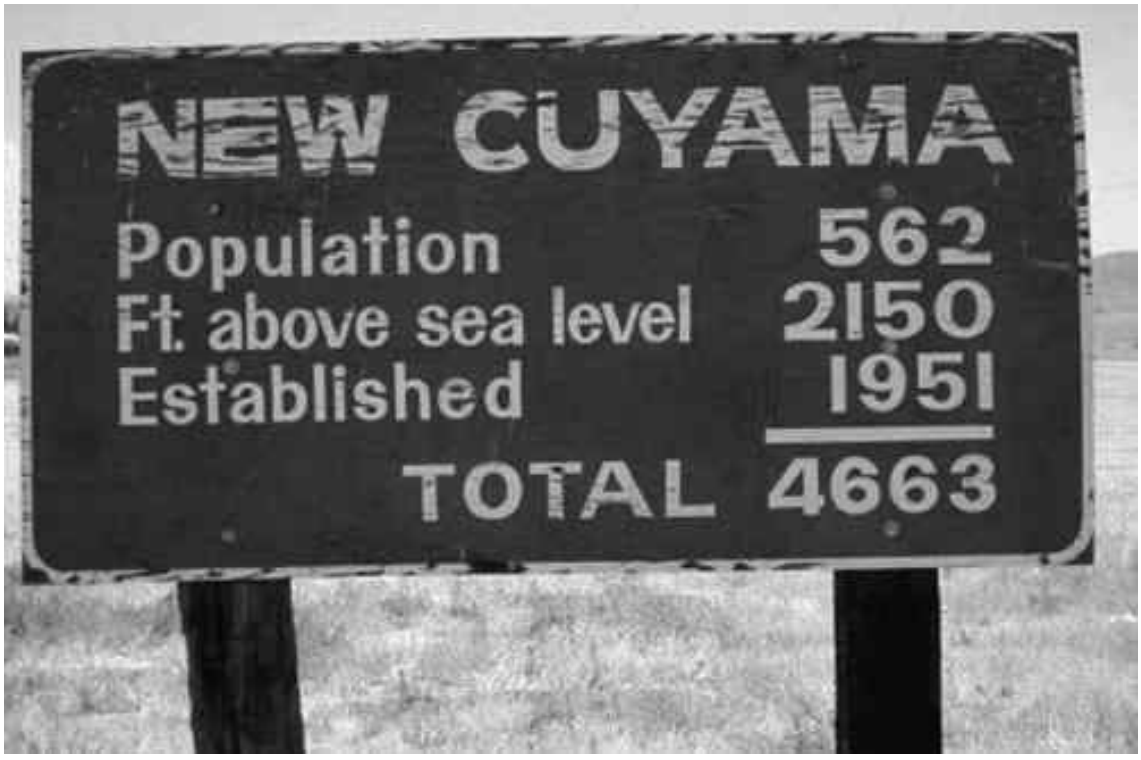
Moreover, as soon as you get to 1 and 0 as two successive final digits, the sequence of final digits will repeat.

It is time to write down the final digits of the first 60 Fibonacci numbers:

1	1	2	3	5	8	3	1	4	5	9	4	3	7	0
7	7	4	1	5	6	1	7	8	5	3	8	1	9	0
9	9	8	7	5	2	7	9	6	5	1	6	7	3	0
3	3	6	9	5	4	9	3	2	5	7	2	9	1	0

Now we can tackle the problem of finding the last digit of F_{2009}. We note that $2009 = 33 \times 60 + 29$, so the last digit of F_{2009} is the same as the last digit of F_{29}, which the above table shows is 9. $\square$

NEW CUYUMATH



EINSTEIN ON EUCLID

If Euclid failed to kindle your youthful enthusiasm, then you were not born to be a scientific thinker.

(Albert Einstein, speaking in Oxford in 1933) $\square$

SHARP CALCULATOR COMPETITION

The Sharp Calculator Competition is sponsored by Sharp Electronics (S.A.) (Pty) Limited, who have donated three calculators as prizes.

These prizes are offered for the three best solutions of the problems below. Solutions may be in English or Afrikaans. Partial solutions will be judged on their merits. The Editor reserves the right to make no award if no satisfactory entries are received, and his judgement will be final. The winner of an nth prize in a previous competition will only be eligible for an mth prize if $m < n$.

Only high schools students may compete. Entrants must give their full name and home address, age, school and grade. Solutions must be certified by a mathematics teacher at the school as the entrant's own unaided work.

The closing date for receipt of entries is 31 August 2009, and the results will be announced in our October 2009 edition.

SHARP

1. Solve: $(x+9)(x-3)(x-7)(x+5) = 385$.
2. Let a, b and c be the lengths of the sides of a triangle with perimeter 2. Prove that
$$a^2 + b^2 + c^2 + 2abc < 2.$$
3. 2009 whole numbers are arranged in a circle so that for any five consecutive numbers the sum of three of them is twice the sum of the other two. Prove that all the numbers are zero. □

THE LAST SENIOR WRANGLER

This year marks the 100th anniversary of the last time that the University of Cambridge publicly announced its top maths student as the "Senior Wrangler". The last Senior Wrangler was Percy John Daniell, who achieved the title in 1909 and went on to make a name for himself in physics, mathematics and statistics. In a series of papers in the 1920s Daniell introduced a new form of integral. Now known as the *Daniell integral*, it is a valuable tool in probability theory. □

MILLIONS AND TRILLIONS

How many millions are there in a trillion? In a CARAVAN survey carried out in the Unites States in April this year, 1001 adults (501 men and 500 women) were asked: " How many times larger is a trillion than a million?" The responses were

One Thousand Times:	18%
Ten Thousand Times:	12%
One Hundred Thousand Times:	21%
One Million Times:	21%
Ten Million Times:	17%
Don't Know:	12%

□

CLICKER QUIZ

The gadget pictured above is a *clicker*, which was once used during the summer months in South Africa (but not in England).

What was it used for, when did it become obsolete and why?

The first correct answer opened on 1 September will win a free 2010 subscription to *Mathematical Digest*. □

SHARP SOLUTIONS AND RESULTS

1. *Prove that, if*
$$\frac{ac-b^2}{a-2b+c}=\frac{bd-c^2}{b-2c+d}$$
then both sides are equal to
$$\frac{ad-bc}{a-b-c+d}.$$

Set $\dfrac{ac-b^2}{a-2b+c}=k=\dfrac{bd-c^2}{b-2c+d}$.
Multiplying the first equation out gives
$$ac-b^2=k(a-2b+c)\Leftrightarrow ac-k(a+c)=b^2-2kb.$$
Now add k^2 to each side and factorise:
$$(a-k)(c-k)=(b-k)^2.$$
Similarly, the second equation gives:
$$(b-k)(d-k)=(c-k)^2.$$
Multiplying these two equations together, and cancelling common factors, gives
$(a-k)(d-k)=(b-k)(c-k)$
$\Leftrightarrow ad-ak-dk+k^2=bc-bk-ck+k^2$
$\Leftrightarrow ad-bc=k(a-b-c+d)$
$\Leftrightarrow k=\dfrac{ad-bc}{a-b-c+d}$.
This problem comes from W W Sawyer's classic *Prelude to Mathematics* (Penguin Books, 1955). It was from that source, too, that we derived the "solution" above.

The problem, however, is wrong. Setting $a=1, b=c=2, d=4$, we see that the common value of the two fractions (which we denoted by k) is 2, but $ad-bc=0$, so that the third fraction is not equal to k.

Our thanks to David Allison, formerly of the UCT Department of Mathematics and Applied Mathematics, for pointing out the flaw.

But if the problem is wrong, where does the proof go wrong? It is in the phrase "cancelling common factors", which is always a dangerous thing to do without considering the possibility that either of the factors could be zero. Which they are, when $a=1, b=c=2, d=4$, and $k=2$.

So a correct statement of the problem should add the condition that a, b, c and d are different.

2. *Find all positive integers m and n satisfying the equations*

$$m^2 - 4mn + 6n^2 - 2m - 20n = 29.$$

Putting the equation in the form

$$m^2 - (4n+2)m + 6n^2 - 20n - 29 = 0$$

and solving (as a quadratic in m), we obtain:
$m = \frac{1}{2}(4n + 2 \pm \sqrt{16n^2 + 16n + 4 - 24n^2 + 80n + 116}$
$= 2n + 1 \pm \sqrt{30 + 24n - 2n^2}$
$= 2n + 1 \pm \sqrt{102 - 2(n-6)^2}$.
So we require that $2(n-6)^2 \leq 102$, i.e $(n-6)^2 \leq 51$. Since n is a positive integer, this is equivalent to $-7 \leq n - 6 \leq 7$, or $1 \leq n \leq 13$. We now simply plug in these values of n, for which $102 - 2(n-6)^2$ is a perfect square.

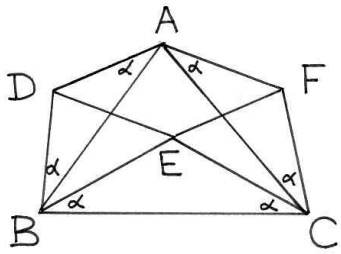
$n = 5:$ $\quad m = 11 \pm \sqrt{100} = 21$ or 1
$n = 7:$ $\quad m = 15 \pm \sqrt{100} = 25$ or 5
$n = 13:$ $\quad m = 27 \pm \sqrt{4} = 29$ or 25.

So the possible values of (m, n) are

$$(1,5), (5,7), (21,5), (25,7), (25,13), (29,13).$$

This problem was taken from the classic Victorian maths textbook, Hall and Knight's *Higher Algebra*, published in 1887.

3. *Three similar isosceles triangles ABD, BCE and CAF are constructed on the sides of triangle ABC as shown. Prove that $ADEF$ is a parallelogram.*



Let α denote the base angles of the three similar isosceles triangles.

We apply the cosine rule in triangle DBE:
$DE^2 = DB^2 + BE^2 - 2DB.BE\cos B$ (since $\angle DBE = \angle ABC$)
$= (\frac{1}{2}AB\sec\alpha)^2 + (\frac{1}{2}BC\sec\alpha)^2 - 2(\frac{1}{2}AB\sec\alpha)(\frac{1}{2}BC\sec\alpha)\cos B$
$= \frac{1}{4}\sec^2\alpha(AB^2 + BC^2 - 2AB.BC\cos B)$
$= \frac{1}{4}\sec^2\alpha.AC^2$
$= AF^2$.

So $DE = AF$, and similarly $EF = DA$.
So $ADEF$ is a parallelogram.

SHARP

Greg Jackson (Bishops) and Henry Thackeray (St Alban's College) submitted correct entries (and detected the flaw in problem 1). But as former First Prize winners they can earn only an Honourable Mention. David Munjeri also earns an Honourable Mention for his solution of problem 2. □

JOURNALISTS AND MATHS

> This wasn't an easy book to read for someone like me who's allergic to long division.

(Samantha Bartlett, reviewing Marcus Chown's *Quantum theory cannot hurt you* in the *Cape Times*, 20 June 2008.)

> I've often wondered what I'd do if I didn't write. The corporate world is out, because I'd be asked to shave regularly and wear sensible shirts. Advertising was a possibility, but I'm just not hip enough to come up with all those big ideas. Besides tending bars, this basically leaves things that demand an understanding of science and maths, both of which vanished from my timetable after Grade 10.

(Thomas Falkiner, motoring columnist of the *Sunday Times*, 22 June 2008.) □

MACHINE TRANSLATION

The 19th Pan African Mathematics Olympiad will take place in Pretoria, South Africa from 19–26 April 2009.

Les 19th L'olympiade africaine du mathématiques du casserole aura lieu à Pretoria, Afrique du Sud du 19 au 26 avril 2009.

(PAMO 2009 website) □

CHEWING GUM BOOSTS MATHS

> A team at Baylor College of Medicine in Houston found that students who chewed gum during maths class had higher scores on a standardised maths test after 14 weeks and better grades at the end of the term than students who did not chew gum.
>
> (*The Times*, 24 April 2009)

Following up this brief newspaper report revealed some interesting facts about the study. The first is that it was sponsored by the makers of Wrigley chewing gum, who have set up the "Wrigley Science Institute", the aim of which is to prove that chewing gum helps the chewers to "stay focused".

The Baylor researchers studied 108 students in a Texas school. About half were given Wrigley's sugar-free gum to chew during class, while doing homework or writing tests. The other half went without.

After 14 weeks, the gum chewers reportedly showed a three percent increase in their maths scores on the Texas Assessment of Knowledge and Skills achievement test. The increase was small, but statistically significant. There was, however, no difference in their maths scores in another assessment, the Woodcock Johnson III Tests of Achievement.

Scientists who had nothing to do with the study commented that it has long been believed that chewing gum reduces stress and helps the chewer to stay alert. Other scientists commented that similar eating activities would probably have the same effect. And merely doing something normally forbidden in the classroom could be a stimulus.

All this seems to be an excellent subject for a school project. It is hardly a new idea. The picture below shows 13-year-old Miranda G. with her school project that won her an award at a Washington Science Fair in 2004.

But instead of investigating the possible effects of chewing gum, the project could investigate other snacks. Will gnawing sticks of biltong improve maths marks?

□

PRIZEWINNERS AND SOLUTIONS

Square Words (April 2009, p.12):
A *square word* is one in which the letters can be replaced by digits, a different digit for each letter, such that the resulting number is a perfect square, and the digits follow the same ordering as the letters.

Moses Kabwe (grade 9, Ligbron Academy of Technology) found several four-letter examples:

WORD: $6241 = 79^2$ EAST: $1024 = 32^2$ MEET: $4225 = 65^2$

Henry Thackeray (St Alban's College) used a computer to generate a long list of square words. Taking from his list only the longer square words that also have a mathematical connection, we have

CYCLIC:	151321	=	389^2	DEGREE:	345744	=	588^2
VERTEX:	804609	=	897^2	QUARTER:	3705625	=	1925^2
CONSTANT:	27489049	=	5243^2				

Henry and Moses each win a R50 Prize Voucher.

This problem turned out to be a new version of an old problem, according to reader Roland Prukl. He writes:

"Thirty years ago in Johannesburg, at Watermeyer, Legge, Piésold and Uhlmann, we had competitions to find Square Words using Hewlett-Packard desktop computers (HP9845). We did not, however, apply the rule, that the digits follow the same ordering as the letters. I could easily modify my program to implement this rule, but many interesting solutions would then be excluded."

Sort the Sports (April 2009, p.12):
The first correct answer drawn (Carl plays football for Westville) was sent in by Jéan-Leigh Jantjes (grade 9, Groote Schuur High School), winning a R50 Prize Voucher.

Crossing the River (April 2009, p.23):
Sarah Tetlow (Herschel Girls' School) wins a R50 Prize Voucher for her river-crossing schedule:
A and B cross (10 minutes), A returns (5 minutes), C and D cross (20 minutes), B returns (10 Minutes), A and B cross (10 minutes).
Total time: 55 minutes.

Sums and Powers (April 2009. p.30):
The problem was to divide the numbers from 0 to 31 into two sets, with

equal sums, equal sums of squares, equal sums of cubes and equal sums of fourth powers. David Munjeri sent in the answer:
A: 0, 3, 5, 6, 9, 10, 12, 15, 17, 18, 20, 23, 24, 27, 29, 30
B: 1, 2, 4, 7, 8, 11, 13, 14, 16, 19, 21, 22, 25, 26, 28, 31
A Prize Voucher worth R50 will be sent to David — as soon as he sends us his address! Computer-generated solutions were also sent in by Henry Thackeray and Roland Prukl.

This problem can be traced back to an 1851 paper by E. Prouhet, and more than 150 years later it is still generating interest, as seen in a recent article by Chris Bernhardt of Fairfield University, USA (*Mathematics Magazine*, **82**(1), February 2009).

Editor's Hat Prize:
All competition entries that did not win a prize went into the Editor's Hat, and three lucky names were drawn. Samukelisiwe Mbanjwa (Sacred Heart Secondary School), Wilfred Khubana (Mbilwe Secondary School) and Imraan Rahim (Groote Schuur High School) all win R100 Prize Vouchers.

Prize Vouchers may be used to buy any of the publications listed at the back of this edition of *Mathematical Digest*.

Competitions are open to all readers, but only high school students are eligible for prizes. Solutions to different problems must be written on separate A4 sheets, each of which must carry the sender's name, address and school. To save postage, solutions to different problems, and from different readers, may be sent in the same envelope. Solutions may also be submitted by fax or email. □

METHAMATICLA ANAGARMS

Group the ten words below in five pairs, and from the letters of each pair make a mathematical word.

agent carrot bag creep grim her loath mote port real

The first three correct entries opened on 1 September, from readers in grade 9 or lower, will each win a free 2010 subscription to *Mathematical Digest*. □

THE 2009 UCT MATHEMATICS COMPETITION

The annual University of Cape Town Mathematics Competition took place on the UCT campus on 21 April this year, attracting participants from 138 Western Cape high schools. Each school could enter up to five individuals and five pairs, in each grade (8 to 12).

Gold Awards were awarded to the top ten individuals and top three pairs in each grade.

Gold Awards Grade 8: Individuals

1.	Nicholas Cotchobos	Rondebosch Boys' High School
1.	James-John Matthee	Hoërskool Stellenbosch
3.	Nevarr Pillay	Bishops
4.	James Nevin	Bergvliet High School
5.	Jake Budler	Elkanah House
6.	Guy Paterson-Jones	Bishops
7.	Samuel Wolski	Bishops
8.	Lauren Denny	Rustenburg Girls' High School
8.	Megan Higgo	Parel Vallei High School
10.	Taesong Kim	Reddam House Atlantic Sea board

Grade 8: Pairs

1.	Loren Ruth de Koker / Lethu Ntshinga	Herschel Girls' School
2.	Sian Bradbury / Carlé Cillié	Paarl Girls' High School
3.	Jared Gordon / Devon Woodman	Bishops
3.	Nicolene Theron / Ryno Swart	Hoërskool Durbanville

Grade 9: Individuals

1.	Johannes Schoeman	Hoërskool Parel Vallei
1.	Khadija Brey	Wynberg Girls' High School
3.	Nicholas Cheng	Fairmont High School
4.	Aidan Horn	Bishops
5.	Keegan Welken	Elkanah House
6.	Ihsaan Bassier	Rondebosch Boys' High School
6.	Timothy de Wet	Parklands College
8.	Sarah Tetlow	Herschel Girls' High School
9.	Jonathan Black	Bishops

9. Robert Spencer Westerford High School

Grade 9: Pairs

1. Yohan Chun / Hyun-Joo Park Rustenburg Girls' H S
2. Lize-Mari Doubell / Amy Kate Thompson Herschel Girls' School
3. Alistair Border / David Laing Elkanah House

Grade 10: Individulas

1. Ashraf Moolla Rondebosch Boys' High School
1. Sean Wentzel Westerford High School
3. Jin-Su Kim Stellenberg High School
4. Melissa Greeff Herschel Girls' High School
5. Charl du Plessis Stellenberg High School
6. Kira Düsterwald Springfield Convent
7. Rehan Visser Bridge House School
8. Sean Cilliers Parel Vallei High School
8. Jeff Kim Reddam House Constantia
10. Tiaan Munnik Hoërskool D F Malan

Grade 10: Pairs

1. Robert Moerman / Nicholas Santrucek Bridge House
2. Christiaan van der Walt / Herman Wolff Paul Roos Gimnasium
3. Ramsay Collins / Leon Raubenheimer Bishops
3. Tiana du Preez / Lezanne Lötter Hoërskool Stellenbosch

Grade 11: Individuals

1. Greg Jackson Bishops
2. Yu Sun Milnerton High School
3. Sunghun Jung Settlers High School
4. Francois van Niekerk Hoërskool Bellville
5. Ashleigh Avenant Westerford High School
6. Lance Li South African College H S
7. Kieran Davies International School of Hout Bay
7. Pieter Klue Langenhoven Gimnasium
9. Yasmin Skibbe Bloemhof Hoërmeisieskool
10. Min Kyu Lee St George's Grammar School

Grade 11: Pairs

1. Christopher Day / William Hillock Rondebosch Boys' H S
2. Adie Lubbe / Bruce McDougall Bishops
3. Emily-Ariane Fogel / Robyn Steenekamp Reddam House

Grade 12: Individuls

1.	Francois Conradie	De Kuilen Hoërskool
2.	Liam Baker	Mondale High School
3.	Maggie Lu	Herschel Girls' High School
4.	John Nicholas	Jan van Riebeeck Hoërskool
5.	Huisung Choi	Abbott's College Milnerton
6.	Jacobus van der Merwe	Hoërskool Brackenfell
7.	Richard Taylor	Bishops
8.	Carl Eriksen	Rondebosch Boys' High School
9.	Michael Davies	Cedar House
9.	Andrew Robertson	Bergvliet High School

Grade 12: Pairs

1.	Greg Burman / Sean Peters	Rondebosch Boys' H S
2.	Jacques Terblanche / Wiaan Visser	Paarl Boys' High School
3.	Giuseppe Carosini / William Naidoo	Reddam House
3.	Stuart Burmeister / Michael Kloos	Somerset College

On overall performance, the top ten schools were

1	Bishops
2	Rondebosch Boys' High School
3	Herschel High School
4	Parel Vallei High School
4	Westerford High School
6	Reddam College Constantia
7	Rustenburg Girls' High School
8	Paarl Boys' High School
9	Bergvliet High School
10	Paul Roos Gymnasium

A free set of competition papers, with answers, may be obtained by sending a stamped self-addressed B5 (220 mm by 160 mm) envelope to

The Director, UCT Mathematics Competition
Department of Mathematics and Applied Mathematics
University of Cape Town
7700 RONDEBOSCH

Books of question papers of previous years, with full solutions, statistics and lists of award winners, are also available—see the order form at the back of the magazine. For more information, see
`www.mth.uct.ac.za/competition`. □

THE UCT INVITATIONAL MATHEMATICS CHALLENGE

The top contestants in the UCT Mathematics Competition, held on April 21, were invited to take part in a follow-up competition, the UCT Invitational Challenge, held on Saturday 9 May. The papers are given below. Answers may be found on page 31.

Junior Paper(grades 8 and 9):

1. Put plus and minus signs between the digits

$$1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9$$

to make 100. For example:

$$12 + 3 + 4 + 5 - 6 - 7 + 89 = 100$$

$$123 - 4 - 5 - 6 - 7 + 8 - 9 = 100$$

Find four more solutions.

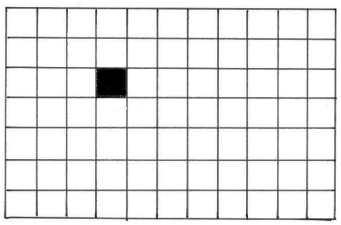
2. Replace the letters in the alphametric

$$DONALD + GERALD = ROBERT$$

by the digits 0, 1, 2, ..., 9 to make the addition correct. You are given that $D = 5$.

3. The lengths of the diagonals of a rhombus are (in cm) two consecutive integers. The area of the rhombus is 210 cm^2. Find the perimeter of the rhombus (in cm).

4. How many rectangles are there in the diagram shown which do not contain the black square?



5. How many litres of a solution that is 15% alcohol do we have to mix with 3 litres of a solution that is 35% alcohol to make a solution that is 21% alcohol?

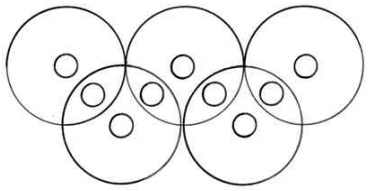
6. Three frogs are sitting on a number line.

The frogs jump according to the following rule: the frog on the extreme left jumps over both other frogs, to land at a point just as far from the frog on the right as before it jumped.

The frogs start at 0, 1 and 2. After the first jump they will be at 1, 2 and 4, and after the second jump they will be at 2, 4 and 7. At what positions on the number line will the frogs be after six jumps?

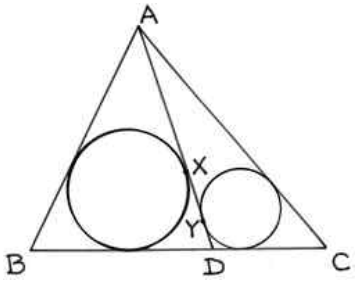
7. If the length of a rectangle is reduced by 5 cm and its width is reduced by 2 cm, then the area is reduced by 66 cm^2 and the figure becomes a square. What was the area of the original rectangle?

8. Place the numbers from 1 to 9 in the small circles, so that the sum of the numbers in each of the large circles is 14.



9. Find the smallest n such that every n-element subset of the set $\{1, 2, 3, \ldots, 1000\}$ contains at least two elements that are relatively prime (i.e. have no common factors greater than 1).

10. In the figure, $AB = 37$, $BD = 23$, $DC = 19$ and $CA = 45$. The inscribed circles of triangles ABD and ADC are tangent to AD at X and Y. Determine the length of XY.



Senior Paper (grades 10, 11 and 12)

1. See Junior paper, question 6, (but after *ten* jumps).

2. See Junior paper, question 3.

3. See Junior paper, question 9.

4. Let f be a real-valued function of a real variable x such that

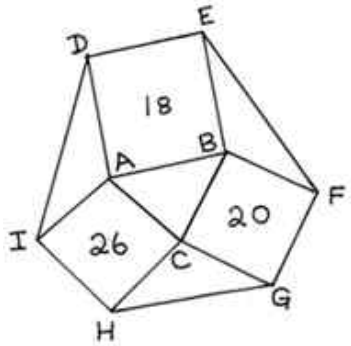
$$f(x) + 3xf(\frac{1}{x}) = 2(x + 1)$$

for all real numbers $x > 0$. Find $f(2009)$.

5. Solve for a, b and c if

$$a^2 + 3 = b + 2c \qquad b^2 + 17 = 3a + 4c \qquad c^2 - 6 = b + a$$

6. Calculate:
$$\frac{1}{1+2}+\frac{1}{1+2+3}+\frac{1}{1+2+3+4}+\ldots+\frac{1}{1+2+3+\ldots+100}.$$
7. A point P lies inside square $ABCD$ such that $AP = 51$, $BP = 26$ and $CP = 20$. Determine DP.
8. See Junior paper, question 10.
9. Find all positive integers m such that both $m - 60$ and $m + 60$ are perfect squares.
10. Squares $ABED$, $BFGC$ and $ACHI$ are constructed outwards on the sides of triangles ABC. The areas of the squares are 18, 20 and 26 respectively. What is the area of the hexagon $DEFGHI$?



Winners:
Grade 8:
1. Sam Wolski (Bishops)
2. Jason Raad (Wynberg Boys' High School)
3. Guy Paterson-Jones (Bishops):

Grade 9:
1. Khadija Brey (Wynberg Girls' High School)
2. Hekkie Breytenbach (Paarl Gymnasium)
2. Mickey Chew (Elkanah House)

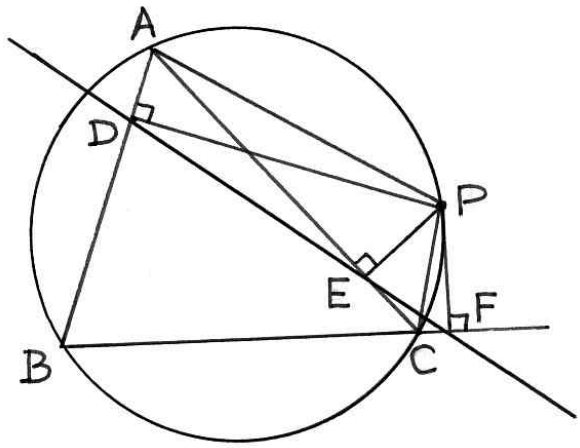
Grade 10:
1. Sean Wentzel (Westerford High School)
2. Charl du Plessis (Stellenberg High School)
3. Melissa Greeff (Herschel Girls' School)
3. Jeff Kim (Reddam House Constantia)
3. Nico du Plessis (Reddam House Constantia)

Grade 11:
1. Lance Li (South African College High School)
2. Greg Jackson (Bishops)
3. Rowan Nicholls (Bishops)

Grade 12:
1. Francois Conradie (De Kuilen High School)
2. Liam Baker (Mondale High School)
3. Matthew Hearn (Bishops)

THE SIMSON LINE

Take any point P on the circumcircle of triangle ABC, and drop perpendiculars from P onto the sides (produced if necessary) of the triangle. Then the feet of the three perpendiculars will always lie in a straight line.



The line is known as the *Simson Line*, after an 18th century Scottish mathematican, Robert Simson. However, there is no evidence that Simson ever proved the result, or even knew it. The statement of the result, and its proof, is found in the work of William Wallace, another Scottish mathematician and contemporary of Simson.

So the Simson Line should really be called the *Wallace Line*. But that would be a bit confusing, because there is another Wallace Line, named after Alfred Russel Wallace, the naturalist credited with Charles Darwin for first expounding the theory of evolution. The Wallace line is not mathematical, but geographical. It is the boundary separating very different animal species in south-east Asia and Australia.

The fact that the three feet of the perpendiculars from P lie on a straight line will be proved by showing that $\angle AED = \angle FEC$.

The key idea is to notice that there are cyclic quadrilaterals in the diagram, namely

- $ABCP$ (it was given that P lies on the circumcircle of triangle ABC)
- $CFPE$ (since $\angle PEC + \angle PFC = 90^\circ + 90^\circ = 180^\circ$)
- $ADEP$ (since $\angle PDA = \angle PEA = 90^\circ$).

Now $\angle AED$
$= \angle APD$ (since $ADEP$ is a cyclic quadrilateral)
$= 90^\circ - \angle PAD$ (right-angled triangle ADP)
$= 90^\circ - \angle PCF$ (exterior angle of cyclic quadrilateral $APCB$)
$= \angle FPC$ (right-angled triangle FPC)
$= \angle FEC$ (since $EFCP$ is a cyclic quadrilateral).

So we have proved that $\angle AED = \angle FEC$.
Since AEC is a straight line, so is DEF: the Simson Line. □

ANOTHER MERSENNE PRIME

A new Mersenne prime was discovered on 12 April this year by a computer belonging to Odd Magnar Strindmo from Melhus, Norway. It is

$$2^{42\ 643\ 801} - 1.$$

This prime is the second largest known prime number, a "mere" 141 125 digits smaller than the biggest ever Mersenne prime found last August:

$$2^{43\ 112\ 609} - 1.$$

See *Mathematical Digest* **154** (January 2009), p. 2, for more details. □

CHALLENGE SOLUTIONS

Junior:

1. $1 + 23 - 4 + 56 + 7 + 8 + 9 = 123 + 4 - 5 + 67 - 89$
   $= 123 + 45 - 67 + 8 - 9 = 123 - 45 - 67 + 89 = 100$
2. 526485 + 197485 = 723970
3. 58 cm
4. 1368
5. 7 litres
6. 20, 33, 54
7. 130 cm^2
8. 5, 9, 2, 3, 4, 7, 1, 6, 8 (other solutions are possible)
9. 501
10. 6

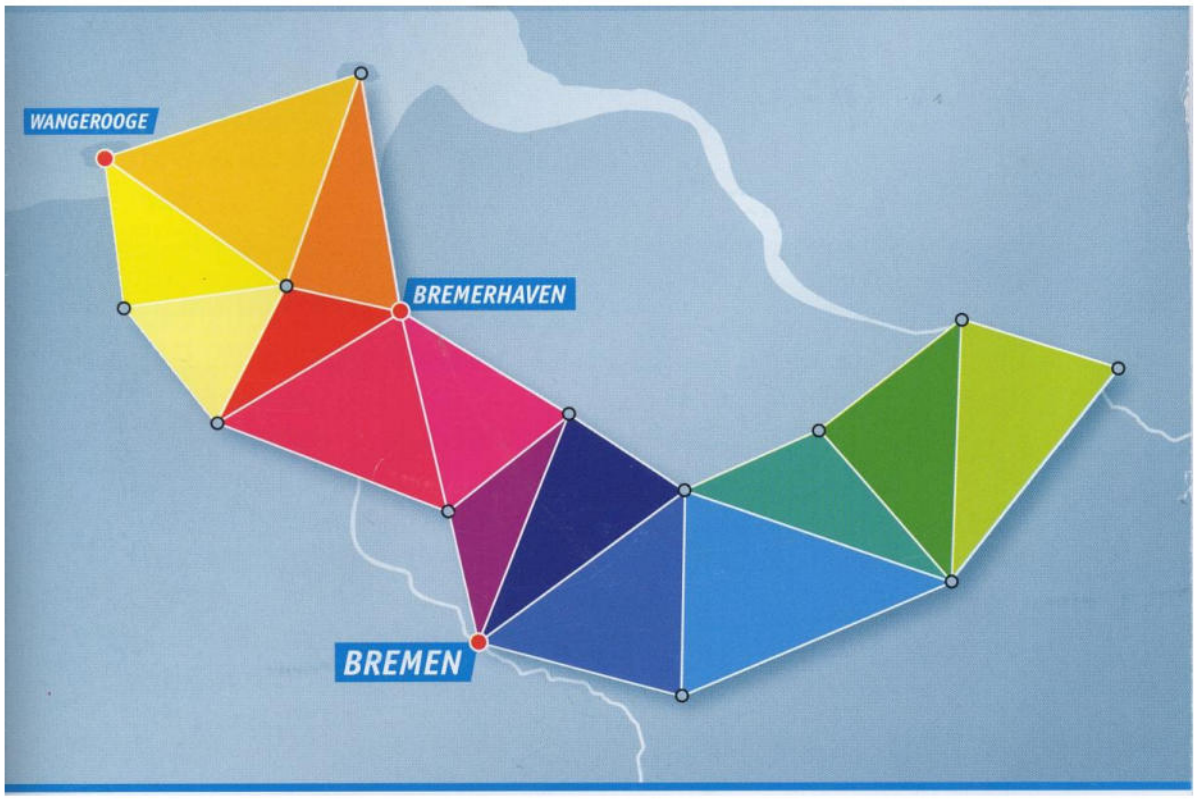
Senior:

1. 143, 197, 376
2. 58 cm
3. 501
4. 1005
5. $a = 2, b = 1, c = 3$
6. 99/101
7. $5\sqrt{93}$
8. 6
9. 61, 109, 229, 901
10. 100

□

mathematical digest

No. 157 October 2009



The Gauss Survey

mathematical digest

Number 157 **October 2009**

At the age of eleven, I began Euclid, with my brother as tutor. This was one of the great events of my life, as dazzling as first love. I had not imagined there was anything so delicious in the world. From that moment until I was thirty-eight, mathematics was my chief interest and my chief source of happiness.

(Bertrand Russell, in his *Autobiography*)

Department of Mathematics and Applied Mathematics
University of Cape Town

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Cover: Our cover features the design used on banners for the International Mathematical Olympiad in Bremen. See pages 1 – 4.

THE GAUSS SURVEY

Nobody in Bremen could have been unaware of the International Mathematical Olympiad (IMO), which took place in the north German city during July this year. Everywhere in the picturesque old town on the river Weser there were banners and posters welcoming the nations of the world to the IMO. The posters featured a map of northern Germany with a colourful string of triangles stretching across it. The same design (featured on our cover) was used on IMO publications and T shirts.

The map on the poster carried a significant mathematical message, for it was originally drawn by the famous mathematician Carl Friedrich Gauss (1777-1855). Mathematicians today recognize Gauss as one of the three greatest mathematicians of all time (the others being Archimedes and Isaac Newton).

Apart from his major achievements in many areas of mathematics, Gauss made important contributions to physics, especially in his work on magnetism. At the age of eighteen Gauss devised the method of least squares, a cornerstone of mathematical statistics. He established his reputation as an astronomer by his calculation of the orbit of the dwarf planet Ceres (discovered on 1 January 1801). His portrait has featured on German stamps and bank notes.

Gauss was also an expert in geodesy and land surveying. In 1820 he was commissioned to survey the Kingdom of Hannover (much of today's German state of Lower Saxony). Gauss did all the surveying himself, recognizing the need for utmost precision. Three of his survey points were venues of significance for the 2009 IMO. Bremerhaven was where the IMO Jury met to select the problems, Bremen was where the papers were written and the island of Wangerooge (a World Heritage site) was where everybody went to relax when all the hard work was done. □

IMO IN GERMANY

The International Mathematical Olympiad (IMO) is the oldest and the biggest of the "intellectual" Olympiads for high schools. The first IMO was held in Rumania in 1959, and the 50th took place this year in Bremen, Germany. For the first time the number of participating countries exceeded 100.

(The astute reader may wonder why this was not the 51st such event, for indeed there are 51 years from 1959 to 2009. The answer is that in 1980 the Olympiad, scheduled for Mongolia, did not take place.)

Each participating country may enter up to six students, who must be at most 19 years old, and not registered at a university. The South African team was: Liam Baker (Mondale High School), Francois Conradie (Hoërskool De Kuilen), Arlton Gilbert (Star College), Greg Jackson (Bishops), Henry Thackeray (St Alban's College) and Sean Wentzel (Westerford High School). The Team Leader was David Hatton (UCT), with Deputy Team Leader Maciej Stankiewicz (UCT). Phil Labuschagne accompanied the team as an Observer, and Professor John Webb (UCT) attended in his capacity as Secretary of the Advisory Board of the IMO.

A record number of 104 countries took part in the 50th IMO, the first time that the number had passed 100. The number of African countries in the IMO is small, but showing an encouraging increase. When South Africa first took part in the IMO in 1992 there were only three other African participants, all from the Mediterranean coast: Algeria, Morocco and Tunisia. After South Africa, Mozambique was the next African country to join, in 2004, and Nigeria joined in 2006. This year there were three new African teams: Benin, Mauritania and Zimbabwe. The Ivory Coast and Senegal sent observers to Bremen, and are expected to send full teams next year. It was unfortunate that Mozambique did not send a team to Bremen.

The Jury, consisting of the Team Leaders of the 104 participating countries, gathered on Friday 10 July in Bremerhaven, on the coast north of Bremen, to set the papers. Three days later the teams arrived in Bremen. A total of 565 students settled into the pleasant campus of Jacobs University. After a rousing Opening Ceremony on 14 July, the hard work began. Two challenging $4\frac{1}{2}$-hour papers were written on the next two days, each paper containing just three questions.

While the teams were then free to socialise, compete in an international football league and go on excursions, the Leaders now had to get to work marking the papers. At the same time, a team of sixty German mathematicians, many of whom had won medals in the IMO in their schooldays, marked photocopies of the same papers. Then they got together with the Team Leaders and came to an agreement on marks for each question. After two days of intensive work the job was done, and the Jury met to agree on the final scores and award medals.

To celebrate the 50th Anniversary of the IMO, six of the world's most prominent mathematicians had been invited to the IMO to meet the students and give short presentations on their work. All of them had won Gold Medals in their IMO days. Bèla Bollobàs (Hungary) took part in the first three IMOs (1959-1961). Làszlò Lovasz (Hungary) took part in four IMOs (1963-1966) and is today President of the International Mathematical Union. Stanislas Smirnov won Golds for the Soviet Union in 1986 and 1987. Then there were three Fields Medallists: Timothy Gowers (United Kingdom, 1981), Jean-Christoph Yoccoz (France, 1973 and 1974) and Terence Tao, who was only 14 when he won a Gold Medal for Australia in 1988, and won a Fields Medal in 2006. The session was chaired by the President of the German Mathematical Society, Günter M. Ziegler, himself an IMO Gold Medallist in 1981.

Each of the speakers gave short accounts of what to them were the most important areas of modern mathematics, and reflected on how the ability to solve difficult IMO problems related to current research issues. During the programme there were breaks during which the students mobbed the speakers. It was a memorable occasion.

The next day was set aside for an excursion for everybody to Wangerooge Island, a seaside resort and World Heritage site on the North Sea coast.

According to the IMO rules, up to half the contestants receive medals, and the Gold, Silver and Bronze medals are awarded in the ratio 1:2:3. Those

who don't win a medal, but have solved at least one problem correctly, win an Honourable Mention.

When the final results were announced China came out first in the team ranking, with 221 points out of a possible 252. Japan was second with 212 points, its best performance at an IMO ever. Russia came third with 203 points. The rest of the top ten countries were South Korea (188), North Korea (183), USA (182), Thailand (181), Turkey (177), Germany (171) and Belarus (167).

South Africa was ranked 43rd, with Liam Baker and Henry Thackeray winning Bronze Medals. Francois Conradie, Arlton Gilbert, Greg Jackson and Sean Wentzel all won Honourable Mentions.

Next year the International Mathematical Olympiad will take place in Astana, the capital of Kazakhstan.

For the full IMO results, go to the website `www.imo-official.org`.

The 2009 International Mathematical Olympiad was organized by Bildung and Begabung. Their permission to use the logo on page 2 and the design on the front cover is acknowledged with thanks. □

LITERARY QUIZ

> "The knowledge I had in mathematics gave me great assistance in acquiring their phraseology, which depended much upon that science, and music; and in the latter I was not unskilled. Their ideas were perpetually conversant in lines and figures. If they would, for example, praise the beauty of a woman, or any other animal, they describe it by rhombs, circles, parallelograms, ellipses, and other geometrical terms, or by words of art drawn from music, needless here to repeat. I observed in the king's kitchen all sorts of mathematical and musical instruments, after the figures of which they cut up the joints that were served to his majesty's table."

From which travel book was this extract taken, and who wrote it? The first correct answer opened on 1 December will win a Prize Voucher worth R50. □

IMO PROBLEMS

Day 1

Problem 1. Let n be a positive integer and let $a_1, \ldots, a_k$ $(k \geq 2)$ be distinct integers in the set $\{1, \ldots, n\}$ such that n divides $a_i(a_{i+1} - 1)$ for $i = 1, \ldots, k-1$. Prove that n does not divide $a_k(a_1 - 1)$.

Problem 2. Let ABC be a triangle with circumcentre O. The points P and Q are interior points of the sides CA and AB, respectively. Let K, L and M be the midpoints of the segments BP, CQ and PQ, respectively, and let Γ be the circle passing through K, L and M. Suppose that the line PQ is tangent to the circle Γ. Prove that $OP = OQ$.

Problem 3. Suppose that $s_1, s_2, s_3, \ldots$ is a strictly increasing sequence of positive integers such that the subsequences

$$s_{s_1}, s_{s_2}, s_{s_3}, \ldots \qquad \text{and} \qquad s_{s_1+1}, s_{s_2+1}, s_{s_3+1}, \ldots$$

are both arithmetic progressions. Prove that the sequence $s_1, s_2, s_3, \ldots$ is itself an arithmetic progression.

Day 2

Problem 4. Let ABC be a triangle with $AB = AC$. The angle bisectors of $\angle CAB$ and $\angle ABC$ meet the sides BC and DA at D and E, respectively. Let K be the incentre of triangle ADC. Suppose that $\angle BEK = 45°$. Find all possible values of $\angle CAB$.

Problem 5. Determine all functions f from the set of positive integers to the set of positive integers such that, for all positive integers a and b, there exists a non-degenerate triangle with sides of lengths

$$a, f(b) \quad \text{and} \quad f(b + f(a) - 1).$$

(A triangle is *non-degenerate* if its vertices are not collinear.)

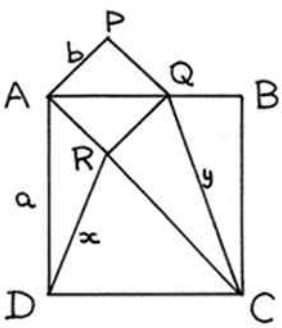
Problem 6. Let $a_1, a_2, \ldots, a_n$ be distinct positive integers and let M be a set of $n-1$ positive integers not containing $s = a_1 + a_2 + \cdots + a_n$. A grasshopper is to jump along the real axis, starting at the point 0 and making n jumps to the right with lengths $a_1, a_2, \ldots, a_n$ in some order. Prove that the order can be chosen in such a way that the grasshopper never lands on any point in M. $\square$

HINGED SQUARES

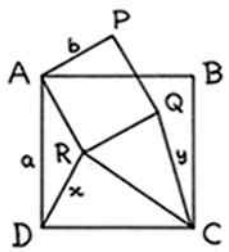
Marcus Bizony
Bishops

The concept of similarity is not really as easy as we sometimes try to make it out to be. Figures can be equiangular without being similar (consider a square and a rectangle) and they can have sides in proportion without being similar (consider a square and a rhombus). In the case of triangles, fortunately, the two concepts, equiangularity and having sides in proportion, are indeed equivalent. By far the more usual method for showing two triangles similar is to show them equiangular; the recognised alternative is to show that the three pairs of sides are in proportion – and then there is always great insistence on the fact that it must be three pairs, not just two.

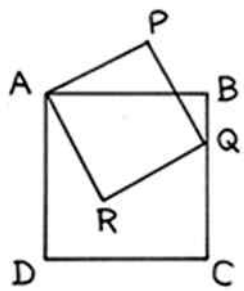
There is however, a third ploy, which is to show two pairs of sides in proportion and the angle included by them equal. The following example uses this idea to good effect.



$ABCD$ is a square of side a; Q is a point on AB so that $APQR$ is a square of side b; $DR = x$ and $CQ = y$. We require to show that $\angle ADR = \angle QCR$.
Clearly $\angle DAR = \angle CAQ$ (both $45°$), while $AD = a$, $AC = a\sqrt{2}$, and $AR = b$ while $AQ = b\sqrt{2}$. So $\triangle ACQ$ is a rotated enlargement of $\triangle ADR$, and the desired result follows. We also see that $y = x\sqrt{2}$.



In fact, if Q is inside square $ABCD$ rather than on the edge AB we still have $\angle ADR = \angle QCR$ and $y = x\sqrt{2}$.



We can also consider what happens if Q is on CB. Is R necessarily on the diagonal DB?

Do we still have $CQ = DR.\sqrt{2}$?
And if Q is outside $ABCD$? $\square$

WHAT DAY WAS IT?

There are various tricks for working out what day of the week it was on a given date. Some of the methods involve memorizing complicated sets of numbers. Here is a method that was published in the *Journal of Recreational Mathematics* some years ago.

To apply the formula, you have to regard the year as beginning on March 1, with January and February counting as the 11th and 12th months respectively of the previous year. Dates are then expressed as follows

Date	**D**	**M**	**Y**
6 September 1963	6	7	63
8 January 1942	8	11	41
1 January 2009	1	11	08

The formula is $T = D + M + Y + \lfloor \frac{4}{5}(2M + 1) \rfloor + \lfloor \frac{1}{4} Y \rfloor$.

(Here the notation $\lfloor x \rfloor$, where x is a positive number, means that you must take only the whole number part of x. For example, $\lfloor 2\frac{4}{5} \rfloor = 2$ and $\lfloor \pi \rfloor = 3$.)

After calculating T, find the remainder R when T is divided by 7. If $R = 0$ the day was a Sunday. If $R = 1$, the day was a Monday, and so on.

For example, on what day did 6 September 1963 fall? Here $D = 6$, $M = 7$ and $Y = 63$. Then

$T = 6 + 7 + 63 + \lfloor \frac{4}{5}(2 \times 7 + 1) \rfloor + \lfloor \frac{1}{4} \times 63 \rfloor = 76 + 12 + 15 = 103$. When 103 is divided by 7 the remainder is 5, so the day was a Friday.

On what day did 8 January 1942 fall? Here $D = 8$, $M = 11$ and $Y = 41$. So $T = 8 + 11 + 41 + \lfloor \frac{4}{5}(2 \times 11 + 1) \rfloor = 60 + 18 + 10 = 88$. When 88 is divided by 7 the remainder is 4, so the day was a Thursday.

On what day did this year start? Here $D = 1$, $M = 11$ and $Y = 08$. So $T = 1 + 11 + 8 + \lfloor \frac{4}{5}(2 \times 11 + 1) \rfloor + \lfloor \frac{1}{4} \times 9 \rfloor = 20 + 18 + 2 = 40$. When 40 is divided by 7, the remainder is 5: a Friday, which is wrong: New Year's Day 2009 was on a Thursday. What has gone wrong?

The formula does not work for dates in this century. Can you work out why? How would you adapt it to work for years after 2000? A Prize Voucher worth R50 will be awarded for the best answer. □

BERTRAND RUSSELL'S PARADOX

Bertrand Russell, the renowned British philosopher, author, pacifist and mathematician, was born in 1872 and lived to the remarkable age of 97, dying in 1970.

The young Bertrand loved mathematics from an early age, despite bad teaching. His private tutor (he never went to school) used to throw books at him when he was unable to recite formulas parrot-fashion.

In 1890 Russell won a scholarship to study mathematics at Trinity College, Cambridge and was ranked Seventh Wrangler in the 1893 Mathematical Tripos. During this time he became increasingly attracted to philosophy and logic.

In 1900 Russell wrote *The Principles of Mathematics*. His aim was to show that mathematics and logic were one and the same thing. He found, however, that it was not easy. There were many difficulties in putting mathematical logic and set theory on a sound footing.

One of the great mathematicians of that time was Gottlob Frege. In 1902, at the end of a monumental work on the foundations of mathematics he had just completed, Frege had to add a sad note.

> "A scientist can hardly encounter anything more undesirable than to have the foundations collapse just as the work is finished. I was put in this position by a letter from Mr Bertrand Russell when the work was almost through the press."

Russell had sent Frege an ingenious paradox, now known as Russell's Paradox, showing that the notion of a *set* is not really as simple as it seems.

Some sets, such as the set of all elephants, are not members of themselves. Clearly, a set of elephants is not an elephant. Other sets, such as the set of all non-elephants, can be regarded as members of themselves, for clearly the set of all non-elephants is not an elephant. Now suppose we denote by S the set of all sets that are not members of themselves. If S is a member of itself, then by definition it must not be a member of itself. Similarly, if S is not a member of itself, then by definition it must be a member of itself.

If that seems too mind-bending, there is a popular version of Russell's Paradox, given in terms of the village barber who advertises that he shaves every man in the village who does not shave himself. Does the barber shave himself? If he does, then he shouldn't, according to the advertisement. If he doesn't shave himself, then he should.

Russell planned to follow *The Principles of Mathematics* with a second volume, but it soon became clear that a separate and independent work was needed. He embarked on a famous collaboration with Alfred North Whitehead, which was to culminate in 1910 in the publication of *Principia Mathematica*, in three massive volumes. Russell and Whitehead each had to contribute £50 towards its publication costs. Thus Russell came to speak of his income of "minus £50" for ten years' work.

In his autobiography, Russell says that the strain of writing *Principia Mathematica* was so great that his intellect never quite recovered. After 1910 Russell's mathematical activity ceased, with one exception.

Russell vigorously opposed the Great War. His activism lost him his fellowship at Trinity, and he was imprisoned for his pacifist activities. In prison he wrote *Introduction to Mathematical Philosophy*. It gave him some wry satisfaction to know that the prison censors had to read every word of his manuscript before it could be sent off to the printers.

Bertrand Russell once had a nightmare, which he related to his friend and fellow-mathematician G.H. Hardy. He was on the top floor of the University Library, about 2100 A.D. A library assistant, carrying an enormous bucket, was taking books down, glancing at them, and then restoring them to the shelves or dumping them into the bucket. Taking down one of the volumes of *Principia Mathematica*, he looked at it for a few moments, puzzled by the strange symbols. Then he closed the volume, balanced it in his hand, and hesitated ... .

Bertrand Russell was one of the great men of the 20th century: author, philosopher and humanitarian. After the Second World War he campaigned vigorously for nuclear disarmament, leading protest marches even in his old age. In 1950 he was awarded the Nobel Prize for Literature "in recognition of his varied and significant writings in which he champions humanitarian ideals and freedom of thought". His nightmare about his mathematical legacy is today seen to be unfounded. Because of its importance in artificial intelligence and theoretical computer science, there is today a resurgence of interest in mathematical logic. □

THE FINE STRUCTURE CONSTANT

The *fine structure constant* is one of the most intriguing numbers to be studied in modern physics. It combines three fundamental concepts: the speed of light (crucial to relativity), Planck's constant (the heart of quantum theory) and the magnitude of the charge of the electron.

To be precise: the fine structure constant is the ratio of the square of the charge of the electron divided by the speed of light c multiplied by Planck's constant $\hbar$. Each of the three constants has to expressed in terms of specific units: the speed of light is 3×10^8 metres per second, Planck's constant is 1.05×10^{-34} joule-seconds and the charge of the electron is 1.60×10^{-19} coulomb. However, even if you use a different measurement system (inches, ounces and years, or some unheard-of system devised by a civilisation in another part of our galaxy), when you put them together the units cancel out and the fine structure constant comes out the same: it is *dimension-free*.

Perhaps because if is so important to physics, the first letter of the Greek alphabet, α, is used to denote the fine structure constant.

$$\alpha = 7.297352570(5) \times 10^{-3}$$

This number determines the strength of the electromagnetic force. More accurately, it governs how strongly a charged particle will interact with that force (i.e. absorb or emit photons). So the fine structure constant is a basic physical constant that determines the nature of the Universe. If it were a different value, electrons wouldn't orbit atomic nuclei in the way they do—and might not orbit at all. If α were larger, you could not distinguish between matter and radiation. If it were smaller, particles would be unaffected by electromagnetic interaction.

Because it is so important, and dimension-free, and because $1/\alpha$ is so close to 137 (its value is 137.035999070(98)), the fine structure constant has from the start excited numerical speculation. Some of the world's greatest physicists, from Eddington to Feynmann, mused on its numerological significance. Wolfgang Pauli was a great physicist and Nobel prize-winner. He formulated the "exclusion principle", predicted the existence of the neutrino before it was experimentally detected — and became fascinated by the numerical significance of α. But his mental instability sent him to consult the analyst C G Jung (who knew nothing about physics). Pauli

and Jung became friends, and together they delved into the hidden meanings of the number 137, a prime number, with digit-sum 11, the *Tao*. This was surely meaningful! Jung was an expert in the Kabbalah, the study of numerical meanings of words in the Bible, and found it enormously significant that the four Hebrew letters of *Kabbalah* have values 5, 30, 2 and 100, which add up to 137.

Numerology is a fascinating, futile, illogical but unstoppable pursuit. The hospital room in which Wolfgang Pauli breathed his last was number 137.

Reference:
Deciphering the Cosmic Number, by Arthur J Miller.
(Reviewed in *New Scientist*, 25 April 2009) □

PRIME PROBLEM

Consider the polynomial

$$p(n) = n^6 + 3091$$

for successive integer values of n:

$p(1) = 3092$, which is composite (divisible by 2)
$p(2) = 3155$, which is composite (divisible by 5)
$p(3) = 3820$, which is composite (divisible by 10)
$p(4) = 7287$, which is composite (divisible by 7).

Carrying on in this way, it is found that $p(n)$ is composite for more and more values of n. (This is, of course, obvious for odd values of n, for then $p(n)$ is even. So all one has to do is test even values of n.)

Is it true that $p(n)$ is composite for *all* values of n?

The numbers quickly get out of the range of a scientific calculator, so we challenge our readers to use their computers to find the first value of n for which $p(n)$ is prime.

A R50 Prize Voucher will be awarded for the first correct answer opened on 1 December. □

WHO WAS EUCLID?

Euclid is renowned as the author of the most successful textbook ever written. His textbook of basic mathematics, the *Stoichia*, or *Elements*, was written around 300 B.C. and survived as a school textbook into the 20th century. It is said that it was published in more editions and more translations than any other book except the Bible. The name of Euclid has become a synonym for geometry.

However, the author of the element remains almost totally unknown. In some books he is confused with another Euclid: Euclid of Megara, a philosopher and pupil of Socrates about 100 years earlier. The geometer Euclid is assumed to have worked in Alexandria, founded by Ptolemy I, who ruled Egypt from 323 to 385 B.C. This fact is deduced from the anecdote that Ptolemy once impatiently asked Euclid if there was an easier way to study geometry than via the elements. Euclid is said to have replied loftily "O King, there is no royal road to geometry."

It's a good story, but unreliable, because the same exchange is said to have taken place between the future Alexander the Great and his mathematical tutor, Menaechmus, some years earlier.

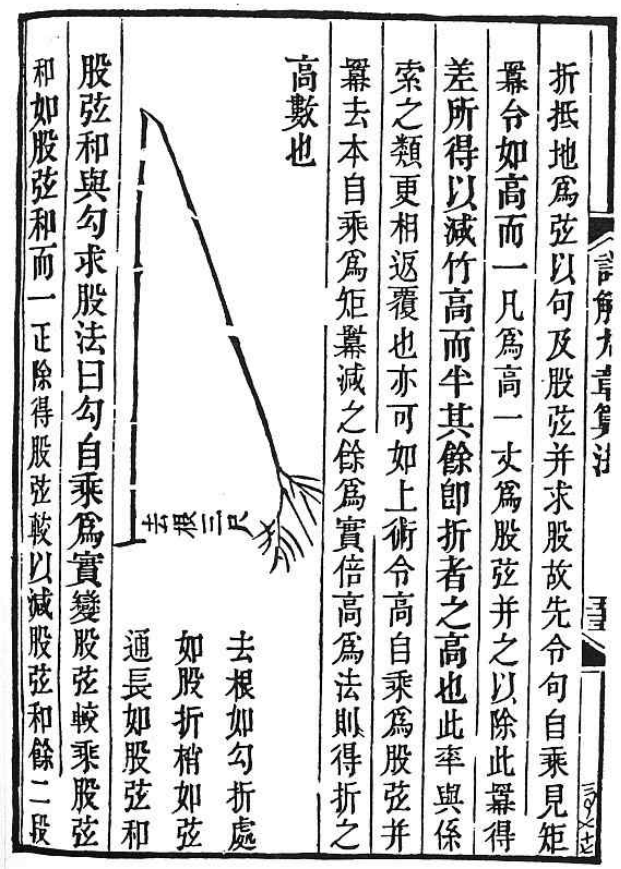
Another Euclidean anecdote relates how a student, after learning Proposition 1 of book 1 of the Elements, asked Euclid what he would gain from learning geometry. Turning to a slave, Euclid said sarcastically "Give him a penny, for he must profit by what he learns."

So the person who was Euclid remains a mystery. He has been portrayed by artists and sculptors in imaginary portraits and statues over the years. The statue shown is in the Oxford Museum of Natural History. Some historians are even prepared to admit the possibility that Euclid never existed, and that the Elements was written by a committee of Greek mathematicians who used the name Euclid as a collective pseudonym. □

THE BROKEN BAMBOO

The most influential ancient Chinese mathematical work was *Jiu Zhang Suanshu*, or the *Nine Chapters on the Mathematical Art*. The date and author are unknown, although the book it is believed to have been written in the second or third centuries A.D.

One of the most famous problems in the *Nine Chapters* is that of the broken bamboo, which was published in many subsequent books. The version below appeared in a book written in 1261.


折抵地爲弦以句及股弦并求股故先令句自乘見矩
冪令如高而一凡爲高一丈爲股弦并之以除此冪得
差所得以減竹高而半其餘卽折者之高也此率與係
索之類更相返覆也亦可如上術令高自乘爲股弦并
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A bamboo stalk 10 *syaku* high is broken at a point Q so that the top of the stalk falls over and touches the ground at a point T. The distance from the root of the bamboo P to T is 3 *syaku*. Find the distance PQ.

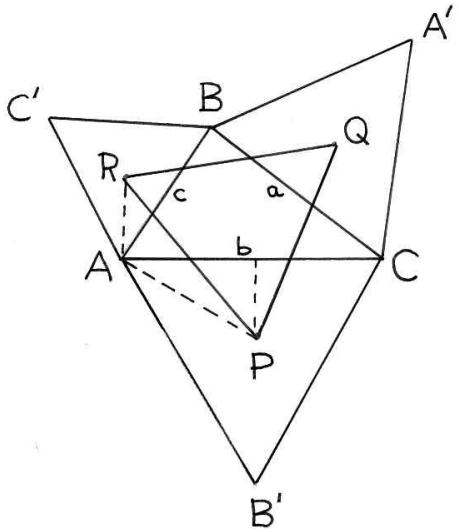
The best solution, from a reader in grade 10 or lower, will win a Prize Voucher worth R50. □

NAPOLEON'S THEOREM

The French Emperor Napololeon Bonaparte (1769-1821) did a great deal to advance science and mathematics in France during the early 19th century. He is also credited with discovering and proving the following theorem in Euclidean geometry.

> If equilateral triangles are constructed on the sides of any triangle, then the centroids of the three equilateral triangles will themselves form an equilateral triangle.

(The centroid of a triangle is the point of intersection of the three medians, and is the same as the point of intersection of the three internal angle bisectors.)



In the figure, ABC is the original triangle, with sides of length a, b and c. Triangles $AB'C$, $CA'B$ and $BC'A$ are contructed as shown, with centroids P, Q and R respectively.

We shall prove that triangle PQR is equilateral by calculating the length of PR, using trigonometry, and showing that it is symmetric in a, b and c.

First of all, we note that $\cos\angle PAC = \frac{1}{2}b/AP$. Since $\angle PAC = 30°$ and $\cos 30° = \frac{1}{2}\sqrt{3}$, it follows that $AP = b/\sqrt{3}$. Similarly, $AR = c/\sqrt{3}$.

Furthermore, $\angle PAR = \angle PAC + \angle CAB + \angle BAR = 60° + \angle A$.

Using the cosine rule in triangle PAR, we have

$PR^2 = AP^2 + AR^2 - 2AP.AR\cos\angle PAR$

$= \frac{1}{3}\big(b^2 + c^2 - 2bc\cos(60^\circ + \angle A)\big)$

$= \frac{1}{3}\big(b^2 + c^2 - 2bc(\frac{1}{2}\cos A - \frac{\sqrt{3}}{2}\sin A)\big)$

$= \frac{1}{3}(b^2 + c^2 - bc\cos A + \sqrt{3}bc\sin A)$.

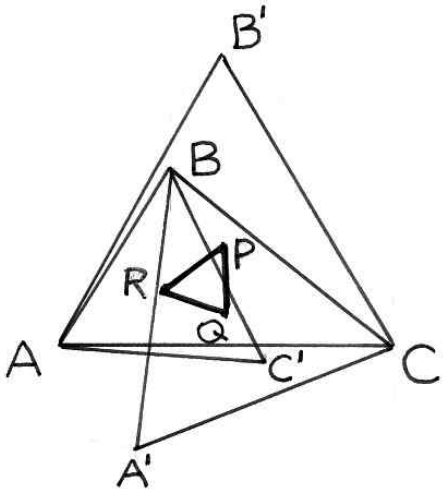
Applying the cosine rule to triangle ABC gives $bc\cos A = \frac{1}{2}(b^2+c^2-a^2)$. We also note, by the sine formula for the area of a triangle, that $bc\sin A = 2S$, where S is the area of triangle ABC. Making these substitutions gives

$PR^2 = \frac{1}{3}\big(b^2 + c^2 - \frac{1}{2}(b^2 + c^2 - a^2) + \sqrt{3}.2S\big) = \frac{1}{6}(a^2 + b^2 + c^2 + \frac{2}{\sqrt{3}}S)$.

This expression is symmetric in a, b and c, so PQ^2 and QR^2 will turn out to be just the same.

So $PQ = QR = RP$, showing that triangle PQR is equilateral.

If instead of constructing the three equilateral triangles *outwards* on the sides of triangle ABC, they were constructed *inwards*, i.e. flipped over, we would once again get an equilateral triangle. The proof above can be easily adapted to this new case. The two equilateral triangles thus obtained are called the outer and inner Napoleon triangles.



Prove that the difference between the areas of the inner and outer Napoleon triangles is equal to the area of the original triangle ABC. The best solution will win a R50 Prize Voucher.

There are many other ways of proving Napoleon's Theorem, and a surprising range of generalizations of the result. We leave it to our enterprising readers to discover these for themselves. $\square$

SHARP CALCULATOR COMPETITION

The Sharp Calculator Competition is sponsored by Sharp Electronics (S.A.) (Pty) Limited, who have donated three calculators as prizes.

These prizes are offered for the three best solutions of the problems below. Solutions may be in English or Afrikaans. Partial solutions will be judged on their merits. The Editor reserves the right to make no award if no satisfactory entries are received, and his judgement will be final. The winner of an nth prize in a previous competition will only be eligible for an mth prize if $m < n$.

Only high schools students in grades 11 or lower may compete. Entrants must give their full name and home address, age, school and grade. Solutions must be certified by a mathematics teacher at the school as the entrant's own unaided work.

The closing date for receipt of entries is 1 December 2009, and the results will be announced in our January 2010 edition.

SHARP

1. Solve the alphametric
$$\sqrt{ABCD} = AD$$
where A, B, C and D are four different non-zero digits.

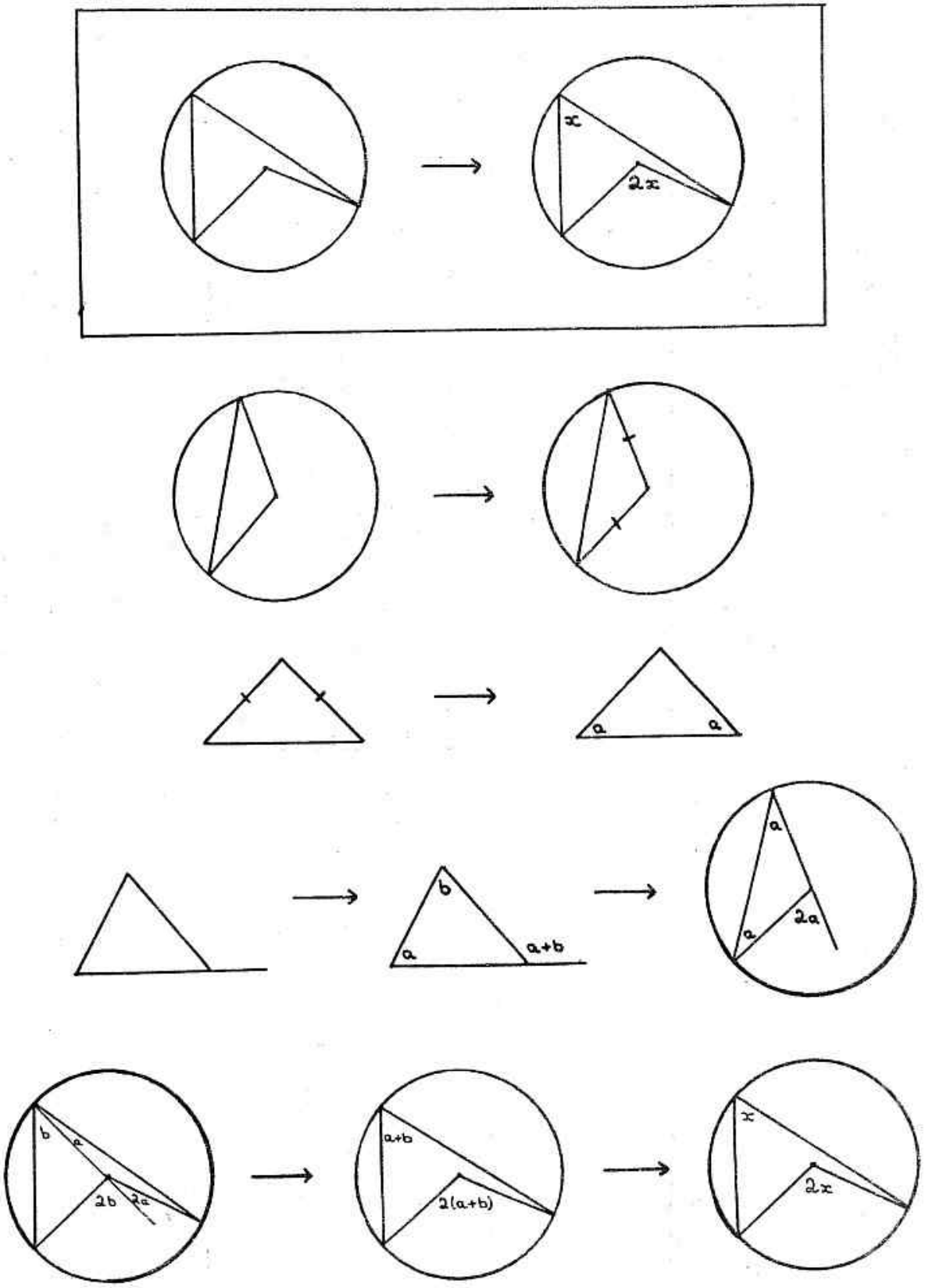
2. Prove that, if a, b, c and d are natural numbers such that
$$\sqrt{a} + \sqrt{b} + \sqrt{c} = d,$$
then a, b and c are perfect squares.

3. Prove that
$$\frac{3}{5} \cdot \frac{7}{9} \cdot \frac{11}{13} \cdots \frac{4n-1}{4n+1} < \sqrt{\frac{3}{4n+3}}$$
for all natural numbers n. □

DID YOU KNOW?

1.8015336×10^{12} is the speed of light in furlongs per fortnight. □

THEOREM WITHOUT WORDS



(From: *The Psychology of Learning Mathematics,* by R R Skemp) □

SHARP SOLUTIONS AND RESULTS

1. *Solve:* $(x+9)(x-3)(x-7)(x+5) = 385$.

   $(x+9)(x-3)(x-7)(x+5) = 385$
   $\Leftrightarrow \big((x+9)(x-7)\big)\big((x-3)(x+5)\big) = 385$
   $\Leftrightarrow (x^2+2x-63)(x^2+2x-15) = 385$
   $\Leftrightarrow (X-63)(X-15) = 385 \quad$ (setting $X = x^2+2x$)
   $\Leftrightarrow X^2 - 78X + 945 = 385$
   $\Leftrightarrow X^2 - 78X + 560 = 0$
   $\Leftrightarrow (X-8)(X-70) = 0$
   $\Leftrightarrow x^2+2x-8 = 0 \quad$ or $\quad x^2+2x-70 = 0$

   The first quadratic factorizes to give $(x+4)(x-2) = 0$, from which we get two solutions: $x = -4$ and $x = 2$.

   Using the quadratic formula to solve the second equation, we obtain two more solutions:

   $$x = \frac{-2 \pm \sqrt{2^2 - 4(2)(-70)}}{2} = -1 \pm \sqrt{71}.$$

   Source: The classic Victorian textbook, Hall and Knight's *Higher Algebra* (page 101, number 18).

2. *Let a, b and c be the lengths of the sides of a triangle with perimeter 2. Prove that*

   $$a^2 + b^2 + c^2 + 2abc < 2.$$

   If p is the perimeter of the triangle, then p is greater than each of $2a$, $2b$ and $2c$. So

   $0 < (p-2a)(p-2b)(p-2c)$
   $= p^3 - 2(a+b+c)p^2 + 4(ab+bc+ca)p - 8abc$
   $= p^3 - 2p^3 + 2\big((a+b+c)^2 - a^2 - b^2 - c^2\big)p - 8abc$
   $= p^3 - 2(a^2+b^2+c^2)p - 8abc.$

   Now put $p = 2$. This gives:

   $$0 < 8 - 4(a^2+b^2+c^2) - 8abc$$

   from which it follows that

   $$a^2 + b^2 + c^2 + 2abc < 2.$$

   Source: The German magazine $\overset{\text{Die}}{\sqrt{\text{WURZEL}}}$, May 2009.

3. *2009 whole numbers are arranged in a circle so that for any five consecutive numbers the sum of three of them is twice that of the other two. Prove that all the numbers are zero.*

   Denote the numbers by $a_1, a_2, \ldots, a_{2009}$. Consider any five consecutive numbers. Since the sum of any three of them is twice the sum of the other two, the sum of the five numbers is divisible by 3. Since $a_1 + a_2 + a_3 + a_4 + a_5$ and $a_2 + a_3 + a_4 + a_5 + a_6$ are both divisible by 3, it follows that a_1 and a_6 leave the same remainder when divided by 3. The same is true for a_6 and a_{11}, a_{11} and a_{16}, and so on, up to a_{2001} and a_{2007}, a_{2007} and a_3 (the numbers are in a circle). Going round the circle we deduce that all the numbers leave the same remainder on division by 3.

   But if five consecutive numbers, which have the same remainder when divided by 3, have a sum that is divisible by 3, then that common remainder is divisible by 3. So all five numbers are divisible by 3. Hence all the numbers in the circle are divisible by 3.

   Divide all the numbers by 3. The new numbers have the same property as the original numbers (that for any five, some three have a sum which is twice the sum of the other two). So the quotients are all divisible by 3. Divide by 3 again: we deduce that all the original numbers are divisible by any power of 3. So all the original numbers are divisible by any power of 3, which is impossible unless all the original numbers are zero.

   Source: Belarus Mathematical Olympiad, 2007.

SHARP

No prizeworthy entries were received. □

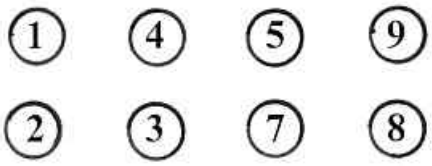
MATHS AND THE ECONOMY

When the going gets tough, the tough get their maths books out. Many of the diehard optimists on Wall Street have been beaten to a pulp by now, but those still standing have fallen back on a nifty piece of calculus. The second derivative, they say, is turning positive. That means that although the economy is spiralling down, it is doing so more slowly.

(*The Economist*, February 21–27, 2009) □

PRIZEWINNERS AND SOLUTIONS

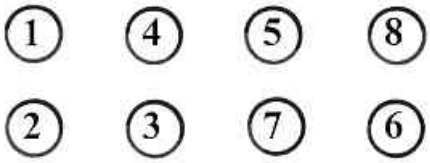
Counter Problem (July 2009, p. 5):
Eight numbered counters have been arranged in two rows, as shown



In this arrangement, the sums of the numbers in the two rows are different. Can you switch a counter in the first row with a counter in the second row, so that the sums in the two rows are the same?

Several readers asserted that the numbers add up to 39. If they were arranged into two rows of four, with equal sums, the sum of all eight numbers would be have to be even. So, they said, there is no solution.

They overlooked the solution below:



Count Down (July 2009, p.7):
Can you put the standard arithmetical symbols between the numbers

10 9 8 7 6 5 4 3 2 1

so that the calculation gives exactly 2009? Cenèa Whiteley (Umzinto Secondary School) submitted

$$10 \times 987 - 6^5 - (43 \times 2 - 1)$$

winning a copy of Steve Olsen's *Count Down*.

Methamaticla Anagarms (July 2009, p. 23):
Nicholas Cotchobos (Rondebosch Boys' High School) was able to pair off the ten words and rearrange the letters to form mathematical words:

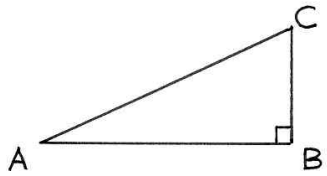
loath + grim = algorithm, bag + real = algebra, her + mote = theorem, agent + creep = percentage, carrot + port = protractor.

His entry wins him a free 2010 subscription to *Mathematical Digest*. □

PYTHAGORAS IN 3-D

The Theorem of Pythagoras says that, in a right-angled triangle ABC with $\angle B = 90°$,

$$AB^2 + BC^2 = AC^2.$$



There is a three-dimensional version of this famous result. Instead of a triangle, consider a tetrahedron $OABC$ with three faces mutually perpendicular to each other at O. Then the sum of the squares of the areas of the three mutually perpendicular faces is equal to the square of the area of the fourth face.

The 3D Theorem of Pythagoras is not difficult to prove. Not surprisingly, it will use the well-known 2D version.

Set up the tetrahedron $OABC$ in a three-dimensional coordinate system with the origin at O and the three vertices A, B and C on the x-, y- and z- axes respectively. Let $OA = a$, $OB = b$ and $OC = c$, as shown.

Let P be a point on AB such that PC and AB are perpendicular, and let $OP = d$ and $PC = e$.

It is convenient to use the notation $|XYZ|$ to denote the area of a triangle XYZ.

Then $|AOB|^2 + |BOC|^2 + |COA|^2$

$$= (\tfrac{1}{2}\sqrt{a^2+b^2}.d)^2 + (\tfrac{1}{2}bc)^2 + (\tfrac{1}{2}ca)^2$$

$$= \tfrac{1}{4}(a^2+b^2)d^2 + \tfrac{1}{4}(a^2+b^2)c^2$$

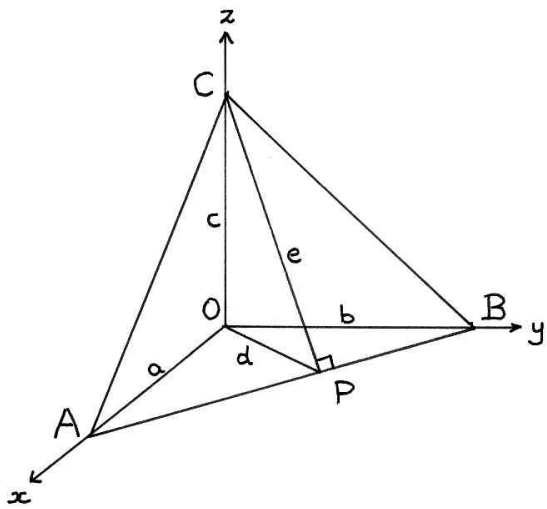
$$= \tfrac{1}{4}(a^2+b^2)(d^2+c^2)$$

$$= \tfrac{1}{4}(a^2+b^2)e^2$$

$$= \tfrac{1}{4}AB^2e^2$$

$$= (\tfrac{1}{2}AB.e)^2$$

$$= |ABC|^2.$$



□

THE TWENTIETH INTER-PROVINCIAL MATHEMATICS OLYMPIAD

The first Interprovincial Mathematical Olympiad (IPMO) took place in 1990, with just two teams taking part: Western Province and Natal. The 20th IPMO took place this year on the afternoon of Saturday 12 September, with ten competing provinces: Boland, Eastern Province, Free State and Northern Cape, Gauteng North, Gauteng South, KwaZulu-Natal Coast, KwaZulu-Natal Midlands, KwaZulu-Natal North Coast, North-West Province and Western Province.

A "province" does not necessarily coincide with any of the official nine provinces of the country, and in fact is closer to sports leagues, such as the Currie Cup.

Each province enters two teams of ten: Junior (grades 8 and 9) and Senior (grades 10, 11 and 12), and may also enter B and C teams at each level. This year 20 Junior teams and 19 Senior teams took part.

Teams meet at different centres around the country. With sponsorship from the Actuarial Society of South Africa, financial support is available to all participating provinces to provide T-shirts, certificates and refreshments for their teams.

The first part of the Olympiad is a one-hour individual problem paper, consisting of 20 problems in multiple-choice format. After a break for refreshments and a discussion of team strategy, the second part of the Olympiad takes place. This is a paper consisting of ten quite difficult problems, each requiring only an answer. A team works together and submits just one set of answers.

Papers are marked at once, and teams scores are phoned, faxed or e-mailed through to the IPMO National Organiser, Prof. Peter Dankelmann, at the University of KwaZulu-Natal. When all scores are in, the rankings are sent back to the provinces at about 5 p.m.

This year's IPMO saw Western Province emerge clear winners in each division. The top three Junior teams were Western Province, with 1606 points, followed by KwaZulu-Natal Coast (1454) and Free State and Northern Cape (1374). The scores of the top Senior teams were: Western Province (1607), Gauteng South (1380) and Gauteng North (1303).

Full details, including the question papers and solutions, may be found on the website `www.samf.ac.za/ipmo`.

Information on how to take part in the Interprovincial Mathematical Olympiad may be obtained from:

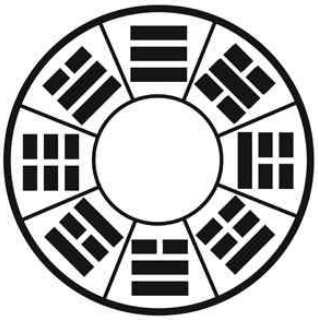
Prof. Peter Dankelmann
Department of Mathematics
University of KwaZulu-Natal
Tel (031) 260-1017
Email: dankelma@ukzn.ac.za

□

THE 2009 UCT MATHEMATICS COMPETITION

A full report on the 2009 UCT Mathematics Competition has just been published. It comprises all the question papers (grades 8 to 12) set in this year's competition, full solutions, a list of all the prizewinners and detailed statistics and graphs.

An invaluable source of ideas for the classroom, the book includes a large number of unusual and challenging problems very different from the routine exercises found in textbooks and standard school examinations. It thus provides essential training material for anybody preparing for maths competitions and Olympiads. For those planning to continue with mathematics at university level, it offers useful problem-solving experience and insights into the sort of flexible thinking required in university maths courses. In particular, working through the grades 11 and 12 papers is excellent preparation for university entrance examinations or placement tests.



Copies of *The 2009 UCT Mathematics Competition* cost R40 each (including VAT, packing and postage). Use the form at the back of the magazine to order copies. □

BOOK REVIEW

Aenigmata, by Lloyd Tooke and Lionel Waker, 9LivesPublishing, 2009. ISBN: 978-0-620-42491-2

Take a gentle, cerebral, version of *Dungeons and Dragons*, add a dash of H.E. Dudeney's *The Canterbury Puzzles*, the quirky logic of Lewis Carroll and the intrigue of *Lord of the Rings* and you have this little book, just 15 cm square and half a centimetre thick. It will take you on a medieval journey to rescue a beautiful princess. On the way, you meet a series of warlords and wizards who demand answers to riddles and solutions to puzzles before allowing you to pass.

At the Isthmus of Intrigue, the traveller meets two men on horses, riding very slowly. They tell you

> *We are great horsemen and love to race our steeds. Our father has died and his final instructions were that we were to ride across the isthmus and whoever's horse got there last would be heir to his lands and wealth. We've been stalling for days and see no way out of this situation. Help us and you can have the pick of our horses.*

You solve the puzzle and now have a horse. Later, at a T-junction, you meet an apprentice wizard and two other men. The apprentice says:

> *You need to turn here and one of the roads leads to great danger. I have with me identical twins, one of whom always tells the truth and the other always lies. What question would you ask in order to discover whether you should turn left or right?*

Having worked out which road to follow, you later arrive at Lava Lake, where another apprentice wizard bars your way. He says:

> *Welcome to Lava Lake. Only every 45 minutes is the lava solid enough to walk across, and then only for a minute. Here are two pieces of string; each will take exactly one hour to burn, but not at a consistent rate. Use these strings and your matches to work out exactly when the next opportunity to cross will be. A hot-headed guess will result in, well, a hot head.*
>
> The apprentice then calmly walks across the solid lake and once on the other side, to illustrate the danger, tosses a small stone onto

the lava. It sinks, as does your heart as you realise you have no time to waste.

Can you solve any of these puzzles? Send your solutions to the Editor. The best solutions will win a copy of *Aenigmata*.

The final problem, which will win you the hand of the beautiful princess (no incentives for female readers), is really tough. You will need to read the whole book in order to crack its well-hidden code. □

SUMS OF SQUARES

Which numbers can be expressed as the sum of two squares? Let's start by listing all possible sums of two squares systematically.

$0^2 + 0^2 = 0$	$0^2 + 4^2 = 16$	$5^2 + 5^2 = 50$
$0^2 + 1^2 = 1$	$1^2 + 4^2 = 17$	$0^2 + 6^2 = 36$
$1^2 + 1^2 = 2$	$2^2 + 4^2 = 20$	$1^2 + 6^2 = 37$
$0^2 + 2^2 = 4$	$3^2 + 4^2 = 25$	$2^2 + 6^2 = 40$
$1^2 + 2^2 = 5$	$4^2 + 4^2 = 32$	$3^2 + 6^2 = 45$
$2^2 + 2^2 = 8$	$0^2 + 5^2 = 25$	$4^2 + 6^2 = 52$
$0^2 + 3^2 = 9$	$1^2 + 5^2 = 26$	$5^2 + 6^2 = 61$
$1^2 + 3^2 = 10$	$2^2 + 5^2 = 29$	$6^2 + 6^2 = 72$
$2^2 + 3^2 = 13$	$3^2 + 5^2 = 34$	
$3^2 + 3^2 = 18$	$4^2 + 5^2 = 41$	and so on.

Here are some questions which should strike you as you study this list.

- Which numbers appear more than once on the right hand sides of the above sums? Well, 25 does for a start. Are there any others?
- Can a number appear more than twice on the right hand side?
- Which numbers never appear on the right? In other words, can you predict, without trying all the possibilities, whether a number can be expressed as a sum of two squares?
- The list shows that $5(= 1^2 + 2^2)$ and $8(= 2^2 + 2^2)$ are both sums of two squares. Notice also that $5 \times 8 = 40$ is also a sum of two squares. Can you prove that, in general, if two numbers are expressible as the sum of two squares, then so is their product? □

IOI IN BULGARIA

The 2009 International Olympiad in Informatics (IOI) took place in Plovdiv, the second largest city in Bulgaria. This was the 21st IOI; the first took place in Pravetz, Bulgaria in 1989.

Teams of four from 83 countries took part. The South African team was:
Francois Conradie (De Kuilen High School, Kuils River, Western Cape)
Schalk-Willem Krüger (Ferdinand Postma High School, Potchefstroom)
Graham Manuell (De La Salle Holy Cross College, Johannesburg)
Kosie van der Merwe (Brackenfell High School, Western Cape)

The Team Leader was Peter Waker, Manager of the Standard Bank/CSSA Computer Olympiad, assisted by Max Rabkin, a postgraduate student at the University of Cape Town and a former IOI medallist.

The team was chosen from the results of the 2008 Standard Bank/Computer Society of South Africa Computer Olympiad, which attracted 33 000 entrants. Three additional training camps were run by the South African Computer Olympiad Trust.

The participants spent five-hour days writing computer programs to solve eight problems. Francois Conradie took the lead among the South Africans by scoring 266 (out of 400) on the First Day. On the next competition day he added another 179 points; sufficient to earn him a bronze medal.

The outright winner of the IOI was 14-year-old Henadzi Karatkevich, from Belarus.

Team Leader Peter Waker commented: "The problems were extremely challenging this year, and some participants from other countries had to go home with a zero score. Obtaining a bronze under those circumstances is an excellent achievement."

While in Bulgaria, the teams had plenty of opportunities to relax. There was a visit to the old "Old City" of Plovdiv, go-kart racing and an excursion to the Black Sea.

For more information about the South African Computer Olympiad and participation in the IOI, contact:
Peter Waker (Manager: Standard Bank/CSSA Computer Olympiad)
Phone: (021)448 7864 Fax: (021)447 8410 Email: info@olympiad.org.za
Website: www.olympiad.org.za □

COMPUTER OLYMPIAD RESULTS

The results of the 2009 South African Computer Olympiad were announced at an Awards Dinner in Cape Town on Monday 28 September. From an initial entry of over 34 000, just fourteen finalists were present. They had been invited to Cape Town to write the final round of the Olympiad over the previous weekend.

Bronze Medals were won by Francois Conradie (De Kuilen High School), Schalk-Willem Krüger (Ferdinand Postma High School) and Sean Wentzel (Westerford High School).

A Silver Medal was won by Graham Manuell (De La Salle Holy Cross College).

For the first time in the history of the Computer Olympiad, there was a tie for first place. So two Gold Medals were awarded, to Kosie van der Merwe (Brackenfell High School) and Gwylim Ashley (Oakhill School).

The runners-up were

Kieren Davies (International School of Hout Bay)
Charl du Plessis (Stellenberg High School)
Arlton Gilbert (Star College, Durban)
Melvin Musehani (Pretoria Technical High School)
Vaughan Newton (Bridge House)
Awadi Rathugamage (St John's College)
Robert Spencer (Westerford High School)
Benjamin Swart (Bellville High School).

The Computer Olympiad is a project of the Computer Society of South Africa, and is sponsored by Standard Bank. □

WATT'S WRONG?

As iron consumes 1 500 watt an hour, as much energy as ten 100 watt light bulbs.

(*Southern Suburbs Tatler,* 24 September 2009) □

SALARY SURPRISE

Suppose you are offered two jobs.

Job A offers a salary of R100 000 per annum, with increases of R4 000 per annum.

Job B offers the same initial salary of R100 000 per annum, but with increases of R1 000 every six months.

Which should you take? At first sight, Job A looks better, since a single increment of R4 000 looks better than two increments of R1 000. However, it is worthwhile to look at the numbers in detail.

In the first year, Job A will pay you a total of R100 000. But Job B will pay R101 000: R50 000 in the first six months of the year, and R51 000 in the second half of the year. So at the end of the first year Job B has paid a total of R101 000, and is therefore R1 000 ahead of Job A.

In the second year, Job A will pay you a total of R104 000. Job B will pay R52 000 in the first half-year, and R53 000 in the second half-year, for a total of R105 000 for the year. So at the end of the second year Job B is once again ahead by R1 000.

In the third year, Job A pays R108 000, but Job B pays R54 000 + R55 000 = R109 000, and is again R1 000 ahead of Job A.

The pattern continues. Every year Job B will be ahead of Job A by R1 000 per annum. □

BACON QUOTE

All science requires mathematics. The knowledge of mathematical things is almost innate in us. This is the easiest of sciences, a fact which is obvious in that no one's brain rejects it; for laymen and people who are utterly illiterate know how to count and reckon.

(Roger Bacon, *Opus Maius*, 1267) □

DID YOU KNOW?

Numeraria was the Roman Goddess of Mathematics. Romans prayed to her mainly to make them rich. □

IG NOBEL PRIZE IN MATHEMATICS

Hyper-inflation expert Gideon Gono, governor of Zimbabwe's Reserve Bank, has been awarded the prestigious Ig Nobel award for Mathematics. He has been honoured as one "who imbued his compatriots with a sophisticated understanding of large numbers by printing the national currency in denominations ranging from 1 cent to 100 trillion dollars as the country's inflation rate soared to 231 million per cent".

The 2009 Ig Nobel awards ceremony took place at Harvard University on 1st October, with the coveted prizes handed out by real Nobel laureates. Gideon Gono did not attend the ceremony.

The award of the Ig Nobel Prizes is an annual exercise in irreverence to recognize achievements that "cannot, or should not, be repeated". They are given to scientists and others whose results first make people laugh, and then make them think. The announcement of the Ig Nobel Prizes is hosted by the Harvard satirical journal *Annals of Improbable Research*. Said to be "second most important event on the scientific calendar", it is timed to take place shortly before the real Nobel Prizes are announced in Stockholm. □

NUMBER REFLECTIONS

Emmanuel Setlhapelo, of Kismet Combined School in North West Province, was doing his homework one night when he noticed that the time was 21:12. The idea of number reflections then "exploded in my head" he writes. What happens if you subtract a number from its reversal?

He noticed that $21 - 12 = 9$.

If you start with 51 and repeat the process you eventually get 9:

$$51 - 15 = 36, \quad 63 - 36 = 27, \quad 72 - 27 = 54, \quad 54 - 45 = 9.$$

Take any two-digit number, reverse it and subtract (larger minus smaller). If the difference is a two-digit number, repeat the process. Does the process always end with a 9?

What happens if you start with a 3-digit number? Emmanuel's idea is a good topic for an classroom investigation or individual project. □

MATHS AND CHESS

> "The best players of such puzzle games as chess and draughts are not mathematicians, though it is just possible that often they may have undeveloped mathematical minds."

Professor Dirk Laurie (University of Stellenbosch) sent in this quotation, and asked the following questions:

1. In what famous book on mathematical puzzles, published in 1907 and reissued in 1919, does the above sentence appear?

2. Why was the timing of the sweeping statement expressed in it particularly unfortunate?

The first correct answer opened on 1 December will win a R50 Prize Voucher. □

TELLING AGES

Alan, Brian, Colin and Damon were asked their ages by their teacher. They each made two statements: one true and the other false.

Alan said:	"Brian is 15. Colin is 12."
Brian said:	"Colin is 13. Damon is 15."
Colin said:	"Damon is 17. Alan is 13."
Damon said:	"Alan is 14. Brian is 16."

The maths teacher found that there was not enough information to work out the boys' ages, but she was able to determine which boy was the youngest. Can you? A Prize Voucher worth R50 will be awarded for the best solution. □

THE WORLD'S BEST MATHS HOWLERS

An aziom is something so obvious that you don't have to see it.

Infinity is where things happen that don't.

Parallel lines do not meet unless produced to eternity.

Algebra was the wife of Euclid.

A polygon is a dead parrot.

An ellipse is what happens when the Moon gets behind the Sun.

A point has position but no magnetism.

90° is the boiling point of a right-angled triangle.

A circle is a polygon with no corners and only one side.

The basic postulates of geometry are pencil and paper.

One horsepower is the amount of energy it takes to drag a horse 550 feet in one second.

Momentum is what you give a person when they are going away.

A Newton is the weight of an apple.

In geometry we learn how to bisex angels.

A seven-sided polygon is called a hooligan.

An eight-sided polygon is called an oxygen.

An angle is a triangle with only two sides.

If an angle is greater than 90° it is obscene.

A parallelogram with all its sides equal is called a rumpus.

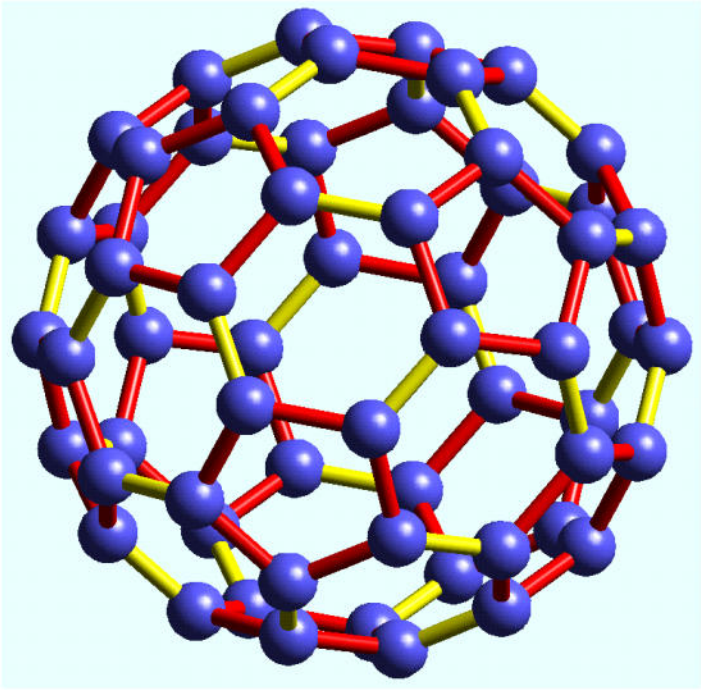
The commutative law of addition is when you add a column of numbers from top to bottom and then from bottom to top and get two different answers.

A circle that cuts through each vertex of a triangle is called a circumcised circle.

Send your favourite maths howlers to the Editor. Prize Vouchers worth R50 will be awarded for the best submissions. □

mathematical digest

No. 158 January 2010



The Buckyball

mathematical digest

Number 158 **January 2010**

While the individual man is an insoluble puzzle, in the aggregate he becomes a mathematical certainty. You can, for example, never foretell what any one man will be up to, but you can say with precision what an average number will be up to. Individuals vary, but percentages remain constant. So says the statistician.

(Sherlock Holmes, in *The Sign of Four*)

Department of Mathematics and Applied Mathematics
University of Cape Town

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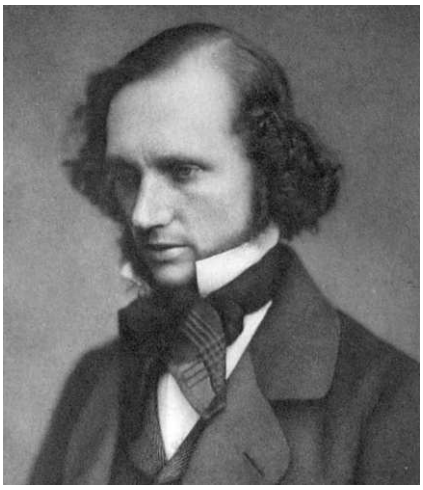
Cover: The carbon-60 molecule, known as *buckminsterfullerene* (*buckyball* for short), was discovered in 1985. You can read more about it on page 2.

KELVIN: SECOND WRANGLER

At Cambridge University the final examination in Mathematics is called the Mathematical Tripos. Students who achieve a First Class in the Tripos are dubbed *Wranglers*, and the top student is named *Senior Wrangler*. After that, the students are named Second Wrangler, Third Wrangler, etc.



One of the greatest physicist of the 19th century was William Thomson, later Lord Kelvin, who attended Cambridge in the 1840s. He was a brilliant mathematics student, and heavily tipped to win the accolade of Senior Wrangler. Legend has it that on the day the results came out, he said to his manservant imperiously "Oh, just run along to the Senate House to see who is Second Wrangler." The servant returned with the reply "You, sir!"

It is a good story but untrue. The Tripos results were always read out in public in the Senate House of the University, and attracted a large crowd. From a letter to his father written on the day after the announcement, it is clear that Thomson was there when the results were read out.

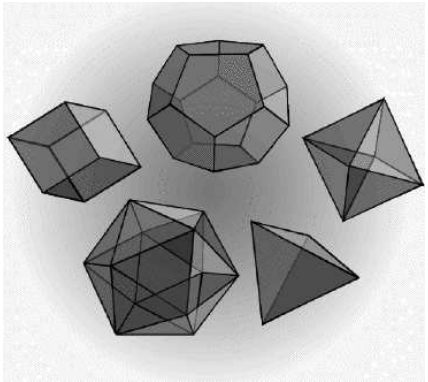
Another good story is told of how it happened that William Thomson was ranked below the Senior Wrangler (one Stephen Parkinson). The story is that the final exam included the proof of a theorem that Thomson himself had discovered and published while still a student. Because he had created the proof, he had not bothered to memorise it, and spent a lot of time in the exam working it out from scratch. Parkinson knew the result and had memorised the proof, so knocked it off very quickly.

This story is also untrue. In his two-volume biography of Kelvin, Sylvanus P. Thompson says Thomson put his failure down to bad examination technique. In a crucial paper, he said, "I spent nearly all my time on one particular problem that interested me, about a spinning top being let fall on to a rigid plane—a very simple problem if I had tackled it in the right way. But I got involved and lost time on it, and wrote something that was not good, and there was no time left for the other questions."

A week after the Tripos results came out it was announced that William Thomson had been placed first in a different Cambridge examination, for the coveted Smith's Prize. Parkinson was second. □

THE FOOTBALL, THE BUCKEYBALL AND THE FAT MAN

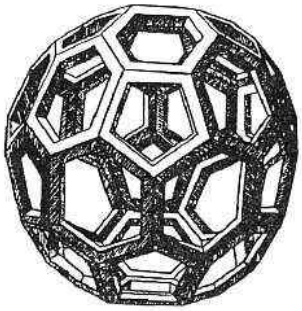
The design of the standard football is based on a geometrical figure known as a *truncated icosahedron*. The icosahedron (the word is comes from the Greek, and means "twenty-sided") consists of twenty equilateral triangles, all of the same size, joined together along their edges so that five triangles meet at each vertex. It is one of the five regular polyhedra known to the Greeks. Shown, clockwise from the left they are: cube, dodecahedron, octahedron, tetrahedron, icosahedron.



One of the great achievements of Euclid's work on geometry was the proof that there are only five regular polyhedra.

If you take an icosahedron, divide each edge into three equal lengths, and cut off pentagonal pieces at each vertex, you arrive at what is known as a *truncated icosahedron*. ("Truncate" just means "cut"; the original Latin meaning of the word was "to maim, or mutilate").

With all the corners cut off the icosahedron, we have a new shape with 32 faces: 20 hexagonal faces (the original twenty triangular faces, with their corners cut off, and twelve pentagonal faces (from cutting off each of the twelve corners). At each vertex of the new polyhedron, two hexagons and one pentagon meet.



This is the design of the standard football. Inflating the ball makes the faces bulge out into a sphere. This design has been known for many centuries, as the woodcut by Leonardo da Vinci shows.

The truncated icosahedron is also the shape of the carbon 60 (C_{60}) molecule, first synthesised in 1985 by chemists Robert Curl, Harold Kroto and Richard Smalley, who named it *Buckminsterfullerene*. That mouthful was coined because the shape of the molecule reminded the chemists of the geodesic

domes pioneered by the American architect Richard Buckminster Fuller, shown below in front of one of his domes.

The discoverers of buckminsterfullerene were awarded the Nobel Prize in Chemistry in 1996. The name was soon shortened to "fullerene", and more colloquially "buckyball".

While the football and the buckyball have the same shape, they are rather different in size. The diameter of a football is 22 cm and the fullerence molecule measures about one nanometre (10^{-9} metres, or one-millionth of a millimetre). So the (linear) ratio of their sizes is 220 million to one. In volume terms, it would take 10^{25} buckyballs to fill a football.

The same shape was used in designing the detonation devices for the first atomic bombs. In order to set off a chain reaction in a mass of plutonium, it must be compressed into critical mass, at which nuclear fission starts. If it is not held together long enough, the fission merely blows the plutonium apart into non-critical pieces and the chain reaction stops. Explosive charges directed inwards on the mass of plutonium must hold the plutonium together long enough for it to go critical. The positioning of the explosive charges is crucial, and it was found that charges set at the centres of the faces of a truncated icosahedron worked well. Fat Man, the nickname of the bomb dropped on Nagasaki in 1945, was detonated in this way. □

THE TURING TEST

Alan Turing (1912–1954) was a British mathematician who played a leading role in the Second World War in cracking the German codes, thus giving the British important advance notice of German military plans. He designed an early form of computer to do the job. After the War he made major contributions to mathematical logic and its application to computer science.

In 1950 he asked the question "Can Machines Think?", and proposed a test that has become a cornerstone of artificial intelligence. A human judge engages one human and one machine in a conversation, via a computer keyboard and screen. If the judge cannot tell which is which, the computer is said to have passed the Turing Test.

In 1952 tragedy struck. Alan Turing was convicted of homosexuality (then a crime) and shortly afterwards committed suicide. The story is movingly told in Hugh Whitemore's 1986 play, "Breaking the Code".

In September last year, when the 70th anniversary of the outbreak of the Second World War was being commemorated, British Prime Minister Gordon Brown paid special tribute to Alan Turing.

> Turing was a quite brilliant mathematician, most famous for his work on breaking the German Enigma codes. It is no exaggeration to say that, without his outstanding contribution, the history of the Second World War could have been very different. He truly was one of those individuals we can point to whose unique contribution helped to turn the tide of war. The debt of gratitude he is owed makes it all the more horrifying, therefore, that he was treated so inhumanely.
>
> In 1952, he was convicted of "gross indecency"– in effect, tried for being gay. His sentence – and he was faced with the miserable choice of this or prison – was chemical castration by a series of injections of female hormones. He took his own life just two years later.

> It is thanks to men and women who were totally committed to fighting fascism, people like Alan Turing, that the horrors of the Holocaust and of total war are part of Europe's history and not Europe's present. So on behalf of the British government, and all those who live freely thanks to Alan's work, I am very proud to say: we're sorry. You deserved so much better.

The BBC Radio 4 programme *Saturday Live* featured the official apology to Alan Turing for his 1952 conviction. The programme's resident poet, Matt Harvey, penned this short but poignant poem to mark the occasion:

heres a toast to Alan Turing
born in harsher, darker times
who thought outside the container
and loved outside the lines
and so the code-breaker was broken
and we're sorry
yes now the s-word has been spoken
the official conscience woken
very carefully scripted but at least
it's not encrypted
and the story does suggest
a part 2 to the Turing Test:
1. can machines behave like humans?
2. can we? □

FANCY FACTORS

Each of the following polynomials can be factorized into two or more polynomials of lower degree, with integer coefficients. And none of the factors are linear!

$$1 + x^4 + x^5$$
$$1 + x^7 + x^8$$
$$1 + x^6 + x^7 + x^8 + x^9$$

Send your factorizations to the Editor. Can you find more examples like these? The best entry will win a Prize Voucher worth R50. □

STEPHEN HAWKING RETIRES

Steven Hawking is probably the most famous of all living scientists. His book, *A Brief History of Time*, published in 1988, stayed on the New York Times best seller list for a record-breaking 237 weeks. At the end of September 2009 he retired from his position as Lucasian Professor of Mathematics at the University of Cambridge.

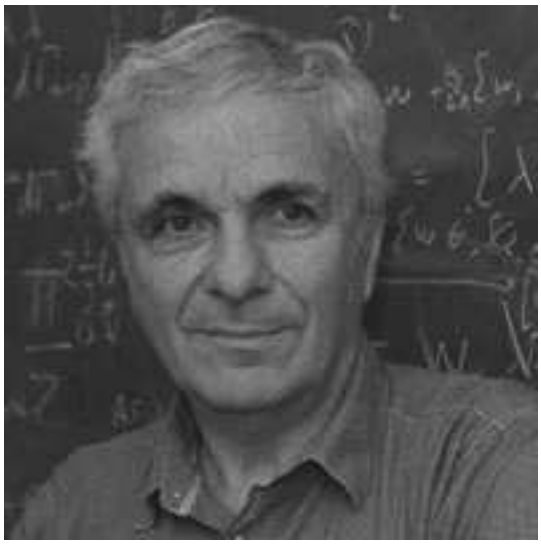
The position of Lucasian Professor at Cambridge was founded in 1663 as a result of a gift from Henry Lucas, the Member of Parliament for Cambridge at the time. Lucas left instructions in his will that a piece of land should be purchased that would provide an annual income of one hundred pounds to support the professorship.

The position has had many famous incumbents, including Isaac Newton (1669-1702), the computer pioneer Charles Babbage (1828-1839) and the theoretical physicist Paul Dirac (1932-1969). Stephen Hawking was elected to the position in 1979.

He is known for his work in cosmology and quantum gravity, and especially for his description of black holes. With Roger Penrose he analyzed singularities in the framework of general relativity, and predicted that black holes should emit radiation.

From a young age he was diagnosed with motor neurone disease, a neuromuscular dystrophy that has progressed over the years and has left him almost completely paralyzed. His mathematical work has continued unabated throughout.

His books and public appearances have made him an international celebrity and he has received many honours over the years. Just before his retirement he was awarded the Presidential Medal of Freedom, the highest civilian award in the United States.



Michael Green, a distinguished theoretical physicist, has been elected the 18th holder of the Lucasian Professorship of Mathematics. He is one of the founders of string theory, which he pioneered from the early seventies onwards. Apart from original research in the area, he co-authored, with Edward Witten and John Schwarz, a book which for many years was the only string theory textbook around.

String theory is an attempt to reconcile quantum mechanics with relativity by arguing that subatomic particles are strings vibrating through space and time, in a universe of 10 or 11 dimensions. The goal of string theory is to produce a Theory of Everything. □

MATH QUIZ

1. Which famous mathematician was born on Christmas Day?
2. Which mathematician gave his name to a unit of magnetism?
3. A famous living mathematician was born exactly three hundred years (to the day) after the death of a famous pioneer physicist. Who is the mathematician?
4. Who was the physicist?
5. Who said "O King, there is no Royal Road to Geometry".
6. Who was the King?
7. Which famous mathematician was killed in the sack of his home town by an invading army?
8. Who was the General in command of the army?
9. Whose "Last Theorem" was finally proved in 1994?
10. Who proved it?

A Prize Voucher worth R50 will be awarded for the first reader sending in the correct answers. □

FLORENCE NIGHTINGALE: MEDICAL STATISTICIAN

Florence Nightingale was one of the most influential women of the 19th century. She did more than any other person in Britain to raise the standard of medical treatment in hospitals, and make nursing and honourable profession. She would not have succeeded in her mission without a solid background of mathematics.

Florence was born in 1820 and as a young girl was fascinated by mathematics, which she insisted on studying instead of the traditional feminine pursuits of needlework and dancing. Her mother disapproved, but her father engaged James Joseph Sylvester, later a prominent mathematician, as one of her tutors. Before her involvement with nursing, she tutored young children in arithmetic and geometry.

After a year as unpaid superintendent of a London "establishment for gentlewomen during illness", the Secretary of War, Sidney Herbert, recruited Florence Nightingale and 38 nurses for service in the Scutari hospital during the Crimean War (1853–1856). The Crimean War is today notorious for military blunders; incompetence and mismanagement extended to the military hospitals. When Florence Nightingale and her nurses arrived in Scutari they found a nightmare of chaos and misery. Food was inadequate. Soldiers were more likely to die of preventable diseases like cholera than from military action. Nearly half the wounded died in hospital because of the unsanitary conditions. Florence Nightingale's reports back to Britain brought action. By the end of the Crimean War the hospitals were well-run and efficient, with mortality rates no greater than civilian hospitals in England. At the personal level her practice of making late-night rounds of hospital wards to minister to injured soldiers led to her being dubbed "The Lady with the Lamp". The American poet Henry Wadsworth Longfellow immortalised her in his 1857 poem "Santa Filomena":

Lo! in that hour of misery
A Lady with a Lamp I see
Pass through the glimmering gloom,
And flit from room to room.

Florence Nightingale returned to England amid huge public acclaim, determined to continue her campaign to reform public health. To this end she had a potent new weapon, an area of mathematics just beginning to be developed: statistics.

The idea that social phenomena could be measured and understood through mathematics was in the 19th century nothing short of revolutionary, and was regarded with deep suspicion by the political establishment.

Florence Nightingale did more than collect and classify data: her brilliant use of graphs and colourful pie charts went a long way to persuade everybody, from the general public through politicians and army generals all the way to Queen Victoria, of the urgency of the need for hospital reform. Her work with medical statistics was so impressive that in 1858 she was elected a member of the new Statistical Society of England, and in 1874 became an honorary member of the American Statistical Association.

During the American Civil War, she was approached by the United States government to give advice on army health. Her comprehensive statistical examination of health and sanitation in the Indian countryside made her the leading figure in the introduction of improved medical care and public health service in India.

She died in 1910. □

CROSSNUMBER PUZZLE

1	2	3	4
6			
6			
7			

Across
1. prime
5. square
6. square
7. cube of a prime

Down
1. cube
2. square of a prime
3. square
4. prime

The first correct solution opened on 1 March will win a R50 Prize Voucher.

HOW MANY FOLDS?

How many times can a piece of paper be folded in half? If you start with a standard A4 sheet of paper you will find that you can fold it six times in half. It is then the size of a postage stamp and too thick to fold again. Try a larger sheet, say from a newspaper. This time you may manage to fold it seven times.

Each time you make a fold, the wad of paper doubles in thickness. An ordinary sheet of paper is 0.1 mm thick, so after 8 folds the wad will be $2^8 \times 0.1 = 256 \times 0.1 = 25.6$ mm thick.

If you fold a piece of A4 paper in alternate directions, it keeps the same ratio of length to breadth, becoming successively A5, A6, A7, etc in size. That is because the standard paper sizes are designed to have a length-to-breadth ratio of $\sqrt{2}$ to 1.

If you start with an A0 sheet (which is exactly one square metre in area, and measures approximately 119 cm by 84 cm), after 8 folds the wad of paper will be about the size of a pack of playing cards and impossible to fold again.

But it's not just the simple doubling of thickness with every fold that makes the folding difficult. The folds themselves form bands of strength across the paper and make successive folds more difficult.

The popular view is that 8 folds are the maximum one can achieve with any sheet of paper, however large it is at the beginning.

In 2002 a Californian high school student Britney Gallivan was challenged by her mathematics teacher to take on paper-folding as a project for extra credit. Could she fold a piece of paper in half more than 8 times? Britney accepted the challenge.

When folding in half several times, there are two ways: in the same direction every time, or in alternate directions. Britney realised that instead of folding the paper in alternate directions, it would be easier to start with a long narrow strip of paper and fold it in half in the same direction every time, halving the length with each fold. Toilet paper (single-ply) was the obvious experimental material.

Starting with a single piece of toilet paper 4000 ft (that's about three-quarters of a mile, or 1220 metres) in length, Britney got to work. With her parents' help, she laid out the toilet paper roll in a shopping mall in their home town, Pomona, and set about folding it carefully in half. After seven hours, they had folded the strip an astonishing twelve times, as shown on the right. Clearly a thirteenth fold was not possible.

Britney also derived approximate formulas for the amount of paper needed to make a given number of folds. For folding in the same direction the length L of paper needed is given by the formula

$$L = \frac{\pi t}{6}(2^n + 4)(2^n - 1)$$

where the thickness of the paper is t and the number of folds is n.

If, however, you start with a square piece of paper of thickness t and width W, and fold in alternate directions, an approximate formula is

$$W = \pi t 2^{(3/2)(n-1)}.$$

Britney got the extra credit from her maths teacher, and a world record in paper folding. □

UNIT QUIZ

1. What unit in physics is named after a river in Scotland?
2. What unit in meteorology is named after a Royal Navy Admiral?
3. What unit in radioactivity has the name of an African river?
4. What unit of depth is derived from an Anglo-Saxon word meaning "embrace"?
5. What unit of speed is named after an Austrian physicist?

The first correct set of answers opened on 1 March will win a Prize Voucher worth R50. □

THE GOLDBACH CONJECTURE

On 7 June 1742 a Prussian lawyer and mathematician, Christian Goldbach, wrote a letter to Leonhard Euler, the most prominent mathematician of the day. He posed a simple problem:

> Can every even integer greater than 2 be expressed as the sum of two prime numbers?

Euler wrote back, saying that he was convinced that the observation was true, but could not prove it. To this day, nobody has been able to prove what is now known as the *Goldbach Conjecture*.
It is very easy to check the conjecture for small numbers.

$$\begin{aligned}
4 &= 2+2,\\
6 &= 3+3,\\
8 &= 3+5,\\
10 &= 7+3 = 5+5,\\
12 &= 5+7,\\
14 &= 3+11 = 7+7,\\
16 &= 3+13 = 5+11,\\
18 &= 5+13 = 7+11, \qquad \text{etc.}
\end{aligned}$$

This is a job that can easily be handed over to a computer, and Goldbach's Conjecture has been found to be true for all even integers up to 10^{18}.
Though the conjecture has received a good deal of serious attention over the last 250 years, no proof has yet been found. On the other hand, no counter-example (that is, an even number that is *not* expressible as the sum of two prime numbers) has been found either.
Some even numbers can be expressed as the sum of two primes in more than one way (for example, 10, 14, 16 and 18, as noted above). There are no fewer than 12 different ways of writing 120 as the sum of two primes. On the whole, as n gets larger, the number of ways of expressing $2n$ as the sum of two primes increases. Each way of expressing a number as the sum of two primes is called a *Goldbach partition*.

The number of Goldbach partitions of $10, 10^2, 10^3, 10^4, 10^5$ have been calculated: they are, respectively, 2, 6, 28, 127, 810. These numbers are pretty strong experimental evidence that the Goldbach Conjecture is true, but they do not, of course, constitute a proof.

Frustrated by not being able to crack the Goldbach conjecture, mathematicians have come up with variations. Two results are worth mentioning.

In 1973 the Chinese mathematician Chen Jingrun showed that every sufficiently large even number can be written as the sum of either two primes, or as the sum of a prime number and a *semiprime* (a number which is the product of exactly two primes, such as 15 or 77). Here, *sufficiently large* means that the result is true for all but a finite number of cases.

In 1995, Olivier Ramaré, a French mathematician who teaches at the Université des Sciences et Technologies de Lille, proved that every even number can be expressed as the product of at most six primes.

Because the Goldbach Conjecture is so easy to state, it has generated a lot of interest. To promote the 1992 novel *Uncle Petros and Goldbach's Conjecture* by Apostolos Doxiadis, the book's publisher offered a million dollar prize for a proof submitted before April 2002. The prize was never claimed. □

MEGAWATT MUDDLE

> Burning coal emits approximately one ton of carbon dioxide for every megawatt of electricity generated.
>
> (*Business Report*, 10 December 2009)

What is obviously wrong with this sentence is that a quantity (a ton of carbon dioxide) is equated to a rate of production of energy (a megawatt), without specifying a period of time over which the energy was produced. It's rather like saying that a car travelling at 100 km/h will use 10 litres of petrol, which tells you nothing about the car's efficiency unless you are given the distance travelled or the duration of the journey.

The article in *Business Report* was derived from a Deloitte report, *Energy Predictions for 2010*, and a quick google check showed that the newspaper reporter had quoted Deloitte correctly.

Can any reader turn this piece of bad maths into something meaningful? What was the Deloitte report trying to say? A R50 Prize Voucher will be awarded for the most helpful suggestions.

(Thanks to David Allison for bringing this Megawatt Muddle to our attention.) □

NIM

Nim is a very old game that can be found in many different versions and cultures, even as far away as China. It is a two-player game of strategy in which players take turns removing stones from distinct heaps, according to a couple of simple rules. The player forced to take the last stone is the loser.

There are many versions of the game, including changing the final result to declare the player who takes the last stone as the winner.

Nim is a game of strategy. No luck is involved, and at every stage both players have full information of the state of play. The game cannot go on for ever, and cannot end in a draw. So there has to be a winning strategy for one of the players.

A complete analysis of the game was developed in 1901 by Charles Bouton, a Harvard University maths professor, who also first gave it the name of Nim. The name of the game comes from an old Teutonic word for "take", *niman*, from which we today have the German *nehmen* and Afrikaans *neem* (both mean "take").

Here is a simple version of the game, with just two piles of stones. The rule for taking stones is that a player must take at least one stone from either pile, but may also take two stones, both from the same pile or one from each pile. The winner is the player who takes the last stone.

5	✓	×	✓	✓	×	✓
4	✓	✓	×	✓	✓	×
3	×	✓	✓	×	✓	✓
2	✓	×	✓	✓	×	✓
1	✓	✓	×	✓	✓	×
0		✓	✓	×	✓	✓
	0	1	2	3	4	5

To decide which player has the winning strategy, and what it is, may not be easy to work out at first. It will, of course, depend on the number of stones in each pile. The key to finding the winning strategy is given by the diagram above.

On the horizontal and vertical axes the numbers give the stones in the two piles. The square (m, n) represents the state of the game when there are m stones in the first pile and n in the second. The square on the bottom left corner is shaded, indicating the end of the game. The game starts at some position on the grid, and each moves takes the players closer to the shaded square.

The key to discovering the winning strategy is to work backwards. Clearly the positions $(1, 0), (1, 1)$ and $(0, 1)$ are winning positions for the player whose turn it is. Mark them with a ✓on the grid. Now mark with a $\times$ all positions from which a player is forced to move to a position marked with a ✓, giving the other player the win. Work up the grid in this way, marking the winning and losing positions. If the square (m, n) is marked with a ✓, it offers a win to the player who has the move: just move to a square marked with a $\times$.

Now change the rules: with the same moves as before, the player who takes the last stone *loses*.

A Prize Voucher worth R50 is offered for the best analysis of the losing version of Nim.

The game can also be played with three piles of stones. In this case it could be analysed with a three-dimensional grid.

Nim features in the classic French film *Last Year at Marienbad* (1961), in which it has some enigmatic significance.

Is there significance in the observation that **NIM** rotated through 180° gives **WIN**? □

SHOPPING PROBLEM

John went to the tuckshop and bought three items, which together cost R6.42. He then noticed that when he *multiplied* the prices of the three items together, the product also came to 6.42.

What were the prices of the three items? A Prize Voucher worth R50 will be awarded for the first correct entry opened on 1 March. □

SHARP CALCULATOR COMPETITION

The Sharp Calculator Competition is sponsored by Sharp Electronics (S.A.) (Pty) Limited, who have donated three calculators as prizes.

These prizes are offered for the three best solutions of the problems below. Solutions may be in English or Afrikaans. Partial solutions will be judged on their merits. The Editor reserves the right to make no award is no satisfactory entries are received, and his judgement will be final. The winner of an nth prize in a previous competition will only be eligible for an mth prize if $m < n$.

Only high schools students may compete. Entrants must give their full name and home address, age, school and grade. Solutions must be certified by a mathematics teachers at the school as the entrant's own unaided work.

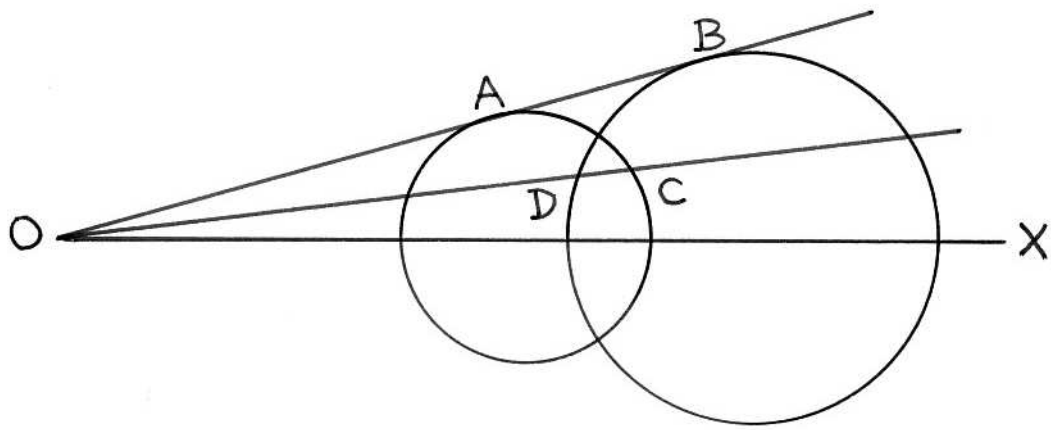
The closing date for receipt of entries is 28 February 2010, and the results will be announced in our April 2010 edition.

SHARP

1. For which positive integers n is $\sqrt{\dfrac{9n-1}{n+7}}$ a rational number?

2. Can 19^{19} be represented as the sum of a cube and a fourth power?

3. In the figure below, the centres of the two circles lie on the line OX, and the line OAB is tangent to the circles. Another line from O cuts the circles at C and D.



Prove that $ABCD$ is a cyclic quadrilateral. □

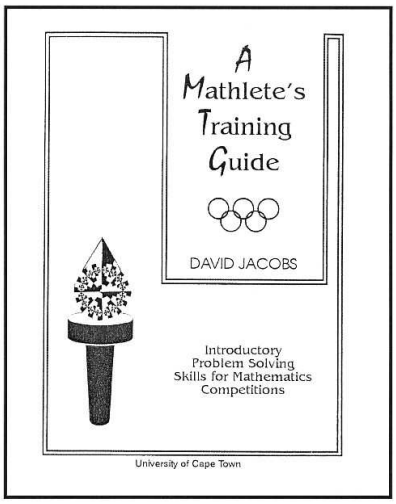
MATHS COMPETITION BOOKS

Mathematics Competitions and Olympiads are playing an increasingly important part in mathematics education. Any high school student who plans to continue studying mathematics at tertiary level (and that means all fields of science, engineering and commerce) is strongly advised to take maths competitions seriously, especially with the general downgrading of the content and softening of the examination level of Senior Certificate mathematics.

Unlike stereotyped school examinations, maths competitions include original and unusual questions that require individual and off-beat thinking. The best way of preparing for a maths competition is to get hold of question papers of previous years and tackle the problems.

A good range of competition books are available, at very reasonable prices. The annual University of Cape Town Mathematics Competition attracts over 6000 participants from Western Cape schools, and every year the full set of papers, with solutions, statistics and names of individual winners and schools are published. Books of question papers from 1995 to 2009 are available.

For a readable introduction to competition problem-solving, David Jacobs' successful *A Mathlete's Training Guide* is recommended. It is targeted at grades 8 to 10. Details and prices are at the back of this magazine. After a successful competition and Olympiad career while at school, David ran a series of problem-solving workshops for grade 8 – 10 pupils in Athlone schools. This book grew out of those workshops.



Two books of problems from the South African Mathematics Olympiad are also available. In these books the problems have been grouped into themes, so you do not get the papers from previous years to practise on.

Enjoying Mathematics is a collection of junior level problems (grades 8 and 9) from papers set from 1996 to 2001. *The Love of Mathematics* covers grades 10 to 12 problems from 1995 to 2005. The books may be ordered from the South African Mathematics Foundation, Private Bag X173, Pretoria 0001. □

SHARP SOLUTIONS AND RESULTS

1. *Solve the alphametric*
$$\sqrt{ABCD} = AD,$$
where A, B, C and D are different digits, from 1 to 9.

Since $0 < D < 10, 1000A + 100B + 10C + D < (10A + 10)^2$

$\Rightarrow 100A^2 + 200A + 100 > 1000A + 100B$

$\Rightarrow A^2 - 8A + 1 > B$

$\Rightarrow A^2 - 8A + 16 > B + 15$

$\Rightarrow (A - 4)^2 > 16$

$\Rightarrow A = 9.$

Since $(AD)^2$ ends in D, $D = 1$, 5 or 6. Checking the three possibilities shows:

$91^2 = 8281$, which is not possible since 8 is repeated, and the digits must be different;

$95^2 = 9025$, which is not possible since all the digits have to be non-zero;

$96^2 = 9216$: the solution.

Source: American Regions Mathematics League Contest 2001.

2. *Prove that, if a, b, c and d are natural numbers such that*
$$\sqrt{a} + \sqrt{b} + \sqrt{c} = d,$$
then a, b and c are perfect squares.

We first show that if a and b are natural numbers such that $\sqrt{a}+\sqrt{b}$ is also a natural number integer, then both a and b are perfect squares. For if $\sqrt{a} + \sqrt{b} = n$, then $a^2 = (n - \sqrt{b})^2$, from which it follows that $\sqrt{b} = (n^2 + b - a^2)/2n$: a rational number. Since we know that b is an integer, it must be a perfect square. Similarly, a is a perfect square.

Now we return to the original problem. From $\sqrt{a} + \sqrt{b} + \sqrt{c} = d$, it follows that $(\sqrt{a}+\sqrt{b})^2 = (d-\sqrt{c})^2$. So $a+b+2\sqrt{ab} = d^2+c-2d\sqrt{c}$. By the above reasoning, ab and c are perfect squares, and similarly bc and a, and ca and b are perfect squares. So a, b and c are all perfect squares.

3. *Prove that, for all natural numbers n,*
$$\frac{3}{5} \cdot \frac{7}{9} \cdot \frac{11}{13} \cdots \frac{4n-1}{4n+1} < \sqrt{\frac{3}{4n+3}}.$$

We shall prove this result by mathematical induction.

When $n = 1$ the left hand side is $\frac{3}{5}$ and the right hand side is $\sqrt{\frac{3}{7}}$. Since $(\frac{3}{5})^2 = \frac{9}{25} < \frac{9}{21} = \frac{3}{7}$, the result is true for $n = 1$.

Now let us suppose that the result is true for some natural number m, i.e.

$$\frac{3}{5} \cdot \frac{7}{9} \cdot \frac{11}{13} \cdots \frac{4m-1}{4m+1} < \sqrt{\frac{3}{4m+3}}.$$

Multiply each side by $\dfrac{4m+1}{4m+3}$:

$$\frac{3}{5} \cdot \frac{7}{9} \cdot \frac{11}{13} \cdots \frac{4m-1}{4m+1} \cdot \frac{4m+1}{4m+3} < \sqrt{\frac{3}{4m+3}} \times \frac{4m+1}{4m+3}.$$

If we can now prove that

$$\sqrt{\frac{3}{4m+3}} \times \frac{4m+1}{4m+3} < \sqrt{\frac{3}{4m+7}},$$

we will have established that the inequality is true for the natural number $m + 1$.

Now $\sqrt{\dfrac{3}{4m+3}} \times \dfrac{4m+1}{4m+3} < \sqrt{\dfrac{3}{4m+7}}$

$\Leftrightarrow (4m+1)^2(4m+7) < (4m+3)^3$

$\Leftrightarrow N^2(N+6) < (N+2)^3$ (putting $N = 4m+1$)

$\Leftrightarrow N^3 + 6N^2 < N^3 + 6N^2 + 12N + 8$, which is clearly true.

So we have establish that

IF the inequality is true for some natural number m,

THEN it is also true for the next natural number, $m + 1$.

But we established at the beginning that the inequality is true when $n = 1$.

Therefore it is also true when $n = 2$.

Therefore it is also true when $n = 3$, and when $n = 4$, and when $n = 5$, and so on.

This method, called *mathematical induction*, establishes that the inequality is true for all natural numbers n.

Source: The German magazine $\sqrt[Die]{Wurzel}$, 2009

SHARP

Sharp scientific calculators are awarded to:

First Prize: Sean Wentzel (Westerford High School)
Second Prize: Kira Düsterwald (Springfield Convent)
Third Prize: Francois van Niekerk (Bellville High School).

Former First Prize Winner Greg Jackson (Bishops) wins an Honourable Mention for his solutions. □

PRIZEWINNERS AND SOLUTIONS

Literary Quiz (October 2009, p. 4):
The traveller who observed the mathematical precision of the cooks in the kitchen of the King of Laputa was Lemuel Gulliver, as told in *Gulliver's Travels*, by Jonathan Swift. Emmanuel Setlhapelo (grade 9, Kismet Combined School) wins a R50 Prize Voucher for his answer.

What Day is it? (October 2009, p.7:
The formula given for working out the day of the week for a given date used only the last two digits of a year, and is valid only for the 20th century. Greg Jackson (grade 11, Bishops) wins a R50 Prize Voucher for his proposed adjustment to the formula: just subtract 1 for dates in this century, giving the formula

$$T = D + M + Y + \lfloor \frac{4}{5}(2M+1) \rfloor + \lfloor \frac{Y}{4} \rfloor - 1.$$

Maths and Chess (October 2009, p. 30):
The statement that the best chess players are not mathematicians was made in H.E. Dudeney's *Canterbury Puzzles*, which was published in 1907 and reissued in 1919. Dudeney could not have been more mistaken. From 1894 to 1921 the world chess champion was Emanuel Lasker, an eminent mathematician whose 1902 Ph.D. thesis is today regarded as one of the foundations of modern algebra. Emmanuel Setlhapelo wins another R50 Prize Voucher for identifying the quotation.

Telling Ages (October 2009, p. 30:
There are two possibilities for the ages of the boys (Alan, Brian, Colin, Damon) from the information given: (14, 15, 13, 17) or (13, 16, 12, 15). In either case, Colin is the youngest boy. Emmanuel Setlhapelo wins a third R50 Prize Voucher for his answer. □

MATHEMATICAL TALENT SEARCH

The Mathematical Talent Search is a free problem-solving correspondence course for high schools, aimed at identifying and developing mathematical talent.

To enrol in the course, solve the problems below and send the answers to

Mathematical Talent Search
P O Box 34034
7707 RHODES GIFT

Give your name, age, postal address, school and grade, and include a stamped self-addressed envelope so that the next round of problems can be sent to you.

1. What is the value of $(1-\frac{1}{2}) \times (1-\frac{1}{3}) \times (1-\frac{1}{4}) \times \ldots \times (1-\frac{1}{2010})$?
2. In how many ways can 4 boys be paired with 4 girls for a dance?
3. A hungry spider is sitting on the floor in a corner of a rectangular room 5m long, 4m wide and 3m high. She sees a fly on the ceiling in the furthest corner. What is the length of the shortest route for the spider to crawl to reach the fly?
4. Find a 12-digit number for which every two consecutive digits are distinct two-digit prime numbers.
5. Four people are trying to cross a bridge in the dark. It takes each person a different amount of time to cross the bridge—it takes the first person 1 minute, the second 2 minutes, the third 11 minutes and the fourth 16 minutes. They must cross the bridge in pairs, using the only flashlight, and travelling at the slower speed of the two. Since there is only one flashlight one person must come back across the bridge with the flashlight so that the next pair can cross. What is the shortest time needed to get everyone across the bridge?

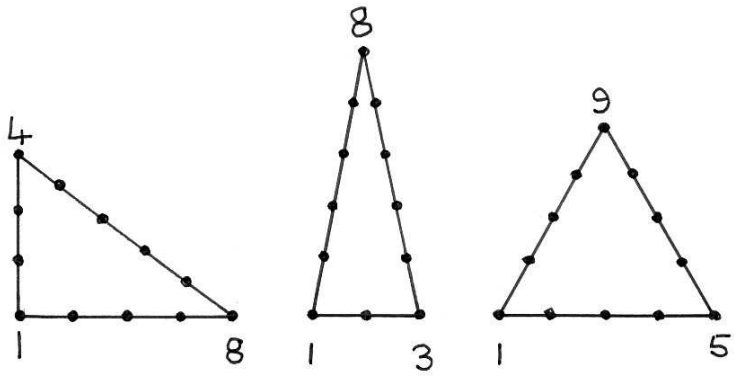
The Mathematical Talent Search is a project of the South African Mathematics Foundation to choose teams to represent South Africa at the Pan African Mathematics Olympiad and the International Mathematical Olympiad.

For more information, visit the website www.samf.ac.za/TalentSearch. □

KNOTTED TRIANGLES

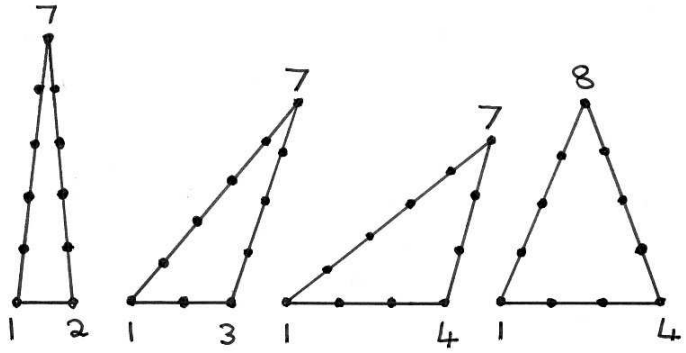
Suppose you a loop of string with twelve equally spaced knots on it, numbered from 1 to 12. If you choose knots 1, 4 and 8, the loop can be pulled tight to form a 3–4–5 triangle.

If you take hold of three equally-spaced knots (4 units apart) then the stretched loop will form an equilateral triangle. If you take hold of knots 1, 3 and 8, the loop can be pulled tight to form an isosceles triangle.



So with 12 knots on a loop you can form three different triangles.

If the loop had 11 equally-spaced knots, you would be able to form four different triangles, with sides (5, 5, 1), (5, 4, 2), (5, 3, 3) and (4, 4, 3).



What about loops with other numbers of knots? With just two knots on the loop, you cannot form a triangle, of course. With three knots, and equilateral triangle is the only possibility. With four knots, you again cannot form a triangle (not a proper one, that is, because sides of 1, 1 and 2 don't make a proper triangle).

An interesting subject for investigation is to work out how many triangles can be formed with other numbers of knots on the loop. This would make a good expo project, because the loops and triangles can be attractively displayed in a systematic way. □

PRIME STACKS

The prime numbers form the sequence

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, \ldots$$

Suppose we add up the successive terms and record the cases when the sum is itself prime.

Adding up just one term gives (of course) the sum 2, which is prime.

The sum of the first two terms is $2 + 3 = 5$, which is prime.

Since all primes after the first are odd, the sum of any odd number of terms after the first is always even, so we continue by adding terms two at a time.

The sum of the first four terms is $2 + 3 + 5 + 7 = 17$, which is prime.

The sum of the first 6 terms is $2+3+5+7+11+13 = 41$, which is prime.

The sum of the first 8 terms is $2+3+5+7+11+13+17+19 = 77 = 7\times 11$, which is not prime.

The sum of the first 10 terms is $2+3+5+7+11+13+17+19+23+29 = 129 = 3 \times 43$, which is not prime.

The sum of the first 12 terms is $2 + 3 + 5 + 7 + 11 + 13 + 17 + 19 + 23 + 29 + 31 + 37 = 197$, which is prime.

The sum of the first 14 terms is $2 + 3 + 5 + 7 + 11 + 13 + 17 + 19 + 23 + 29 + 31 + 37 + 41 + 43 = 281$, which is prime.

A reader (who prefers to remain anonymous) calls the prime sums *stacked primes*, and the last prime in the sum is called a *stacking prime*.

So the first six stacked primes are

$$2, 5, 17, 41, 197, 281$$

and they correspond to the first six stacking primes

$$2, 3, 7, 13, 37, 43.$$

You have to do a lot of work to get to the next stacked prime: it is 7699, obtained by adding up the primes up to 281.

Adding up primes in this way, and looking for prime sums, suggests several interesting questions. How do they continue? The number 281 appears on both lists. Are there any other prime numbers on both lists? $\square$

WATT'S WRONG

With electricity consumption much in the news, the use of a nonsensical unit "kilowatts per hour" in the media is widespread, and has been frequently featured in *Mathematical Digest*.

While journalists working under pressure to keep deadlines can be excused for their innumeracy, the same cannot be said of mathematics educators. It was rather disturbing to see the same error in a sample Senior Certificate Mathematical Literacy paper, published in a *Sunday Times 2009 Matric Q&A Guide.*

In paper 2, Question 5.1.1, the following was asked:

"The CFL bulb uses 15W of electricity per hour that it burns. How many kilowatts does it burn in one hour?"

This should have been "The CFL bulb uses 15W of electricity. How many kilowatt-hours does it use in one hour?"

The answer also got the use of units wrong:

" 15W/h ÷ 1 000 W = 0,015 kW/h"

Here "W/h" (watts per hour) is again wrong, and the division is by 1000, not 1000W. The units in the equation do not match up. The correct line should be

"15W ÷ 1 000 = 0,015 kW"

The next line also had the incorrect "kW/h" unit. "Hourly running cost = 0,015kW/h × 65,35c/kWh" should be "Hourly running cost = 0,015kW × 65,35c/kWh".

By analogy with motion, a kilowatt is a speed (a rate of energy use) while a kilowatt-hour is the amount of energy, analogous to distance. So a "kilowatt per hour" would be analogous to acceleration of energy. This has no relevance in household use.

Acceleration of energy does have importance in electricity generation, though instead of "kilowatts per hour" electrical engineers will measure energy acceleration in "megawatts per minute". The unit comes into importance when a power station generator is being brought on stream. The generator has to be brought up to full power output slowly, so as not to damage the turbines. In a 300 megawatt power station the rate would typically be

10 megawatts per minute, and so it would need 30 minutes to bring the generator up to full power.

Readers are invited to send in cases of this common misuse of energy units. Prize Vouchers worth R50 will be awarded for any examples that we publish. □

PROBLEM OF THE YEAR

Using the digits 2, 0, 1 and 0, in that order, and the basic operations of arithmetic, $(+, -, \times, \div, \sqrt{})$ and brackets, can you represent the numbers 1, 2, 3, ...?

For example:

$$\begin{aligned} 1 &= 2+0-1+0 = 20 \div 10 \\ 2 &= 2+0\times 1\times 0 = (2+0)(0+1) \\ 3 &= 2+0+1+0 \end{aligned}$$

It seems difficult to get 4 in this way, so we will allow some extra operations and symbols. Exponential notation (eg $2^0 = 1$), putting two numbers together (eg 20) and recurring decimal notation (eg $.\dot{9} = 1$) are also permitted. You may also use factorial notation: $n! = 1 \times 2 \times 3 \times \ldots \times n$, with the convention that $0! = 1$.

Now we can get a little further:

$$\begin{aligned} 4 &= 2+0+1+0! \\ 5 &= (2+0!)! - 1 + 0 = 2+0!+1+0! \\ 6 &= 2\times(0!+1+0!) = (2+0!)(1+0!) \\ 7 &= (2+0!)! + 1 + 0 = (2+0!)! + (1\times 0!) \\ 8 &= (2+0!)! + 1 + 0! = 2^{(0!+1+0!)} \\ 9 &= 2^0 \div (.\dot{1}) + 0 \end{aligned}$$

How far can you extend the list? This problem has proved popular over the years, but this year,with only the digits 0, 1 and 2 to work with it looks rather more difficult than usual. Still, we rely on the ingenuity of our readers to come up with imaginative solutions.

Prize Vouchers worth R50 will be awarded for the three best lists sent by readers in grade 9 or lower. Where possible, more than one way of representing a number should be given. □

HERON'S FORMULA

Heron's formula, named after the Greek mathematician Heron of Alexandria, gives a way of calculating the area of a triangle when the lengths of the three sides are known.

The standard form of the formula is

$$A = \sqrt{s(s-a)(s-b)(s-c)} \tag{1}$$

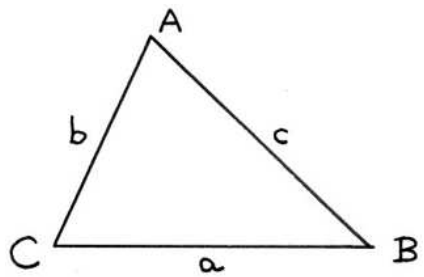
where a, b and c are the lengths of the three sides of the triangle and $s = \frac{1}{2}(a+b+c)$ (the "semi-perimeter" of the triangle).

The formula is found in Heron's book *Metrica*, written in 60 A.D.

In *Mathematical Digest* **155** (April 2009), we showed how to use the Theorem of Pythagoras to derive Heron's Formula. It's an essential item for the Olympiad Toolbox.

If you don't know Heron's Formula, you could still work out the area of a triangle from the lengths a, b and c of its three sides by first using the cosine rule

$$c^2 = a^2 + b^2 - 2ab\cos C$$



to calculate $\cos C$, then working out the sine of $\angle C$ (remembering that $\sin^2 C + \cos^2 C = 1$), and finally using the sine formula for the area of the triangle:

$$A = \frac{1}{2}ab\sin C.$$

This approach can also be used to provide a different proof of Heron's Formula. Try it for yourself.

There are other forms of Heron's Formula which look a little more complicated, but can sometimes be useful.

$$A = \frac{1}{4}\sqrt{(a+b+c)(a+b-c)(b+c-a)(c+a-b)} \tag{2}$$

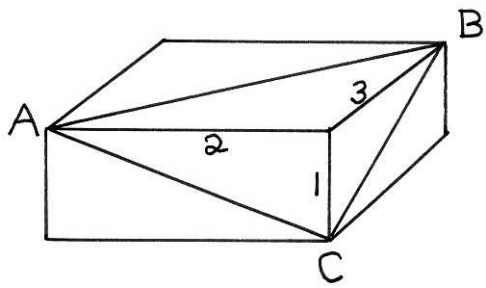
$$A = \frac{1}{4}\sqrt{2(a^2b^2+b^2c^2+c^2a^2)-(a^4+b^4+c^4)} \tag{3}$$

$$A = \frac{1}{4}\sqrt{(a^2+b^2+c^2)^2-2(a^4+b^4+c^4)} \tag{4}$$

It is not difficult to see that equations (1) and (2) are the same, and that (3) and (4) are the same. But it requires a little more effort to show that (2) and (3) are the same. It's a useful exercise in algebraic agility.

Sometimes these alternative forms of Heron's Formula are more convenient than the original form for working out the area of a triangle.

For example, suppose that we have a $1 \times 2 \times 3$ box as shown. What is the area of triangle ABC?



By Pythagoras' Theorem, $AB = \sqrt{1^2 + 2^2} = \sqrt{5}$, $BC = \sqrt{2^2 + 3^2} = \sqrt{13}$ and $CA = \sqrt{3^2 + 1^2} = \sqrt{10}$. Substituting these numbers into formula (1) with $s = \frac{1}{2}(\sqrt{5} + \sqrt{10} + \sqrt{13})$ immediately promises to be messy with all those awkward square root signs. Formula (2) is not much better.

But if we use either formula (3) or formula (4) the square roots disappear and the answer comes out very quickly. Try it and see. $\square$

MTHMTCL WRDS WTHT VWLS

Vwls r nt ncssr fr th prpr ndrstndng f nglsh r vn mthmtcs. T prv tht ths sttmnt s crrct, tr t wrk t th rgnl twnt wrds blw, whs vwls hv bn mttd.

HPTNS CSCNT PRTRCTR VLM SBTRCTN

CNTR GMTR CBC LGRTHM SMPTT

BLLN VRTX PLNML PRLLLGRM SLTN

HPRBL CMPSSS QTN NN RTNL

Snd yr lst f cmplt wrds t th dtr. Th frst crrct nswr pnd n 1 Mrch wll wn Prz Vchr wrth R50. $\square$

THE ISLAMIC CALENDAR

Calendars are either solar (based on the movements of the Sun) or lunar (based on the movements of the Moon). The Western calendar is solar. Its basic unit is the year, of 365 or 366 days. The years are counted from the assumed birth of Christ, 1 A.D., which stands for the Latin *Anno Domini*, "the Year of the Lord". The Western calendar was originally set up by Julius Caesar over two thousand years ago, and then adjusted in 1582 by Pope Gregory.

The Islamic calendar is lunar, based on 12 lunar months, each of a little more than 29 days, giving a year of 354 or 355 days. The calendar is used in Muslim countries to determine the proper days on which to celebrate Islamic holy days and festivals. The years are counted from the *hijra*, or migration of the Prophet Mohammed from Mecca to Medina, which occurred in 622 A.D. in the Western Calendar. The count is sometimes written A.H., from the Latin *Anno Hegirae*, "the year of the hijra".

The Islamic year 1430 A.H. ran from 28 December 28 2008 to 17 December 17, 2009. The current year is therefore 1431 A.H. or 2010 A.D.

Because the Islamic year is 11 days shorter than the Western year, the beginning of the Islamic year moves backwards through the Western year, making a full cycle in about 34 years. It can happen that an Islamic year will lie entirely inside a Western calendar year. This happens every 33 or 34 years, and last happened when 1429 A.H. fell inside 2008 A.D.

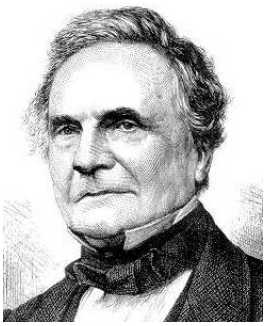
The difference between the Gregorian year number and the Islamic year number (currently 579) decreases by one about every 34 years. The year count in the two calendars will be the same in 20 874 A.D./A.H.

For a rough conversion from A.H. to A.D., multiply the Islamic year by 0.97, and then add 622 to get the Gregorian year. To go the other way, subtract 622 from the Gregorian year and multiply by 1.03.

The rule is rather like the way one converts temperature measurements from degrees Celsius to degrees Fahrenheit: multiply by 9/5 and add 32. To go the other way, from Fahrenheit to Celsius, subtract 32 from the Fahrenheit temperature and multiply by 9/5. The two rules are similar in mathematical form, but different in function. For temperatures, the conversion is exact. For the two calendars the conversion may be wrong by a unit, since the two years start and end at different times. □

POETRY AND MATHS

There is a story that the mathematician Charles Babbage (the 19th century computer pioneer) wrote the following letter to the poet Alfred, Lord Tennyson, about a couplet in Tennyson's *The Vision of Sin*.



Every minute dies a man,
Every minute one is born.

"I need hardly point out to you that this calculation would tend to keep the sum total of the world's population in a state of perpetual equipoise, whereas it is a well-known fact that the said sum total is constantly on the increase. I would therefore take the liberty of suggesting that in the next edition of your excellent poem the erroneous calculation to which I refer could be corrected as follows:

Every minute dies a man
And one and one-sixth is born.

I may add that the exact figures are 1.167, but something must, of course, be conceded to the laws of metre."

It's a good story, and whether it is true is not all that important. It must, however, be noted that in later editions of the poem Tennyson changed "minute" to "moment", but did not bow to Babbage's demands for statistical accuracy. □

PERCENTAGE ERROR

[A] poll of 2072 British adults revealed that 52% find the "queen's English" most appealing when calling a call centre.

The Scottish accent came second at 34% and the northern Geordie twang third at 26%, followed by accents from Yorkshire and Wales.

The least popular accents were those from Liverpool or Birmingham, which tied at 9%.

(*Sunday Times*, 6 December 2009) □

FIBONACCI FORMULAS

The Fibonacci sequence

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$$

is formed, after the first two terms, by adding the two previous terms. The rule, precisely stated, is:

$$a_1 = 1, \quad a_2 = 1 \quad \text{and} \quad a_{n+1} = a_n + a_{n-1} \quad \text{for all} \quad n > 1.$$

This type of rule is known as a *recurrence relation*, and is the well-known form in which Fibonacci numbers are encountered at school.

What is less well-known is a rule for generating the same sequence, using a very different recurrence relation that uses only one preceding term.

The rule is

$$a_1 = 1 \quad \text{and} \quad a_{n+1} = \frac{1}{2}(a_n + \sqrt{5a_n^2 + 4 \times (-1)^n}) \quad \text{for all } n \geqslant 1.$$

There seems to be no connection between the two formulas.

While it is clear that the the first formula will always give whole numbers, after starting off with 1 and 1, it is by no means obvious that the second formula will always output whole numbers, because of the square root operation. Nevertheless, it is not difficult to work out the first few terms generated by the second rule. They all seem to be the usual Fibonacci numbers.

But just because a pattern starts out in a familiar way, generating Fibonacci numbers, can we be sure that it continues to do so all the time, and does not throw up other, non-Fibonacci, numbers? What is needed is a careful mathematical proof that the second formula generates all the Fibonacci numbers, and nothing else.

From the original formula

$$a_{n+1} = \frac{1}{2}(a_n + \sqrt{5a_n^2 + 4 \times (-1)^n})$$

we get

$$(2a_{n+1} - a_n)^2 = 5a_n^2 + 4 \times (-1)^n.$$

Squaring out and simplifying this equation gives

$$4a_{n+1}^2 - 4a_{n+1}a_n - 4a_n^2 = 4(-1)^n$$

and (dividing by 4 throughout)

$$a_{n+1}^2 - a_{n+1}a_n - a_n^2 = (-1)^n$$

Replace n by $n+1$ in the above formula. This gives

$$a_{n+2}^2 - a_{n+2}a_{n+1} - a_{n+1}^2 = (-1)^{n+1}$$

Adding the two equations gives

$$-a_{n+1}a_n - a_n^2 + a_{n+2}^2 - a_{n+2}a_{n+1} = 0$$

$$a_{n+2} - a_{n+2}a_{n+1} = a_{n+1}^2 + a_{n+1}a_n$$

Complete the square by adding $\frac{1}{2}a_{n+1}^2$ to both sides:

$$a_{n+2} - a_{n+2}a_{n+1} + \frac{1}{2}a_{n+1}^2 = a_{n+1}^2 + a_{n+1}a_n + \frac{1}{2}a_{n+1}^2$$

$$(a_{n+2} - \frac{1}{2}a_{n+1})^2 = (a_n + \frac{1}{2}a_{n+1})^2$$

Take square roots:

$$a_{n+2} - \frac{1}{2}a_{n+1} = a_n + \frac{1}{2}a_{n+1}$$

which simplifies to

$$a_{n+2} = a_{n+1} + a_n$$

—the first Fibonacci recurrence relation. □

THE FIFA RUGBY BALL

The new Jabulani football was unveiled during the FIFA World Cup draw on 4 December 2009 in Cape Town, watched by viewers all over the world. Viewers watching the event on extra wide screens may have been confused to see FIFA President Sepp Blatter holding up what was very clearly a rugby ball. Had he suddenly changed codes?

Indeed not. In changing the aspect ratio of the picture to fill a wide television screen, the spherical football had been stretched into an ellipsoidal rugby ball. □

mathematical digest

No. 159 April 2010

The Death of Archimedes

mathematical digest

Number 159 **April 2010**

When I was a schoolboy I was not good at arithmetic, but I have since heard it said that certain mathematical quantities when they pass through infinity change their signs from plus to minus – or the other way round. I do not venture to plunge too much into detail of what are called the asymptotes of hyperbolae, but any hon. Gentleman who is interested can find an opportunity for an interesting study of these matters. It may be that this rule may have a novel application and that when the advance of destructive weapons enables everyone to kill everybody else nobody will want to kill anyone at all.

(Winston Churchill, in a speech to the House of Commons, 1953)

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Cover: The mosaic shows a Roman soldier about to kill Archimedes (see *The Death of Archimedes* on page 1). The mosaic is in a museum in Frankfurt, Germany. Once thought to be from a Roman villa, the mosaic is now believed to be a 17th century forgery.

THE DEATH OF ARCHIMEDES

Archimedes was an old man of 75 when the Romans besieged his home city, Syracuse. The siege was part of the Second Punic War, between Carthage and Rome, and Syracuse had sided with Carthage. It turned out to be a bad decision.

The Roman general Marcellus was given the task of capturing Syracuse. For a long time Marcellus and his army were kept at bay by ingenious defensive weapons designed by Archimedes, such as giant catapults. But the Romans eventually breached the defences of Syracuse and sacked the city. Marcellus had given instructions that the life of Archimedes should be spared, but it was not to be. The full story of the siege, including the death of Archimedes, was told by the Roman historian Plutarch in his biography of Marcellus. The translation below is due to the English poet John Dryden.

> But nothing afflicted Marcellus so much as the death of Archimedes; who was then, as fate would have it, intent upon working out some problem by a diagram, and having fixed his mind alike and his eyes upon the subject of his speculation, he never noticed the incursion of the Romans, nor that the city was taken. In this transport of study and contemplation, a soldier, unexpectedly coming up to him, commanded him to follow to Marcellus; which he declining to do before he had worked out his problem to a demonstration, the soldier, enraged, drew his sword and ran him through. Others write, that a Roman soldier, running upon him with a drawn sword, offered to kill him; and that Archimedes, looking back, earnestly besought him to hold his hand a little while, that he might not leave what he was then at work upon inconclusive and imperfect; but the soldier, nothing moved by his entreaty, instantly killed him. Others again relate, that as Archimedes was carrying to Marcellus mathematical instruments, dials, spheres, and angles, by which the magnitude of the sun might be measured to the sight, some soldiers seeing him, and thinking that he carried gold in a vessel, slew him. Certain it is, that his death was very afflicting to Marcellus; and that Marcellus ever after regarded him that killed him as a murderer; and that he sought for his kindred and honoured them with signal favours. □

A SHARP PROBLEM REVISITED

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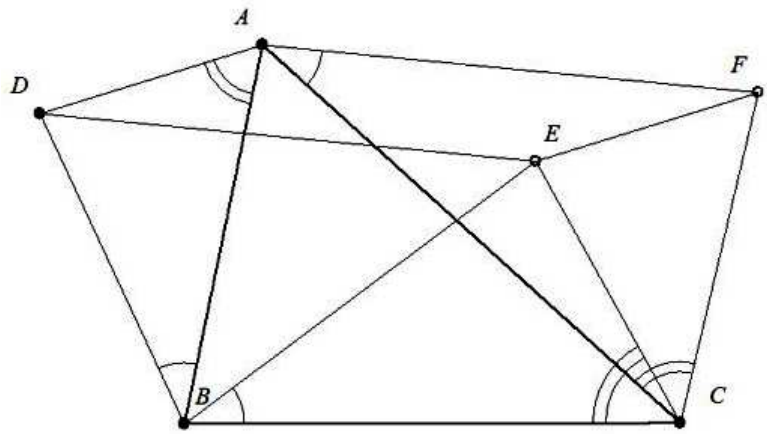
This note refers to the following nice problem, in *Mathematical Digest* 155 (April 2009):

Three similar triangles ABD, CBE and CAF are constructed on the sides of triangle ABC as shown in Fig. 1. Prove that $ADEF$ is a parallelogram.

This problem was solved in *Mathematical Digest* 156 by using the cosine rule. Here we present two alternative solutions and make some historical remarks.



The first proof

Let $\alpha = \angle ABD = \angle CBE$, $\beta = \angle BCE = \angle ACF$ and

$$k = \frac{DB}{AB} = \frac{BE}{BC} = \frac{AF}{AC}.$$

As $\dfrac{BD}{BE} = \dfrac{BA}{BC}$ and $\angle DBE = \angle ABC (= \alpha + \angle ABE)$, it follows that

$$\triangle BDE \sim \triangle BAC.$$

We have

$$\frac{DE}{AC} = \frac{BD}{BA} = k = \frac{AF}{AC},$$

and hence $DE = AF$.

Similarly, $\triangle ECF \sim ABC$, hence we also have $EF = AD$. These prove that $ADEF$ is a parallelogram.

The second proof (by using complex numbers)

We reformulate the problem as follows:

If three directly similar triangles ABD, CBE and CAF are are constructed on the sides of triangle ABC, then $ADEF$ is a parallelogram.

Recall that two triangles are said to be directly similar if one of them can be obtained by translating and rotating the other one, then scaling. If two triangles ABC and $A_1B_1C_1$ are directly similar then

$$\frac{c_1 - a_1}{c - a} = \frac{b_1 - a_1}{b - a},$$

where z represents the affix of the point Z.

Using the above formula, we obtain:

$d = a + \sigma(b - a),\ e = c + \sigma(b - c),\ f = c + \sigma(a - c)$, therefore $a + e = a + c + \sigma(b - c) = d + f$, proving that $ADEF$ is a parallelogram.

This solution suggests that a similar property may hold if similar triangles are constructed on the sides of a convex quadrilateral:

If directly similar triangles ABP, CBQ, CDR, ADS are constructed on the sides of a convex quadrilateral, then $PQRS$ is a parallelogram.

It is not hard to check that this is indeed the case: the affixes of the points P, Q, R, S are given by $p = a+\sigma(b-a),\ q = c+\sigma(b-c),\ r = c+\sigma(d-c),\ s = a + \sigma(d - a)$, hence $p + r = s + q = a + c + \sigma(b + d - a - c)$.

Actually these two problems are quite old ones. Recently we have found a Romanian translation by Professor Mihai Miculiţa of the 1881-1960 collection of *Mathesis*.

In *Mathesis 1881,* p.118 problems 55 and 56, are as follows:

Problem 55. *If, similar with a given triangle, triangles $BB'C$, $CC'A$, $BA'A$ are constructed externally on the sides BC, CA and internally on the third side AB, then the figure $B'CC'A'$ is a parallelogram.*

Problem 56. *If externally on the two opposite sides AB, CD of a quadrilateral $ABCD$ and internally on the other two sides BC, DA similar with a given triangle triangles $AA'B$, $CC'D$, $CB'B$, $AD'D$ are constructed, then the figure $A'B'C'D'$ is a parallelogram.*

The author of these problems is H. Van Aubel.

We finish by reproducing the elegant solution by Pierre Interdonato of Problem 56, from *Mathesis 1881,* p.167, which shows that the second problem can be reduced to the first one. As in the pages of *Mathesis,* the reader is invited to construct the picture for himself.

Draw the diagonal BD. As the triangles $AA'D'$, ABD are similar (see Problem 55), the angle between the lines $A'D'$ and BD is equal to $\angle A'AB$. Similarly, the angle between the lines $B'C', BD$ is equal to $\angle C'CD$. But $\angle A'AB = \angle C'CD$, therefore the lines $C'B', D'A'$ are parallel. By analogy, $B'A'$ and $C'D'$ are parallel; hence the figure $A'B'C'D'$ is a parallelogram. $\square$

FIBONACCI'S BIRD PROBLEM

In 1202 the Italian mathematician Leonard Fibonacci wrote *Liber Abaci* (the Book of the Abacus), a book promoting the use of Hindu-Arabic numerals.

The book contains the famous problem of breeding rabbits, giving rise to the Fibonacci numbers

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \cdots .$$

Not nearly as well known is a problem in the same book about birds:

> A man bought 30 birds, a mixture of partridges, pigeons, and sparrows, for 30 denari. A partridge costs 3 denari, a pigeon 2 denari, and a sparrow $\frac{1}{2}$ denaro. How many birds of each kind did he buy?

(A *denaro* was a medieval Italian coin.)

A R50 Prize Voucher will be awarded for the best solution of this problem, from a reader in grade 10 or lower. $\square$

THE MYSTERY DODECAHEDRA

A *dodecahedron* is a solid object with twelve sides (*dodeca* = twelve, *hedron* = side), and a *regular* dodecahedron is a a dodecahedron with twelve regular pentagonal sides. The dodecahedron has been known for thousands of years, and Euclid showed how to construct one using only ruler and compasses.

The dodecahedron was regarded by Greek philosophers as symbolic of the universe. A silver dodecahedron, with its twelve sides inscribed with the twelve signs of the zodiac, was featured on the cover of *Mathematical Digest* **126**, January 2002. Dating from 400 A.D., it was probably used in fortune-telling.

Nearly 100 bronze dodecahedra have been unearthed in northern Europe. They date from 200 to 400 A.D. They range from 4 cm to 11 cm in diameter and are hollow, with circular holes of different diameters in each face. At each vertex is a small sphere.

What were they used for? One was found with traces of wax inside, so maybe they were simply decorated candlesticks, with holes for candles of different diameters. Or maybe they were used by plumbers to measure the diameter of pipes. Were they twelve-sided dice? One elaborate theory suggests that they were used as a type of sundial. Or perhaps they were just children's toys. Nobody really knows. It is a modern archaeological mystery.

To complicate the issue, a rather similar *icosahedron* (twenty-sided figure) has also been found. Like the mysterious dodecahedra it is bronze, hollow, about 8 cm in diameter and nobody has any idea of what it was used for.

It was stored in a museum in Bonn for nearly forty years, misclassified as a dodecahedron. As one archaeologist drily observed: "That illustrates nicely how people see things differently depending on their background knowledge." □

WYTHOFF'S NIM

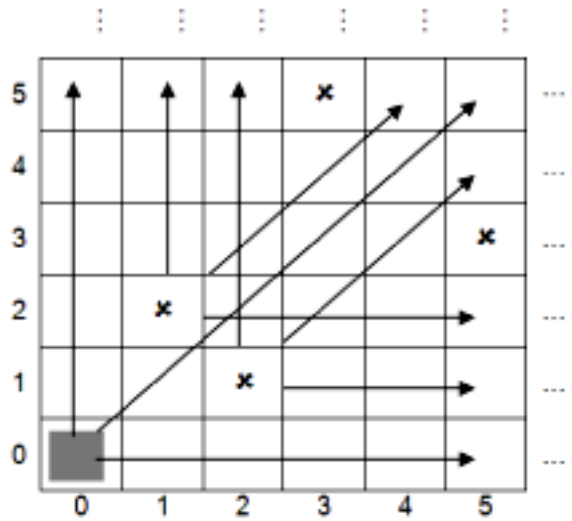
Greg Jackson
Bishops

In *Mathematical Digest* **158** (January 2010), pp.14–15, the game of Nim was discussed. Nim is a two-player game in which the players take turns removing stones from distinct heaps.

Wythoff's Nim (or Wythoff's Game, as it is often called), is a specific version of 2-heap Nim. Players take turns removing stones from one or both heaps. A player must remove at least one stone. If a player chooses to remove stones from both heaps, the number he removes from each heap must be equal. The game ends when a player leaves no stone in either pile after his turn, thus winning the game.

Martin Gardner claims that this is an ancient game of Chinese origin, but the first mathematical analysis of it was published in 1907 by Willem A. Wythoff. The winning strategy depends on how many stones are in each pile.

A a *cold position* is one which is a losing position for the player who is to move (assuming his opponent uses the winning strategy). They can be determined recursively as shown in the diagram.



The horizontal and vertical axes give the number of stones in each heap. The square (m, n) is a state in the game where there are m stones in one heap and n stones in the other. We shall assume that $m < n$ in what

follows, since if (m, n) is a cold strategy, so is (n, m). The lower left corner is shaded to represent the end of the game.

The game has a wonderful link to something apparently unrelated: *Beatty sequences*. In 1926 Samuel Beatty, Professor of Mathematics at the University of Toronto, proposed a problem in the *American Mathematical Monthly*, a result now known as *Beatty's Theorem*.

A *Beatty sequence* B_r is defined for a positive irrational number r as the sequence

$$\lfloor r \rfloor, \lfloor 2r \rfloor, \lfloor 3r \rfloor, \ldots$$

where $\lfloor r \rfloor$ denotes the largest integer not greater than r.

Beatty's Theorem states that if $\frac{1}{r} + \frac{1}{s} = 1$, then B_r and B_s are complementary sequences: when you put them together you get the natural numbers 1, 2, 3, ...

When $r = \phi$, the Golden Ratio, and $s = \phi^2 = \phi + 1$, then these two numbers generate a complementary pair of Beatty sequences, since

$$\frac{1}{\phi} + \frac{1}{\phi^2} = \frac{\phi + 1}{\phi^2} = 1.$$

Amazingly, it turns out that all the cold positions in Wythoff's Nim are the squares with position $(\lfloor n\phi \rfloor, \lfloor (n(\phi + 1) \rfloor)$. The first few cold positions are

$$(0, 0), (1, 2), (3, 5), (4, 7), (6, 10).$$

The Golden Ratio isn't all about aesthetics!

Reference

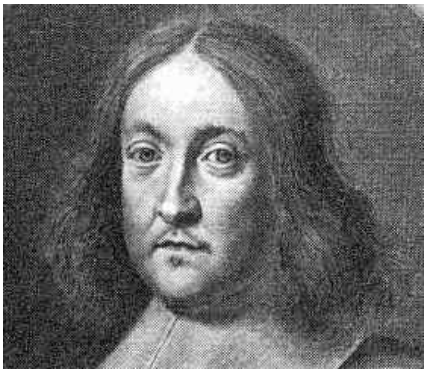
Gardner, M.: *Penrose Tiles to Trapdoor Ciphers* (chapter 8), W.H. Freeman, 1989. □

DIMENSION WISE

> "Dimensionwise the A5 Sportback will swallow more cubic litres than an A4 sedan but not as much as an A4 Avant."

(Motoring journalist Jesse Adams, *Cape Times*, 4 February 2010) □

THE BEAL CONJECTURE



Over 350 years ago the French mathematician Pierre de Fermat stated that the equation

$$A^n + B^n = C^n$$

has no solutions with A, B, C and n positive whole numbers, with $n > 2$.

Fermat said that he had proved what he called "this wonderful result", but did not write down the proof. For centuries mathematicians tried to rediscover Fermat's proof, but without success. The problem became famous as *Fermat's Last Theorem*. It was only in 1994, right at the end of the 20th century, that the result was finally proved by the English mathematician Andrew Wiles.

Meanwhile another Andrew, Texas businessman and number theory enthusiast Andrew Beal, was one of many beavering away at the problem. In the course of his investigations, which included extensive computer use, Beal considered the more general equation

$$A^p + B^q = C^r$$

with the exponents p, q and r all greater than 2. His computer searches found a number of solutions, and Beal noticed that in every case A, B and C had a common factor.

For example, the solution $3^3 + 6^3 = 3^5$ has bases with a common factor of 3 and the solution $7^6 + 7^7 = 98^3$ has bases with a common factor of 7. In general, the identity

$$(a(a^m + b^m))^m + (b(a^m + b^m))^m = (a^m + b^m)^{m+1}$$

for any a, b and m, with $m > 3$, gives infinitely many solutions of Beal's equation, but the bases all have the common factor $(a^m + b^m)^m$.

A solution not in this family, with three different exponents without a common factor, is $27^4 + 162^3 = 9^7$ (but again the bases have a common factor of 3).

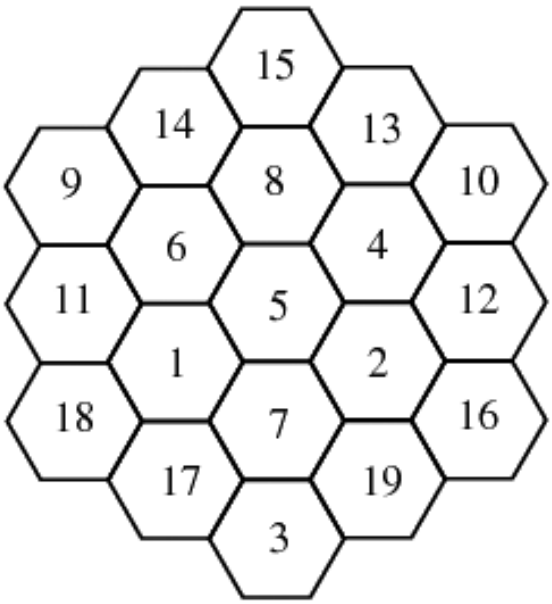
Beal contacted a number of mathematicians and told them about his observations. They agreed that his conjecture was a variation of Fermat's

Last Theorem that nobody had before considered, and in 1993 the *Beal Conjecture* became well known.

Andrew Beal set up a prize of $100 000 for a proof or disproof of his conjecture. The prize is administered by the American Mathematical Society and has not been claimed. □

THE MAGIC HEXAGON

Can you arrange the numbers from 1 to 19 in a hexagonal array so that the sums of the numbers along each of the 15 horizontal or diagonal lines are the same? This is a very difficult problem, which was first proposed (and solved) by a German architect Ernst von Haselberg of Stralsund, Germany in 1887. Since then it has been rediscovered many times.



An interesting feature of the Magic Hexagon is that if you add the the three numbers at the vertices of any small interior triangle you get the number on the edge of the hexagon towards which the little triangle points. For example, the triangle of numbers 8, 4, 5 has sum 17, opposite the triangle's vertex.

Can you find other interesting properties of the Magic Hexagon?

In 1963 Charles W Trigg (USA) proved that, apart from reflections and rotations, there are no other magic hexagons of any size. □

ARCHIMEDES AND THE CROWN

There are many stories about Archimedes, the greatest mathematician of antiquity. The best-known is certainly the story of how Archimedes solved the Problem of the Crown while taking a bath.

Archimedes (284–212 B.C.) lived in the city of Syracuse, which is in Sicily. The story is that Hiero II, king of Syracuse, had commissioned a crown to be made out of pure gold. He had given a mass of gold to the goldsmith, who had duly made it into a crown. But for some reason the king suspected that the goldsmith had cheated by replacing some of the gold with much cheaper silver. The weight of the crown was exactly that of the original lump of gold. The king asked Archimedes if he could figure out how to test the purity of the golden crown, without damaging it in any way.

It is from Vitruvius, a Roman architect and engineer who lived about a hundred years later, that we have the story of the bath. Vitruvius relates that Archimedes hit upon the solution of the problem when he climbed into his bath and noted that the water level rose as he settled down to enjoy a good soak.

Relaxing in his bath, Archimedes reasoned that if he took a piece of gold of exactly the same weight as the crown and submerged it in a bowl of water, the water level would rise. If he then submerged the crown in the same bowl, the water level should rise by exactly the same amount. But if the golden crown had been alloyed with silver, which is a lighter metal, its

volume would be greater than that of the lump of pure gold, so that the water would rise to a higher level.

So pleased was Archimedes at his solution of the problem that he leapt out of his bath and ran, stark naked, through the streets of Syracuse, crying *Eureka! Eureka!* (I have found it!).

It's a good story, but it probably didn't happen that way. First of all, the observation is not really profound, and is not nearly up to the level of Archimedes' other discoveries. Secondly, the method described would in fact be very difficult to implement with any degree of accuracy.

Suppose the crown weighed 1 kilogram. The density of gold is 19.3 g/cc, so 1 kg of gold would have a volume of $1000/19.3 = 51.8$ cc. That's about the size of a small bar of chocolate.

If the crown was of radius 10 cm, Archimedes would have needed a bowl of that radius in which to immerse the crown. The cross-sectional area of the bowl would be $\pi \times 10^2 = 314$ square centimetres, so submerging a volume of 51.8 cc into the bowl would have caused the water level to rise $51.8/314 = 0.165$ cm.

Now suppose that the dishonest goldsmith had replaced 100 grams of the gold given to him for the crown with silver. Silver has a density a little more than half that of gold, at 10.5 gm/cc. So the gold-silver crown would have a volume of $900/19.3 + 100/10.5 = 56.15$ cc.

Submerging this volume into the same container would cause the water level to rise $56.15/314 = 0.178$.

The difference in the levels is $0.178 - 0.165 = 0.013$, a trifle over a tenth of a millimetre. That would have been barely noticeable.

So how did could Archimedes have solved the problem? He would have called upon two laws of physics, both of which he discovered: the Law of the Lever, and the Law of Buoyancy.

The Law of the Lever is the principle on which the classic chemical balance is based. If two weights are suspended from a rod, they will balance if a fulcrum is placed under a point on the rod which divides the rod into two lengths inversely proportional to the two weights.

The Law of Buoyancy, also known today as Archimedes' Principle, which he set out in his treatise "On Floating Bodies", says that if you weigh an object in a liquid, its apparent loss of weight is equal to the weight of the liquid displaced.

With these principles in mind, Archimedes could have solved the problem as follows. Suspend the crown from one end of the balance and an equal weight of pure gold at the other end. With the fulcrum in the middle, they will balance. Now suspend both the crown and the lump of gold in bowls of water. If they still balance, then they have the same volume, and the same weight, so the crown is pure gold. But if the scale tilts in the direction of the gold, then the wreath has lost more weight on immersion, so has greater volume than the gold, and is therefore not pure gold.

By moving the fulcrum of the balance so that the two submerged objects balanced exactly, Archimedes would have been able to measure the ratios of the two volumes. From that he would have been able to calculate exactly the amount of silver used by the fraudulent goldsmith. □

DEMISEMIPRIMES

A *semiprime* is a number which is the product of two prime numbers. Semiprimes are of major importance in commerce, since products of two large primes are the key component in RSA cryptography.

Semiprimes also have relevance in the Goldbach Conjecture (see *Mathematical Digest* **158**, January 2010, pp. 12-13).

A *demisemiprime* is the product of four prime numbers. This is the first time that they have been defined, and they appear to be a concept in need of an application.

But it is appropriate to mention them now, since this year, 2010, is a demisemiprime year:

$$2010 = 2 \times 3 \times 5 \times 67.$$

Nobody seemed to notice that both 1995 ($= 3 \times 5 \times 7 \times 19$) and 2002 ($=2 \times 7 \times 11 \times 13$) were demisemiprime years.

The next year which is demisemiprime is 2030 ($= 2 \times 5 \times 7 \times 29$).

When other demisemiprime years will there be in this century?

What was the first demisemiprime year of the 20th century?

A Prize Voucher worth R50 will be awarded for the the first correct answers opened drawn on 1 June. □

ESKOM ENERGY ERRORS

How many mistakes can you find in the "explanation" below, from the Eskom website?

> **The Watt, kiloWatt and kiloWatt/hour explained**
> Energy released by electricity is measured in watts. In a home every appliance has an electricity rating. This, usually found on the back of the appliance, tells the user how many watts that particular appliance uses in an hour. (If, for example, a lamp uses 100 Watts, it will use this amount of electricity in an hour.) One-thousand Watts (1 000 W) equals one kiloWatt (1kW). When you are charged for electricity, your bill will usually tell you how many units were used during the month. One unit of electricity is exactly equal to 1 000 watts of power used for 1 hour—or kWh, which suppliers use to determine your costs. (Your lamp using 100 Watts will therefore use 0.1 kW in an hour when it's usage is monitored by your home's electricity meter.)

1. In the heading, the unit should be written kilowatt-hour, not kiloWatt/hour. The correct way to write the units is *watt* and *kilowatt* (not *Watt* and *kiloWatt*. However, capital letters are used in the abbreviations: W and kW.
2. The first word of the first sentence is wrong. Electrical *power* is measured in watts. Power is the rate at which energy is generated. Electrical *energy* is measured in kilowatt-hours. Confusing power and energy is like confusing the speed of a car with the distance it travels.
3. An electrical appliance does not use "watts in an hour". It uses watts.
4. The 100W lamp will use 0.1 kWh (not 0.1 kW) in an hour.
5. Spelling errors: "one-thousand" should be "one thousand" and "it's" should be "its". □

QUOTE

Computers are useless. They can only give you answers.

(Pablo Picasso) □

THE INTERNATIONAL DATE LINE

The need for an International Date Line was realised in 1350 by the Bishop of Lisieux, Nicole Oresme, one of the most distinguished mathematicians of his day. He describes a situation of three men, one going around the world eastward in 12 days, one going westward in 12 days, and one remaining at home. When the three get together at the end, they find that they cannot agree what day it is.

When Magellan's crew arrived in Cape Verde in 1522 after their three-year circumnavigation of the globe, they were puzzled to find that it was Thursday there, while their ship's log showed that it was Wednesday.

In Jules Verne's novel *Around the World in 80 Days*, Phileas Fogg makes a bet of £20,000 that he can circumnavigate the world in 80 days. When he gets back after 81 days, he thinks he has lost the bet, but then realises that he forgot to adjust his records when he crossed the International Date Line, and has in fact won.

The paradox of gaining or losing a day when crossing the International Date Line is the theme of an 1867 poem by Bret Harte, *The Lost Galleon*.

In the 19th century in Britain the advent of the railway led to the need for a uniform time system, and in 1847 time in Britain was standardized on London time. Since Britain is not very broad from East to West, there was no need for different time zones.

The first successful transatlantic telegraph cable was laid in 1866, and soon instant communication by telegraph spread across the world. It was realised that agreement had to be reached on a consistent international time system. Conferences in Rome and Washington in 1883-4 agreed to adopt the Greenwich Prime Meridian as the zero point for the measurement of longitude, and Greenwich Mean Time was adopted as the universal standard. When it is midday at Greenwich, it is midnight at longitude 180°. But on either side of the 180° longitude the dates would be different. Thus the International Date Line was born.

Though it happened to be conveniently almost entirely across the uninhabited Pacific Ocean, a few minor adjustments had to be made. The line was bent to make it pass through the Bering Strait between Russia and North America, and further kinks allowed some South Pacific Islands to have the same date as New Zealand.

When Russia sold Alaska to the United States, the territory had to abandon the old Julian Calendar (still in force in Russia) and adopt the Gregorian Calendar, then 12 days ahead. Alaska also had to lose a day as it moved across the Date Line, requiring an eight-day week.

In 1979 the Republic of Kiribati was established, combining a string of 33 small Pacific islands, formerly British colonial possessions. With the International Date Line passing through the new republic, the eastern part of the new republic was a day ahead of the west. A practical problem arose: on only four days of the week could official business be conducted across the country. So in 1995 the President of Kiribati declared that the International Date Line would loop eastward around the island nation, creating by far the biggest kink in the line.

An unexpected spinoff of that decision came just five years later, when Kiribati was internationally recognised as the first country to celebrate the beginning of the Third Millennium. □

MORE MEGAWATT MUDDLES

> Midday summer sun imparts about 1 440 kilowatt hours per square metre. Commercial (high yield vs low cost) solar cells have an efficiency of 11–19%, meaning that one would need 4–6 m^2 of solar panels to capture 1MWh of energy.
>
> (Letter to the *Cape Times*, 24 December 2009)

The first sentence is wrong on two counts. The correct unit to use to measure solar power is kilowatts (not kilowatt-hours) per square metre, and the correct number is 1.440. The writer left out the decimal point.

Allowing for a solar panel efficiency of at most 20%, the statement that one would need 4 to 6 square metres of solar panels to generate 1 megawatt of power is therefore inaccurate by a factor of well over one thousand.

Even the estimate of 1.440 kilowatts per square metre requires qualification. It is solar power as measured above the atmosphere of the Earth, which is reduced by the atmosphere, especially cloud cover.

Furthermore, a solar panel must ideally be at right angles to the sun's rays for maximum effect. If the panel is tilted by an angle of θ off this direction, its effectiveness must be multiplied by $\cos\theta$. □

SHARP CALCULATOR COMPETITION

The Sharp Calculator Competition is sponsored by Sharp Electronics (S.A.) (Pty) Limited, who have donated three calculators as prizes.

These prizes are offered for the three best solutions of the problems below. Solutions may be in English or Afrikaans. Partial solutions will be judged on their merits. The Editor reserves the right to make no award if no satisfactory entries are received, and his judgement will be final. The winner of an nth prize in a previous competition will only be eligible for an mth prize if $m < n$.

Only high school students may compete. Entrants must give their full name and home address, age, school and grade. Solutions must be certified by a mathematics teacher at the school as the entrant's own unaided work.

The closing date for receipt of entries is 1 June 2010, and the results will be announced in our July 2010 edition.

SHARP

1. The number $A_n = 444\ldots443$ consists of n 4's followed by a 3. For which values of n is A_n divisible by 13?

2. Find all real-valued functions f of a real variable such that

$$f(x)f(y) - f(xy) = x + y$$

for all real numbers x and y.

3. In triangle ABC, D is the midpoint of BC and AD is joined. Prove that

$$\frac{2}{\tan ADC} = \frac{1}{\tan B} - \frac{1}{\tan C}.$$

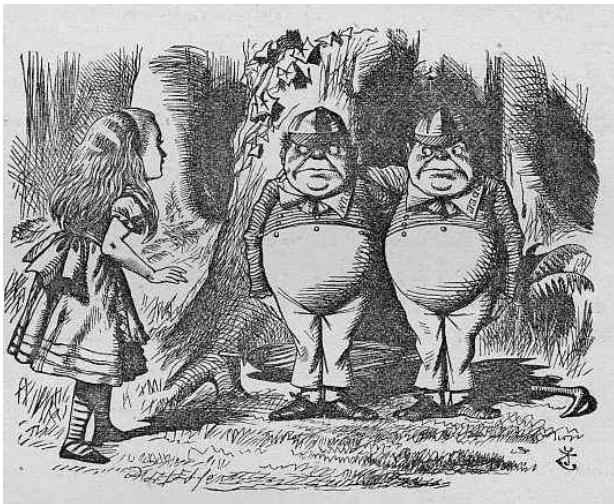
THE LINE

The line is breadthless length. (Euclid, in *The Elements*)

The line is immaterial. (Lady Bracknell, in *The Importance of Being Earnest*, by Oscar Wilde..)

DUM AND DEE

Tweedledum and Tweedledee are identical twins. One of them tells the truth only on Mondays, Tuesday and Wednesdays, and the other tells the truth only on Thursdays, Fridays and Saturdays.



When Alice met them, one of them said "I'm Tweedledum" and the other said "I'm Tweedledee". What day of the week was it? □

PI BY PC

About 2200 years ago Archimedes tackled the problem of calculating the value of π, and showed that

$$3\frac{10}{71} < \pi < 3\frac{1}{7},$$

which is correct to two decimal places. Ever since then, computing the famous constant to more and more decimal places has been a passion among mathematicians and, more recently, computer freaks.

A new record for calculating the digits of π has been set by Fabrice Bellard, a French computer scientist, who has computed π to a spectacular 2.7 trillion places. His calculation took 103 days, with a further 28 days of checking.

That may seem terribly slow compared with the previous record of 2.6 trillion decimal places, which was achieved in a mere 29 hours in Japan. The Japanese pi crunchers, however, had used a high-powered computer. Bellard used a PC which cost under 2000 Euros. □

SHARP SOLUTIONS AND RESULTS

1. *For which positive integers n is $\sqrt{\dfrac{9n-1}{n+7}}$ a rational number?*

   Suppose that $\sqrt{\dfrac{9n-1}{n+7}} = \dfrac{a}{b}$, where a and b are positive integers with no common factor. Then (squaring and solving for n)

   $$n = \frac{7a^2+b^2}{9b^2-a^2} = \frac{7(a^2-9b^2)+64b^2}{9b^2-a^2} = -7 + \frac{64b^2}{9b^2-a^2}.$$

   Since a and b have no common factors, $9b^2-a^2 = (3b+a)(3b-a)$ must be a divisor of 64. These two factors must both be even, and their sum is a multiple of 6.

   We now consider the various possibilities:

   $3b+a = 4$ and $3b-a = 2$. Then $a = b = 1$ and $n = 1$.

   $3b+a = 8$ and $3b-a = 4$. Then $a = b = 2$, which we reject because we assumed that a and b had no common factor.

   $3b+a = 16$ and $3b-a = 2$. Then $a = 7$, $b = 3$ and $n = 11$.

   So there are just two solutions: $n = 1$ and $n = 11$.

   Source: 2009 Hellenic Mathematical Olympiad.

2. *Can 19^{19} be represented as the sum of a cube and a fourth power?*

   The trick here is to consider remainders on division by 13.

   For cubes, the remainders on division by 13 are 0, 1, 5, 8 and 12.

   For fourth power the remainders on division by 13 are 0, 1, 3, 4 or 9.

   Putting these together in all possible ways shows that when a sum of a cube and a fourth power is divided by 13, the remainder could be any whole number *except* 7.

   We now determine the remainder when 19^{19} is divided by 13.

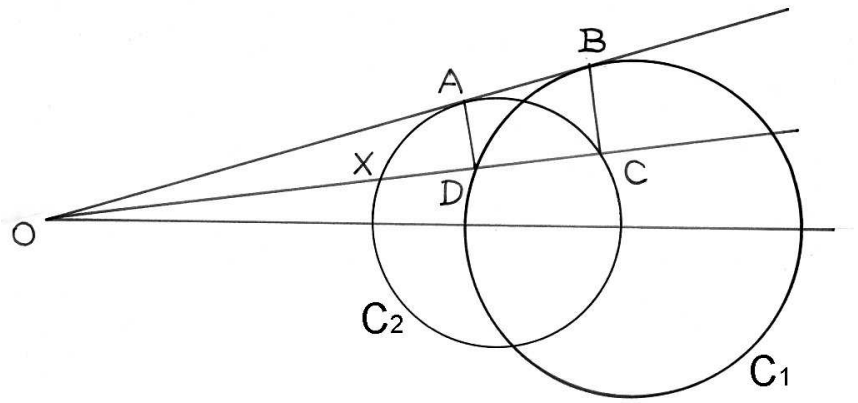
   For successive values of n from 1 to 12, the remainders when 19^n is divided by 13 are

   $$6, 10, 8, 9, 2, 12, 7, 3, 5, 4, 11, 1$$

   after which the list of remainders recurs. So the remainder on division by 13 of 19^{19} is 7.

   So 19^{19} cannot be expressed as the sum of a cube and a fourth power.

3. *In the figure below, the centres of the two circles lie on the line OX, and the line OAB is tangent to the circles. Another line from O cuts the circles in C and D.*
Prove that $ABCD$ is a cyclic quadrilateral.



Call the circles C_1 and C_2 as shown. Then C_1 is an enlargement of C_2 from the point O, and

$$\frac{OB}{OA} = \frac{OD}{OX}.$$

By the tangent-secant theorem $OA^2 = OX \times OC$.

So $\dfrac{OC}{OA} = \dfrac{OA}{OX} = OA \div \dfrac{OA.OD}{OB} = \dfrac{OB}{OD}$.

From this equation, and the fact that triangles OAD and OCB share a common angle at O, triangles OAD and OCB are similar. So $\angle OAD = \angle OCB$. It follows that $\angle BAD + \angle DCB = 180°$, proving that $ABCD$ is a cyclic quadrilateral.

The result holds even if the circles do not intersect.

Source: Mathematical Gazette, March 2003.

No prizeworthy entries were received. □

TEN BOTTLE PROBLEM

A milk crate has space for 25 bottles, in five rows of five each.

Show how to place ten bottles in the crate, so that no row or column or diagonal contains three or more bottles.

A Prize Voucher worth R50 is offered for the first correct solution opened on June 1. □

PRIZEWINNERS AND SOLUTIONS

Mthmtcl Wrds Wtht Vwls (January 2010, p. 27):
Emmanuel Setlhapelo (Kismet Combined School) wins a R50 Prize Voucher for putting the vowels back into the following mathematical words:

HYPOTENUSE	COSECANT	PROTRACTOR	VOLUME
SUBTRACTION	CENTRE	GEOMETRY	CUBIC
LOGARITHM	ASYMPTOTE	BILLION	VERTEX
POLYNOMIAL	PARALLELOGRAM	HYPERBOLA	SOLUTION
COMPASSES	EQUATION	NINE	RATIONAL

Prize Vouchers may be used to buy any of the publications listed at the back of this edition of *Mathematical Digest*.

Competitions are open to all readers, but only high school students are eligible for prizes. Solutions to different problems must be written on separate A4 sheets, each of which must carry the sender's name, address and school. To save postage, solutions to different problems, and from different readers, may be sent in the same envelope. Solutions may also be submitted by fax or email. □

CONTEMPORARY LOVE SONG

R.P. Boas

My love is a computer with a gigabyte of RAM.
It never asks for caviar nor yet Prosciutto ham.
Champagne and gourmet cooking are the farthest from its thoughts;
The nourishment I give it is a modest 40 watts.
It's never watching TV when I want to have a chat,
And when I'm through conversing, I can turn it off like *that.*
It never brings me flowers or takes me to a show;
It never seeks affection, or other quid pro quo.
It doesn't kiss me gently or cuddle me in bed,
But it has such lovely software to give me thrills instead. □

BOOK REVIEW

The First Twenty-Five Years of the SUPERBRAIN, by Diarmuid Early and Des MacHale. x+230 pp. High Perception, 2009.

In 1984 a discussion arose between science and engineering students at the University of Cork in Ireland as to who were better at solving maths problems. To settle the dispute, Des MacHale launched an annual "Superbrain Competition", consisting of a three-hour, ten-question problems paper, open to all Cork students.

The 250 problems of the first 25 years of the competition are now available, with full solutions. Many of the problems can be cracked with just school mathematics and are good training material for Mathematical Olympiads, although few of them are at the very demanding level of the International Mathematical Olympiad (IMO). There are also problems requiring calculus, which is not part of the IMO tradition.

Here are five rather nice SUPERBRAIN problems. Answers can be found on page 31.

1. Ten books are arranged in a row on a shelf. How many ways can this be done if one particular book A must always be to the left, but not immediately to the left, of another book B?
2. In a darts competition, each dart scores 40, 39, 24, 23, 17 or 16 points. How many darts must be thrown to get exactly 100 points?
3. In how many different ways is it possible to pay a punt (100p) using 50p, 10p and 5p pieces only?
4. What, in your opinion is the the next term of the sequence

   $$F, 4, E, S, 9, S, E, 5, E$$

   and why?
5. Prove that $4^{50} + 2^{50} + 1$ is not a prime number.

After 25 years, have the Irish resolved the problem of who are better at maths: science students or engineers? That is one answer that you will, sadly, not find in the book.

The First Twenty-Five Years of the SUPERBRAIN can be ordered through the website `www.highperception.com`. □

THE PYRAMID INCH

The study of the dimensions of the Great Pyramid of Giza, with a view to making astonishing astronomical connections and constructing alternative theories of history, is known as *pyramidology*.

It all began in 1638 when an Oxford Professor of Astronomy, John Greaves, travelled to Egypt to measure the pyramids. His work was continued a hundred years later by John Taylor, who proclaimed that the measurements proved that the ancient architect of the pyramid had used a basic measure that was very close to the British inch: to be precise, 1.00106 British inches. This measure became known as the *pyramid inch*.

It was said that the total length of the four sides of the Great Pyramid came to 36524 pyramid inches, which is one hundred times the length of the year in days. That, it was claimed, could not be a coincidence. It was further claimed that the pyramid inch was exactly one twenty-fifth of the sacred cubit mentioned in the description in the Book of Genesis of the building of the Ark. It followed, of course, that Noah had designed and built the Great Pyramid.

Taylor's interpretations of the measurements of the pyramid enjoyed a tremendous vogue in nineteenth century England. They inspired a religious movement, the British Israelites, who believed that the British race could trace its origins back to the Twelve Tribes of Israel, and that the British Royal Family was directly descended from King David.

One of the most enthusiastic supporters of Taylor's theories and an ardent British Israelite was the Astronomer Royal of Scotland, Charles Piazzi Smyth (1819-1900), who found numerous apparent coincidences between the measurements of the pyramids and the geometry of the earth and the solar system. For example, the polar diameter of the Earth was found to be exactly 500 million pyramid inches.

To Smyth, the existence of the pyramid inch proved that the British Imperial system of inches, feet, yards and miles had divine origins, and that

the rival metric system being promoted by the French was nothing short of sacrilegious atheism. Measurements of the chambers and passages inside the pyramid were regarded as God's messages in stone, because they were found to correspond to important events in history with remarkable accuracy. An attempt to predict the End of the World based on further careful measurements of the pyramid passages was markedly unsuccessful, but that did not deter a succession of further predictions using suitably revised measurements, all of which were just as unsuccessful.

Though the pyramid inch theory had a certain vogue in Victorian England, it lost all its scientific credibility as a result of the visit to Egypt in 1880 by the great British archaeologist Flinders Petrie. Petrie's father had been a good friend of Piazzi Smyth, and a fervent believer in the divine message of the pyramid. Petrie senior must have been profoundly disappointed when his son's accurate measurements of the pyramid showed that it was several feet smaller than previously believed. Petrie concluded that "there is no authentic example, that will bear examination, of the use or existence of any such measure as a Pyramid inch, or of a cubit of 25.025 British inches." Petrie dismissed all the attempts to link the measurements of the pyramid as spurious, and, rather unkindly, dismissed those who pursued pyramidology as "pyramidiots". □

SHOPPING LIST

Four men went shopping. Afterwards Mr Dickson noted that Mr Adamson had spent three times as much as Mr Benson, who had spent four times as much as Mr Clarkson. The men also noted that Adam had spent R550 more than Ben, while Clark had spent R250 more than Dick.

Can you sort out the first names and surnames of the four men, and how much each spent? A copy of *Count Down* by Steve Olsen will be awarded for the first correct entry drawn on on June 1. □

SUNDAY AT HOME

My wife and I all the afternoon at arithmetique, and she is come to do Addition, Subtraction and Multiplication very well.

(From the diary of Samuel Pepys, Sunday 6 December 1663) □

TONGAN TRILLIONS

In the Eighteenth Century the French naturalist Jacques-Julien Houtou de la Billiardiére visited the Pacific island of Tonga, and spent some time recording the language of the Tongan people.

He took detailed records of Tongan words, including words for numbers:

0 (noa)	1 (taha)	2 (ua)	3 (tolu)	4 (fa)	5 (nima)
6 (ono)	7 (fitu)	8 (valu)	9 (hiva)	10 (hongofulu)	

As in many other cultures, the Tongans counted on their fingers and toes.

As de la Billiardière questioned them further, he was amazed to learn that they had words for very large numbers such as ten million (laoalai) and a hundred million (laounoua). The Tongans assured the Frenchman that they needed large numbers to count yams.

De la Billiardière diligently wrote down everything they told him.

Alas, some time later he discovered that they had been pulling his leg. Having run out of fingers and toes for their number words, they had been making up names for larger numbers, using various unmentionable body parts. □

A TRICKY TEST

When researchers posed the following three questions to 3400 students in the United States, only one-third got them all correct. Can you do better?

1. A bottle and a cork together cost R1.10. The bottle costs R1 more than the cork. How much does the cork cost?

2. If it takes 5 machines 5 minutes to make 5 widgets, how long will it take 100 machines to make 100 widgets?

3. In a lake, there is a patch of lily pads. Every day, the patch doubles in size. If it takes 48 days for the patch to cover the entire lake, how long would it take for the patch to cover half of it?

The first correct answers opened on 1 June will win a Prize Voucher worth R50. □

Y2010 BUGS

Ten years ago a huge fuss was made about the Y2K Bug, which doomsayers predicted would cause computers to crash. The problem, was that some computers had recorded the year in a field of just two digits (e.g. 1997 was recorded as just 97). So when 2000 came round, the year would be read as 00 and interpreted as 1900.

Much feverish programming prevented the predicted disaster. But ten years on, as 2009 rolled over into 2010, several computer systems crashed, leaving both software and hardware manufacturers red-faced.

A New Year bug sabotaged credit card transactions at thousands of Australian shops. Systems at the Bank of Queensland said that the year was 2016. The error meant that legitimate credit and debit cards were rejected because they appeared to have expired.

It appeared that the bank's programmers had coded decimal numbers into bytes, and interpreted the two digits of the year, 10, as the hexadecimal 16. It's an easy mistake for a programmer to make, but should have been picked up in systems testing and not got into production. It took several days for the programmers to be called back from their New Year celebrations to fix the mess.

Credit cards linked to German banks also had a problem, but it was caused by an error in the chip implanted in the card. About 30 million chip and pin cards were affected. The French card manufacturer, Gemalto, accepted responsibility for the error, which it was estimated would cost €300m to fix. The company's shares dropped by 4% on the news of its blunder.

SMS messages to Windows Mobile phones displayed a date of 2016 instead of 2010.

SpamAssassin, a major spam filtering product, had a rule to reject emails with dates later than 2009 as too far in the future, but they forgot to remove the rule when the real 2010 came along, so all legitimate email was rejected as if it was spam.

The Apple Newton had a problem with dates after 2010, but it was identified and fixed in mid-2009. Since Newtons came out in 1989 and have not been built since 1998, it's an excusable bug, which affected only a handful of devoted users of the venerable computer. □

VARSITY READINESS TEST

How ready are you for varsity maths? Try the problems below (no calculators!) and then check out the correct answers on page 31. Any mistake you make will flag a weakness in your mathematical background requiring fixing before signing up for first-year university mathematics.

1. How many seconds are there in a day?

   (1) 5 184 000 (2) 1 440 (3) 28 800 (4) 86 400 (5) 14 400

2. Which of the following numbers is the smallest?

   (1) $\frac{3}{5}$ (2) $\frac{4}{7}$ (3) $\frac{2}{3}$ (4) $\frac{5}{8}$ (5) $\frac{3}{4}$

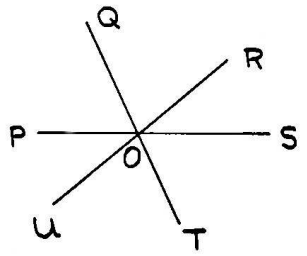
3. $\dfrac{1+3\div 5}{7-(9-11)}$ is equal to

   (1) $\frac{8}{25}$ (2) $\frac{4}{45}$ (3) $\frac{4}{65}$ (4) $\frac{4}{25}$ (5) $\frac{8}{45}$

4. $\dfrac{\frac{1}{2}-\frac{1}{3}}{\frac{1}{3}-\frac{1}{4}} - \dfrac{\frac{1}{2}+\frac{1}{3}}{\frac{1}{3}+\frac{1}{4}}$ is equal to

   (1) $\frac{7}{12}$ (2) $\frac{4}{7}$ (3) $\frac{5}{3}$ (4) $\frac{11}{8}$ (5) $\frac{1}{6}$

5. In the diagram $\angle POR = 120°$ and $\angle QOS = 145°$. What is the size of $\angle TOU$?



   (1) 45° (2) 60° (3) 85°
   (4) 90° (5) 95°

6. A circle has a diameter of $4x + 12$. The area of the circle is

   (1) $4\pi x^2 + 24\pi x + 36\pi$ (2) $8\pi x^2 + 24\pi x + 36\pi$
   (3) $16\pi x^2 + 96\pi x + 144\pi$ (4) $4\pi x^2 + 12\pi x + 36\pi$
   (5) $16\pi x^2 + 48\pi x + 144\pi$

7. Factorize $(x^2 - 5x)^2 - 36$.

   (1) $(x-3)(x-2)(x-6)(x+1)$ (2) $(x+3)(x-2)(x+6)(x+1)$
   (3) $(x-3)(x+2)(x-6)(x-1)$ (4) $(x-3)(x+2)(x+6)(x+1)$
   (5) $(x+3)(x-2)(x-6)(x-1)$

8. $\sqrt{81a^4b^{18} + 36a^8b^{12}}$ is equal to

(1) $3a^2b^6\sqrt{4a^4 + 9b^6}$ (2) $\sqrt{117a^{12}b^{30}}$ (3) $9a^4b^{12}\sqrt{9a^4 + 6b^6}$

(4) $\sqrt{117(a^{12} + b^{30})}$ (5) $9a^2b^9 + 6a^4b^6$

9. The area of an isosceles triangle is 108 cm^2 and its base is 18 cm. What is the perimeter of the triangle?

(1) 12 cm (2) 24 cm (3) 48 cm (4) 63 cm (5) 225 cm

10. If $\left((2^{2n} + 1)(2^{2n} - 1) + 1\right)^{\frac{1}{4}} = 256$ then n is equal to

(1) 2 (2) 4 (3) 6 (4) 8 (5) 10

11. $(a^{-1} + b^{-1})^{-1}$ is equal to

(1) $a + b$ (2) $\dfrac{1}{a^2} + \dfrac{1}{b^2}$ (3) $\dfrac{a}{b} + \dfrac{b}{a}$ (4) $\dfrac{ab}{a + b}$ (5) $a^2 + b^2$

12. $\dfrac{2}{x+1} - \dfrac{3}{x+2} + \dfrac{4}{x-2}$ is equal to

(1) $\dfrac{3x^2 + 15x + 6}{x^3 + x^2 - 4x - 4}$ (2) $\dfrac{3x^2 + 15x - 6}{x^3 - x^2 - 4x - 4}$

(3) $\dfrac{3x^2 - 15x + 6}{x^2 + x^2 + 4x - 4}$ (4) $\dfrac{3x^2 - 15x - 6}{x^3 + x^2 - 4x - 4}$

(5) $\dfrac{3}{x - 4}$

13. The graphs of the equations $3x + 4y = 2$ and $4x - 3y = 11$ are

(1) the same line
(2) two lines of positive gradient
(3) two perpendicular lines
(4) two distinct parallel lines
(5) two lines intersecting at the origin

14. If $x = \dfrac{y+2}{y-3}$, then y is equal to

(1) $\dfrac{3x-1}{x+2}$ (2) $\dfrac{3x+2}{x-1}$ (3) $\dfrac{2x-1}{x+3}$ (4) $\dfrac{x-2}{3x+1}$ (5) $\dfrac{3x+1}{x-2}$

15. A cube has edges of length 6. One of its square faces is $ABCD$, and E is the centre of the cube. What is the volume of the pyramid $ABCDE$?

(1) 54 (2) 36 (3) 108 (4) 72 (5) 216

16. Circle A has radius of π. Circle B has circumference of 8π. Circle C has an area of 9π. List the circles in order, from smallest to largest radius.

(1) A, C, B (2) B, A, C (3) C, B, A (4) B, C, A (5) C, A, B

17. If $\sin\alpha = \dfrac{8}{17}$ then $\tan 2\alpha$ is equal to

(1) $\dfrac{321}{225}$ (2) $\dfrac{16}{15}$ (3) $\dfrac{425}{289}$ (4) $\dfrac{240}{161}$ (5) $\dfrac{143}{97}$

18. The quadratic $x^2 + ax + b = 0$ has equal roots. What is the value of $\dfrac{b}{a^2}$?

(1) 1 (2) 2 (3) $\dfrac{1}{4}$ (4) -4 (5) ± 2

19. If f is a quadratic function such that $f(1) = 2$, $f(2) = 4$ and $f(4) = 14$, determine $f(3)$.

(1) 8 (2) 9 (3) 10 (4) 11 (5) 12

20. $\sin(90^\circ - x)\cos(180^\circ - x) - \tan x \cos(-x)\sin(180^\circ + x)$ is equal to

(1) -1 (2) 1 (3) $\cos 2x$ (4) 0 (5) $-\cos 2x$

21. The lines $y = 2x + 3$ and $y = kx - 3$ intersect above the line $y = 5$. It follows that

(1) $k < -8$ (2) $-8 < k < 0$ (3) $0 < k < 2$
(4) $2 < k < 8$ (5) $k > 8$

22. If $a = 75$, $b = 36\sqrt{5}$, $c = 72$ and $d = 32\sqrt{5}$, which of the following is true?

(1) $a > b > c > d$ (2) $a > c > b > d$ (3) $b > a > c > d$
(4) $b > a > d > c$ (5) $a > b > d > c$

23. If m and n are positive integers such that $6^m + 2^{m+n}3^m + 2^n = 332$, what is the value of $m^2 + mn + n^2$?

(1) 19 (2) 18 (3) 20 (4) 17 (5) 21

24. In triangle ABC, the bisector of $\angle A$ meets BC in D. If $AB = 6$, $BD = 3$ and $DC = 4$, what is the length of CA?

(1) 9 (2) 8 (3) 10 (4) 7 (5) 6

25. Triangle ABC is isosceles with $AB = BC$. Point D is the midpoint of both BC and AE, and $CE = 10$. What is the length of BD?

(1) 8 (2) 5 (3) 10 (4) 6 (5) 7

26. If $3^n + 3^n + 3^n = 9^{2010}$ then n is equal to

(1) 2010 (2) 2007 (3) 2004 (4) 4019 (5) 6029

27. Ann and Bob each throw two coins. What is the probability that Ann throws more heads than Bob?

(1) $\frac{1}{2}$ (2) $\frac{1}{4}$ (3) $\frac{1}{8}$ (4) $\frac{5}{16}$ (5) $\frac{3}{8}$

28. Four points A, B, C and D lie in a plane, with B the midpoint of AC. If $AB = BC = BD = 13$ and $CD = 24$, find the length of AD.

(1) 11 (2) 9 (3) 13 (4) 12 (5) 10

29. For $0° < A < 90°$, $\dfrac{\sin A}{1 - \cos A}$ is equal to

(1) $\sin A - \tan A$ (2) $\dfrac{1 - \sin A}{\cos A}$ (3) $\dfrac{1 + \cos A}{\sin A}$

(4) $\dfrac{\cos A}{1 - \sin A}$ (5) $\dfrac{\cos A}{1 + \sin A}$

30. In triangle ABC, $AB = 8$, $AC = 15$ and $BC = 17$. What is the radius of the inscribed circle?

(1) 3 (2) $2\sqrt{2}$ (3) $1 + \sqrt{3}$ (4) $\sqrt{7}$ (5) $\sqrt{10}$

QUOTE

Photons have mass? I didn't even know they were Catholic.

(Woody Allen) □

ERRATA

Some unfortunate errors crept into the pages of *Mathematical Digest* **158** (January 2010).

On page 13 "every number can be expressed as the product of at most six primes" is obviously wrong (think of $2 \times 3 \times 5 \times 7 \times 11 \times 13 \times 17 = 510510$). This was clearly a case of mathlexia: for "product" read "sum".

In the Sharp rules (page 16), "schools" and "teachers" in paragraph 3 should be "school" and "teacher".

The algebra on page 31 contained a number of errors. Lines 5 to 10 should read:

Adding the two equations gives

$$-a_{n+1}a_n - a_n^2 + a_{n+2}^2 - a_{n+2}a_{n+1} = 0$$

$$a_{n+2}^2 - a_{n+2}a_{n+1} = a_n^2 + a_{n+1}a_n.$$

Complete the square by adding $\frac{1}{4}a_{n+1}^2$ to both sides:

$$a_{n+2}^2 - a_{n+2}a_{n+1} + \frac{1}{4}a_{n+1}^2 = a_n^2 + a_{n+1}a_n + \frac{1}{4}a_{n+1}^2$$

$$(a_{n+2} - \frac{1}{2}a_{n+1})^2 = (a_n + \frac{1}{2}a_{n+1})^2.$$

Our thanks to several alert readers for pointing out these errors. □

TWO AND TWO

In mathematics Woman leads the way!
The narrow-minded pedant still believes
That two and two make four! Why, we can prove—
We women, household drudges as we are—
That two and two make five—or three—or seven—
Or five-and-twenty, as the case demands!

(Princess Ida, in the Gilbert and Sullivan operetta of that name) □

SIX NUMBERS

The six numbers below play an important role in South African life.

42 401 701 724 750 762

The numbers quantify something seen everywhere and every day in the country.

What is it, and what do the numbers signify? The first correct answer opened on June 1 will win a Prize Voucher worth R50. □

SUPERBRAIN ANSWERS

1. 1 814 400.
   There are 10! = 3 628 800 ways of arranging the ten books on the shelf. Half of the arrangements have A to the left of B, and half have B to the left of A.
2. Four 17s and two 16s.
3. 16 ways.
4. The next letter is N.
   Write the numbers as Roman numerals: $F, IV, E, S, IX, S, E, V, E, N$.
5. Use the factorization $x^{4n} + x^{2n} + 1 = (x^{2n} + x^n + 1)(x^{2n} - x^n + 1)$ with $x = 2$ and $n = 25$. □

VARSITY READINESS TEST ANSWERS

(1)	4	(2)	2	(3)	5	(4)	2	(5)	3
(6)	3	(7)	1	(8)	1	(9)	3	(10)	4
(11)	4	(12)	1	(13)	3	(14)	2	(15)	2
(16)	5	(17)	4	(18)	3	(19)	1	(20)	5
(21)	5	(22)	3	(23)	1	(24)	2	(25)	2
(26)	4	(27)	2	(28)	5	(29)	3	(30)	1

mathematical digest

No. 160 July 2010

The Jabulani Football

mathematical digest

Number 160 **July 2010**

Have you noticed that the astronomers and mathematicians are much the most cheerful people of the lot? I suppose that perpetually contemplating things on so vast a scale makes them feel either that it doesn't matter a hoot anyway, or that anything so large and elaborate must have some sense in it somewhere.

(Dorothy L. Sayers)

Department of Mathematics and Applied Mathematics
University of Cape Town

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Cover: The cover features the Jabulani Football, specially designed for the 2010 World Cup in South Africa (see article on page 2)

INTERGALACTIC MATHS

Radio signals from outer space are continually being monitored and examined for signs of intelligent beings in other worlds.

But how would such messages be sent? The obvious answer is mathematics, and it is equally obvious that this answer would occur to other civilizations, who could right now be sending us mathematical messages.

Below is a possible message from outer space, consisting of 26 symbols, which we have represented by the letters of the alphabet for convenience. Can you interpret the message?

- A, B, C, D, E, F, G, H, I, J, K, L, M, N, O, P, Q, R, S, T, U, V, W, X, Y, Z
- AA,B; AAA,C; AAAA,D; AAAAA,E; AAAAAA,F; AAAAAAA,G; AAAAAAAA,H; AAAAAAAAA,I; AAAAAAAAAA,J
- AKALB; AKAKALC; AKAKAKALD; AKAKAKAKALE; BKALC; CKALD; DKALE; BKELG; GLEKB; FKDLJ; JLFKD
- CMALB; DMALC; IMGLB
- CKNLC; HKNLH; DMDLN; EMELN
- JLAN; JKALAA; JKBLAB; AAKALAB; JKJLBN; JKJKJLCN; FNKGLFG
- BPCLF; EPBLJ; FPJLFN
- FQBLC; JQBLE; FNQFLJ
- CRBLI; BRELCB
- JPJLJRBLSLANN; JPJPJLJRCLTLANNN; JPSLT; JPTLJRD
- AQJLU; UQJLAQSLV
- ULWA; UPBLWB; AWDMALWDLDPU; VLWNA; VPCLWNC; VQJLWNNA; VQSLWNNNA; JPEWFGHLEFWGH; SPEWFGHLEFGWH
- GIWIHYHN; TKCYT; ZYCWADAF
- DPZPWNNIBRCQC

The solution will be given in the October edition. □

THE JABULANI BALL

The ball used in the 2010 FIFA World Cup was specially designed for the event by sports manufacturer Adidas, who named it the *Jabulani* ball, from the Zulu word "rejoice".

There are eleven colours on the Jabulani ball, linked to the eleven players in a football team, the eleven official languages in South Africa and (according to the FIFA website) "the eleven South African communities that will welcome the world". It is also the eleventh time (since 1970) that Adidas has produced a special World Cup football.

Having disposed of the numerology, we can examine some serious geometry and physics behind the design of a World Cup football. The classic black and white football of 32 panels (20 hexagons and 12 pentagons) was launched at the World Cup in Mexico in 1970. It was given the name *Telstar,* after an early communications satellite.

Geometrically a *truncated icosahedron*, the Telstar's geometry has connections with Leonardo da Vinci, geodesic domes, the Nagasaki atom bomb and the carbon 60 molecule. (See *The Football, the Buckeyball and the Fat Man,* in *Mathematical Digest* **158**, pages 2-3.)

For the 2006 World Cup in Germany Adidas designed the *Teamgeist* ball. That ball had just 14 panels, and the basic design was that of a truncated octahedron. The surface of the Teamgeist ball was perfectly smooth.

In designing the Jabulani ball for the 2010 World Cup, Adidas used a basic octahedral shape, reducing the number of panels to eight. The Jabulani

ball is said to be even more perfectly spherical than any of its predecessors.

A perfect sphere is not, however, ideal. Golfers have known for a long time that the dimples on a golf ball ensure that it flies further and more accurately when driven than a totally smooth golf ball. The aerodynamic behaviour of a flying ball is surprisingly complex. Small surface irregularities create a layer of turbulence which (though it sounds unlikely) makes the ball fly straighter and further. The 32-panel ball, with 60 vertices and 90 stitched seams, had enough irregularity to ensure predictable behaviour in flight. The smoothness of the Teamgeist ball, with thermally bonded panels instead of stitched seams, made it behave erratically, to the dismay of goalkeepers in the 2006 World Cup.

FIFA regulations are very specific about the size, shape and weight of the official ball. Its water absorption, pressure loss, retention of shape and how it must bounce are carefully spelt out. But there are no rules laid down about the texture of the surface. Taking the tip from the golf ball, Adidas designed the surface of the Jabulani ball with small indentations and grooves to ensure that the ball flies accurately. For goalies, the ball may feel less slippery.

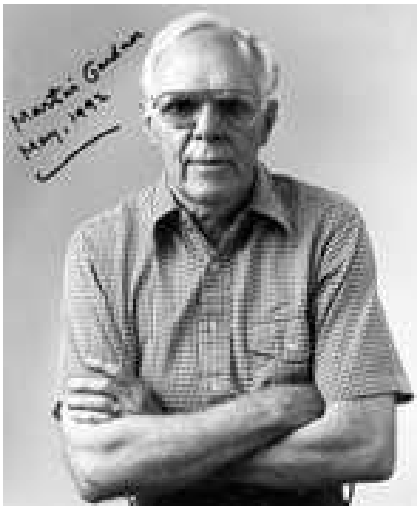
This article was written before the World Cup kick-off. Time will tell how well the Jabulani ball is received by the players. □

NUMBER PATTERN

$$
\begin{aligned}
9 &= 1 + 8 \times 1 \\
98 &= 2 + 8 \times 12 \\
987 &= 3 + 8 \times 123 \\
9876 &= 4 + 8 \times 1234 \\
98765 &= 5 + 8 \times 12345 \\
987654 &= 6 + 8 \times 123456 \\
9876543 &= 7 + 8 \times 1234567 \\
98765432 &= 8 + 8 \times 12345678 \\
987654321 &= 9 + 8 \times 123456789
\end{aligned}
$$

Readers are invited to send their favourite number patterns to the Editor. The best contributions will win R50 Prize Vouchers. □

MARTIN GARDNER (1914-2010)



Martin Gardner, who did more than any other person to popularize mathematics, died on May 22 aged 95.
He was born in Tulsa, Oklahoma, and graduated from college with a degree in philosophy and a wide interest in history, literature and science.
After service in the US Navy in the Second World War he became a freelance writer and edited *Humpty Dumpty*, a children's magazine.

In 1956 he started writing a regular column on recreational mathematics in the monthly magazine *Scientific American*, an assignment that continued for 25 years. The publisher named the column *Mathematical Games*, coincidentally Martin Gardner's initials.

With no mathematical training beyond high school, Gardner ransacked secondhand bookshops in Manhatten to find material, which he put together with great style, employing humour and anecdotes to keep the reader interested. His columns became enormously popular and were anthologized into a series of books that have become essential purchases for school libraries. When he stopped writing the *Mathematical Games* column in 1981, the readership of *Scientific American* dropped dramatically.

Martin Gardner was not just the finest popular writer ever on recreational mathematics. He had many other interests and talents. His *An Annotated Alice* is a wonderful deconstruction of the classic work of Lewis Carroll, uncovering hidden jokes, literary allusions and logical games in the famous children's book. Since its publication in 1960 it has never been out of print. His popular books on physics include *The Ambidextrous Universe* and *Relativity for the Million*.

In *Fads and Fallacies in the Name of Science* Gardner mercilessly skewered pseudoscience, from numerology and health quackery to astrology and UFOs. He was deeply skeptical of the claims of telepathy and clairvoyance and played a major role in debunking the spoon-bending claims of Uri Geller. With Carl Sagan, Isaac Asimov and others he launched the magazine *The Skeptical Enquirer*, in which he wrote a regular column. He was

a founding member of the Committee for the Scientific Investigation of Claims of the Paranormal (CSICOP).

Martin Gardner loved hoaxes, and his columns in the April editions of *Scientific American* frequently featured plausible nonsense. In one column he stated that relativity had been disproved, and went on to provide convincing proof that Leonardo da Vinci had invented the flush toilet. He once wrote, in the *New York Review of Books*, a scathing review of one of his own books on philosophy, *The Whys of a Philosophical Scrivener*, under the pseudonym George Groth.

He was the author of more than 65 books and countless articles, ranging over the fields of science, mathematics, philosophy, literature. He was an accomplished conjuror.

Despite his lack of any professional training, Martin Gardner earned the respect of many prominent mathematicians, with whom he was in constant correspondence. They included Roger Penrose, Ronald Graham, John Conway and Donald Coxeter. All over the world there are today thousands of mathematicians who chose mathematics as a career after Martin Gardner showed them, as teenagers, what fun it all is. □

PI PRODUCTS

$$\frac{\pi}{2} = \frac{2 \times 2 \times 4 \times 4 \times 6 \times 6 \times 8 \times 8 \cdots}{1 \times 3 \times 3 \times 5 \times 5 \times 7 \times 7 \times 9 \cdots}$$

(Discovered by John Wallis in 1655)

$$\frac{2}{\pi} = \sqrt{\frac{1}{2}}\sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2}}}\sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2}}}} \cdots$$

$$\pi = 2 \times \frac{2}{\sqrt{2}} \times \frac{2}{\sqrt{2+\sqrt{2}}} \times \frac{2}{2 + \sqrt{2 + \sqrt{2 + \sqrt{2}}}}$$

(Discovered by François Viète in 1592) □

THE TOP THREE MATHEMATICIANS

Professor Dirk Laurie
University of Stellenbosch

The October 2009 issue of *Mathematical Digest* contains the statement:

> Mathematicians today recognize Gauss as one of the three greatest mathematicians of all time (the others being Archimedes and Isaac Newton).

Since there is no quantifier on the first word, the statement is either almost trivially true ("some mathematicians") or certainly false ("all mathematicians").

I too grew up with that egregious winners' pedestal of mathematicians, which Eric Temple Bell may not have invented but certainly did much to perpetuate when he wrote:

> Archimedes, Newton and Gauss, these three, are in a class by themselves among the great mathematicians, and it is not for ordinary mortals to attempt to range them in order of merit.

In 1978 my father read Michael Hart's book ranking the 100 most influential persons in history and asked me to guess which three mathematicians appear on the list. It was almost a reflex with me to trot out the Bell platitude.

"Zero out of three", he said. The Hart candidates turned out to be Euclid (14), Descartes (49) and Euler (77). Newton is on the list, number 2 in fact, but as a physicist, not as a mathematician.

I immediately pooh-poohed Euclid ("a textbook writer, not a creative mathematician") and Descartes ("he's there as a philosopher") but was stopped in my tracks by the mention of Euler. Why on earth is Euler not in the list received by conventional wisdom?

- The man whose collected works are so voluminous that it took nearly 200 years to publish them.
- The man about whom the haughty Laplace, who referred to God as "that hypothesis", said: "Read Euler. He is the master of us all."

So I set myself two tasks:

1. Design an objective way of rating mathematicians.
2. Find out where the traditional list, excluding Euler, comes from.

Task 1:
Criterion: How many entries in the index of the *Encyclopedia Britannica* (the one on my shelf) refer to the mathematician in question?

Results: 69 Newton, 33 Euler, 33 Riemann 33 Gauss, 29 Euclid, 28 Bernoulli (all three put together), 26 Cauchy, 24 Laplace, 21 Archimedes, 16 Hilbert.

Task 2:
Gauss himself remarked, according to MacTutor: "There have been only three epoch-making mathematicians, Archimedes, Newton, and Eisenstein."

Posterity does not agree with him on Eisenstein, putting Gauss's own name in there instead. Indeed, one has to force oneself to be charitable, to forget Gauss's many claims of having anticipated the work of other mathematicians without bothering to publish, if one is not to conclude that that is precisely what Gauss intended.

The omission of Euler by modern writers, conditioned as they are by decades of bald assertions, may seem to be no stranger than that of Riemann, whose main body of work was done after Gauss died. From Gauss's lips, it is nothing less than a deliberate slight. □

PERCENTAGE ERROR

In his column in *The Times* (29 March 2010), Tony Shapshak wrote (in connection with a certain cellphone charge):

> "It went from R0.25 to R1.25 (or almost 600%)."

The error provoked a letter to the Editor of *The Times* pointing out that an increase from 0.25 to 1.25 is not "almost 600%", but *exactly* 400%.

The letter was not published. Instead, the writer received a personal reply from Tony Shapshak, who wrote:

> "Thanks for pointing it out. I was using public figures and even called a colleague to double check what the increase was. He said it was 575%." □

THE RICHTER SCALE

When earthquakes are reported in the media, their magnitude is usually given by a number on the Richter scale. In January the Haiti earthquake measured 7.0 on the Richter scale, the February earthquake in Chile measured a massive 8.8 while in April China was struck by a 6.9 magnitude earthquake. South Africa's worst earthquake was in 1969, when the little town of Tulbagh was devastated by a 6.5 magnitude tremor.

What do those numbers mean?

In 1935 the geologist C.F. Richter devised a scale to compare the magnitudes of earthquakes in California. Richter designed his scale by analogy with the scale used in astronomy to describe the brightness of stars. He defined the magnitude of an earthquake as the logarithm of the largest amplitude of the quake measured on a Wood-Anderson torsion seismometer at a distance of 100km from the epicentre of the quake. The amplitude is to be measured in microns (1 micron is one-thousandth of a millimetre; the unit is denoted by μm).

Since the logarithm of 1 is zero, an earthquake giving rise to a tremor with an amplitude of 1 μm measures zero on the Richter scale.

The Richter scale usually measures earthquakes in a range from 0 to 10. However, very sensitive instruments can detect tremors of less than 1 μm in amplitude, and because the logarithm function is employed, such tremors will have a negative Richter scale measure. In theory, the magnitude of a zero tremor is negative infinity.

Since the logarithm used in the Richter scale is base 10, a tenfold increase in the seismometer deflection produces an increase of 1 on the scale. That means that an earthquake of magnitude 8 produces a deflection ten times that of an earthquake of magnitude 7.

In terms of the energy of an earthquake, an increase of 0.2 on the Richter scale represents a doubling of energy released. So an increase of 1 unit on the Richter scale represents an increase of energy release by a factor of $2^5 = 32$.

The largest earthquake ever recorded took place in Chile in 1960, and measured 9.5. Though theoretically an earthquake of any magnitude is possible, in practice the amount of stress that can be built up in the earth's crust before being released in an earthquake is rarely so large. $\square$

MATHS REVERSAL

This little problem is best tackled with five Scrabble tiles.

M A T H S

Arrange them in the 2 by 3 grid, with the word MATHS spelt out by the letters, going anti-clockwise. The top centre square of the grid is empty.

M		S
A	T	H

Now move the letters around in the grid, one at a time, each time moving a letter into a vacant square, with the aim of arriving at the configuration below, with MATHS spelt clockwise this time.

S		M
H	T	A

The aim is to achieve the inversion with as few moves as possible.

Send your sequence of moves to the Editor. Just list the letters in the order that they are moved. The best three entries (the shortest) will win R50 Prize Vouchers. □

WORLD CUP PREDICTION

If Swiss mathematician Roger Kaufmann's calculations are anything to go by, it looks like Spain and Brazil are most likely to meet each other at the finals of the World Cup. The mathematician calculates that the Brazilians have 15.4 percent chance of becoming world champions, slightly edging out the Spanish, who have 15.2 percent chance of winning. Kaufmann calculates this using his "dynamic sport analysis" system with data such as the ranking of the team, the number of goals scored and upcoming matches. North Korea stands the smallest chance of lifting the cup, with just 0.1 percent. New Zealand, South Korea, Honduras and Japan all have less than 1 percent chance of winning.

(*Business Report*, 2 June 2010) □

ON TRISECTING AN ANGLE

There is a proof, or so I'm told
By scientists respected,
That certain angles stubbornly
Refuse to be trisected
With compass, straight edge, pen and ink, -
And that's the only ticket,
For Euclid says that other tools
Simply are not cricket.
Yet every year some hardy souls
Will publish refutations
Of scientific treatises
And leaned dissertations.
They send out methods by the score
And hope that by insistence
They will succeed in lowering
The scientists' resistance.
Some hint that mathematicians
(Ugly insinuation)
Claim no solution's possible
Out of sheer desperation.
You think you've found a method, eh?
Which cannot be refuted?
I tell you, sir, I know it's right!
Aha, but is it Euclid?

(Barbara Steinberg, *American Mathematical Monthly*, **51**(7), 1944.) □

PRIME SEQUENCE

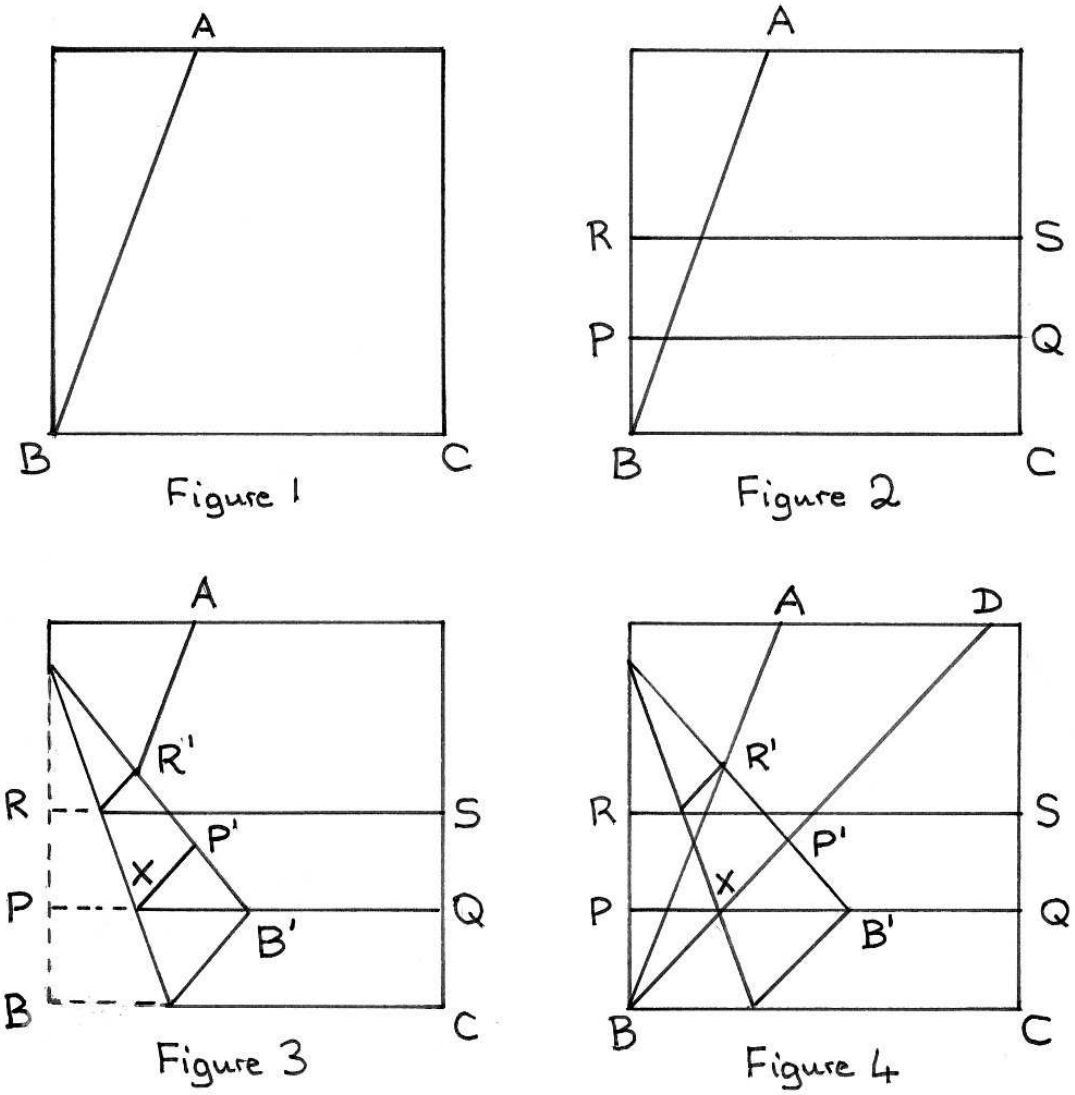
Consider the following sequence of prime numbers:

$$2, 3, 5, 7, 11, 17, 23, 37, 59, 89, 147, \ldots$$

What rule has been used to generate the sequence? A Prize Voucher worth R50 will be awarded for the first correct entry drawn on September 1, giving the rule and the next three terms of the sequence. □

FOLDING A TRISECTION

It is not possible to trisect an arbitrary angle with ruler and compasses. It is, however, possible to produce an accurate trisection of any angle, simply by folding.



Start with a rectangular piece of paper on which the angle ABC is drawn (figure 1).

Fold the paper twice to produce two folds PQ and RS, parallel to BC, with $BP = PR$, and unfold (figure 2).

Now fold the corner of the paper over so that B touches PQ at B' and R touches AB at R'. Denote the new position of P by P' (figure 3).

Without unfolding, fold along the line $P'X$, and unfold (figure 4). Then BX trisects $\angle ABC$. $\square$

THE MONKEY AND THE COCONUTS

The problem of the monkey and the coconuts has been around since 1926, when it first appeared in the American magazine *Saturday Evening Post* .

Back in 1961 it featured in the *Journal of the Science Society* of the University of Cape Town, whose Editor had come across the problem in the classic algebra textbook *A Survey of Modern Algebra* by Garrett Birkhoff and Saunders MacLane. With a little diffidence, the Editor offered his readers the problem, with his own solution.

> "On a desert island, five men and a monkey gather coconuts all day, then sleep. The first man awakens and decides to take his share. He divides the coconuts into five equal shares, with one coconut left over. He gives the extra one to the monkey, hides his share, and goes to sleep. Later, the second man awakens and takes his fifth from the remaining pile: he too finds one extra and gives it to the monkey. Each of the remaining three men does likewise in turn. Find the minimum number of coconuts originally present.
>
> The problem is most easily solved by generalizing: assume that there are m men (and hence m repetitions of the process) and a monkey, and that we begin with x coconuts and end with k. Then, working upwards from k, we can easily show that
>
> $$k = [(m-1)/m]^m x - \sum_{r=1}^{m} [(m-1)/m]^r.$$
>
> Write this as $k = (m-1)^m(x+m-1)/m^m - (m-1)$.
>
> Now unless $m = 1$ (an uninteresting case), m does not divide $(m - 1)$ and $(x + m - 1)$ cannot be zero (since x and m are both positive); therefore, since k must be integral, its minimum value (and hence the minimum value of x) must be obtained with $(x + m - 1)/m^m = 1$, i.e.,
>
> $$x = m^m - m + 1.$$
>
> Thus, in the case $m = 5$ the minimum number of coconuts is 3121."

Perhaps of more interest than this solution is the identity of its author. He was a graduate of the University of Cape Town in both Mathematics and English, and went on to write a number of highly acclaimed novels. His literary awards include two Booker Prizes (1983 and 1999) and the Nobel Prize for literature in 2003. His name? J.M. Coetzee. □

A SHARP PROBLEM REVISITED

Professor Poobhalan Pillay
University of KwaZulu-Natal

Prove that, for all natural numbers n,

$$\frac{3}{5}\cdot\frac{7}{9}\cdot\frac{11}{13}\cdots\frac{4n-1}{4n+1}<\sqrt{\frac{3}{4n+3}}.$$

This was Sharp Problem 3, in *Mathematical Digest* **157** (October 2009). Its solution, on page 18 of *Mathematical Digest* **158** (January 2010) uses induction.

Induction bypasses the process by which a result is obtained, and merely verifies it. Below is a direct proof, which also gives one an idea of where the inequality came from.

For any natural number m, it is clear that $(m-2)(m+2)<m^2$. Substitute, in succession, $m=5,9,13,\ldots$ to get

$$3\cdot 7<5^2,\quad 7\cdot 11<9^2,\quad 11\cdot 15<13^2\quad \ldots.$$

Multiply all these inequalities together:

$$3.7^2.11^2.15^2\ldots(4n-1)^2(4n+3)<(5.9.13.\ldots(4n+1))^2$$

proving that

$$\frac{3}{5}\cdot\frac{7}{9}\cdot\frac{11}{13}\cdots\frac{4n-1}{4n+1}<\sqrt{\frac{3}{4n+3}}.$$

One can begin with $(m-k)(m+k)<m^2$ for different values of k, to come up with other similar inequalities. Of particular interest is the case for $k=1$. We get

$$\frac{1.3.5.\ldots(2n-1)}{2.4.6.\ldots(2n+1)}<\sqrt{\frac{1}{2n+1}}.\quad\square$$

DOUBLE TROUBLE

"Malaria deaths had more than doubled to 4."

(Overheard by Case Rijsdijk on *SAFM News*, 8 January 2010) $\square$

SA IMO TEAM

The team of six to represent South Africa at the 2010 International Mathematical Olympiad (IMO) in Kazakhstan is

Steve Bailey (St David's Marist Inanda)
Greg Jackson (Bishops)
Ashraf Moolla (Rondebosch Boys' High School)
Dylan Nelson (Benoni High School)
Desi Nikolov (Eunice High School)
Sean Wentzel (Westerford High School)

The Team Leader is David Hatton, with Deputy Team Leader Maciej Stankiewicz (both of the University of Cape Town).

The Olympiad will take place in Astana, the capital of Kazakhstan, from July 2 to July 14. A total of 533 students from 97 countries are expected to take part. They will write two $4\frac{1}{2}$-hour papers on 7 and 8 July, and the results will be announced on 13 July.

You can follow the IMO on the website `www.imo-official.org`. □

LIGHT FROM COAL

Did you know that you need about a kilogram of coal to keep a 100 watt light bulb burning for a day?

To work out the energy consumed by a 0.1kW light bulb in 24 hours we multiply 24 by 0.1 to get 2.4kWh.

To find out how much coal a power station needs to provide that amount of energy, we need to know the thermal energy content of coal. Depending on the quality of the coal, a ton of coal has an energy content of about 6000 kilowatt-hours (kWh). That, however, is a theoretical maximum, and coal-fired power stations are at best only 40% efficient. So the electricity generated by a ton of coal is approximately $0.4 \times 6000 = 2400$kWh.

To work out how much coal is needed to keep our 100 watt light bulb glowing for a year, we divide 2.4kWh by 2400kWh/ton to get 0.001 tons, which is 1 kilogram. □

LITERARY QUIZ

1. Who composed the verse below?

   "Ah, but my calculations, people say,
   Have squared the year to human compass, Eh?
   If so, by striking from the calendar
   Unborn tomorrow and dead yesterday."

2. Which famous detective had a brother who was a mathematician and codebreaker? What was the name of the brother?

3. The arch-enemy of the same famous detective was a mathematician, renowned for his work on the binomial theorem. What was his name?

4. Which operatic character boasted

   "I'm very well acquainted too with matters mathematical,
   I understand equations both the simple and quadratical,
   About binomial theorem I'm teeming with a lot o' news —
   With many cheerful facts about the square on the hypotenuse."

5. What other branch of mathematics did he claim to be very good at?

6. Which character, in which Shakespearean play, is described as "a great arithmetician"?

7. Who, in which play by Shakespeare, is said to fight "by the book of arithmetic"?

Answers are on page 30.

LOVE LINE

Anne is in love with Bob, who is in love with Carol, who is in love with David, who is in love with Ethel, who is in love with Frank.

Anne is married and Frank is unmarried.

Is it possible to say with certainty that among these lovelorn people, a married person is in love with an unmarried person?

Think about it, and then turn to the answer on page 28. □

SHARP CALCULATOR COMPETITION

The Sharp Calculator Competition is sponsored by Sharp Electronics (S.A.) (Pty) Limited, who have donated three calculators as prizes.

These prizes are offered for the three best solutions of the problems below. Solutions may be in English or Afrikaans. Partial solutions will be judged on their merits. The Editor reserves the right to make no award if no satisfactory entries are received, and his judgement will be final. The winner of an nth prize in a previous competition will only be eligible for an mth prize if $m < n$.

Only high school students may compete. Entrants must give their full name and home address, age, school and grade. Solutions must be certified by a mathematics teacher at the school as the entrant's own unaided work.

The closing date for receipt of entries is 1 September 2010, and the results will be announced in our October 2010 edition.

SHARP

1. Solve the equations $x^2 - yz = 1$, $y^2 - zx = 2$ and $z^2 - xy = 3$.
2. Find all pairs (a, b) of natural numbers satisfying the equations
$$a^{6a} = b^b.$$
3. The squares of a rectangular grid are coloured alternatively black and white in a chess board fashion. Each square contains an integer. The sum of the numbers in each row and the sum of the numbers in each column is even. Prove that the sum of all the numbers in the black squares is even. □

MATHS AT THE DINNER TABLE

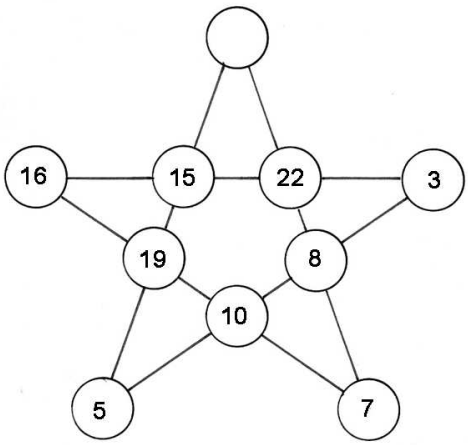
> Nobody, not even in the provinces, should ever be allowed to ask an intelligent question about pure mathematics across a dinner table.

(Oscar Wilde, in a review of *The Principles of the Art of Conversation: A Social Essay*, by J. P. Mahaffy. Macmillan and Co., 1887) □

PENTACLE PROBLEM

In the pentacle below, nine numbers have been placed in the circles, with one circle empty. In this arrangement, the sums of the numbers along the lines are not all the same.

The object is to move the numbers one at a time along the lines into a vacant circle, so that along each of the five lines the numbers have the same sum. Obviously the first move has to be into the vacant circle at the top (either 15 or 22).



How many moves are needed? The best solution will win a R50 Prize Voucher. □

TOURING THE TOWNS

There are 19 different routes from A to B, including those via C.
There are 25 different routes from B to C, including those via A.
There are 31 different routes from C to A, including those via B.

How many different circular routes are there, starting and finishing at the same town?

A Prize Voucher worth R50 will be awarded for the first correct answer opened on 1 September. □

SHARP SOLUTIONS AND RESULTS

1. *The number $A_n = 444\ldots443$ consists of n 4's followed by a 3. For which values of n is A_n divisible by 13?*

   Since $A_{n+1} = 10A_n + 13$, it follows that A_n is divisible by 13 if and only if A_{n+1} is divisible by 13.

   Since $A_1 = 43$, which is not divisible by 13, it follows that none of $A_2, A_3, \ldots$ is divisible by 13.

2. *Find all real-valued functions f of a real variable such that*

$$f(x)f(y) - f(xy) = x + y$$

   for all real numbers x and y.

   Putting $y = 0$ gives $f(x)f(0) - f(0) = x$ (*), which shows that all we need to work out is the value of $f(0)$.

   Putting $x = y = 0$ gives $(f(0))^2 - f(0) = 0$, from which it follows that $f(0) = 0$ or $f(0) = 1$.

   However, putting $x = 1$ and $y = 0$ gives $f(1)f(0) - f(0) = 1$, so $f(0) \neq 0$. So $f(0) = 1$.

   So from equation (*) we see that $f(x) = x + 1$ for all x.

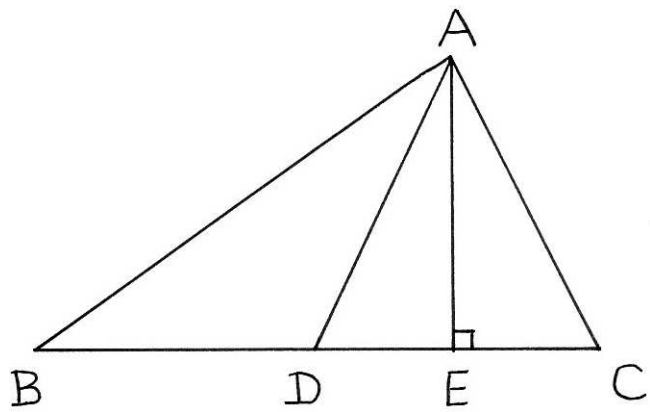
   We finally check that this function does satisfy the equation:

$$(x+1)(y+1) - (xy+1) = xy + x + y + 1 - xy - 1 = x + y,$$

   as required.

3. *In triangle ABC, D is the midpoint of BC and AD is joined. Prove that*

$$\frac{2}{\tan ADC} = \frac{1}{\tan B} - \frac{1}{\tan C}.$$



Drop a perpendicular AE from A on to the line through B and C. Assume E lies to the right of D.

Then $\dfrac{1}{\tan B} - \dfrac{1}{\tan C} = \dfrac{BE}{AE} - \dfrac{CE}{AE} = \dfrac{(BD+DE)-(DC-DE)}{AE}$

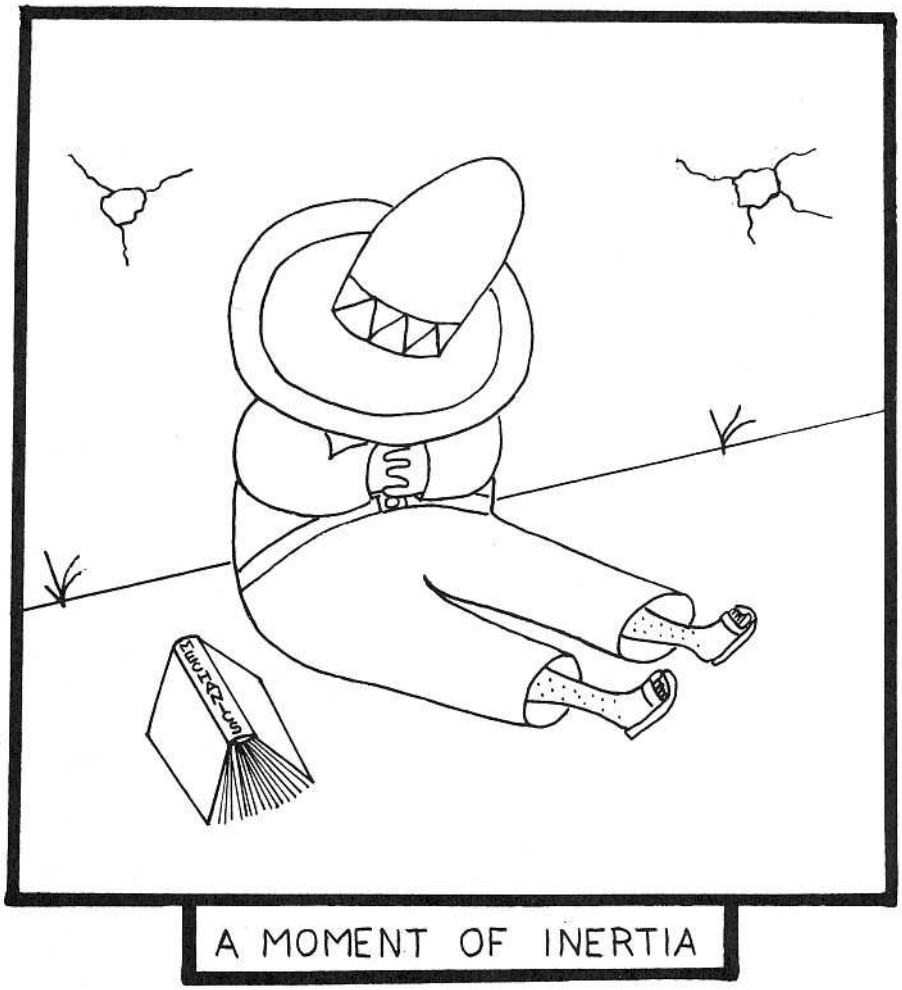
$= \dfrac{2DE}{AE}$ (since $BD = DC$)

$= \dfrac{2}{\tan ADC}$.

If E lies to the left of D, the same argument applies, with minor sign changes.

Problem 1 appeared in the 2007 Dutch Mathematical Olympiad, Round 2, and problem 2 appeared in the 2008 Lithuanian Team Contest.

There were no prizeworthy entries. □



THE 2010 UCT MATHEMATICS COMPETITION

The annual University of Cape Town Mathematics Competition took place on the UCT campus on 29 April this year, attracting over 6500 participants from Western Cape high schools. Each school could enter up to five individuals and five pairs, in each grade (8 to 12).

The question papers were set by a team of local teachers and staff of the UCT Department of Mathematics and Applied Mathematics. Each paper consisted of 30 questions, ranging from rather easy to quite difficult.

Gold Awards were awarded to the top ten individuals and top three pairs in each grade.

Grade 8: Individuals

1	Jonathan Alp	Parklands College
2	Warren Black	Bishops
3	Emma Belcher	Springfield Convent
4	Murray McKechnie	Bishops
5	Robin Visser	St George's Grammar School
5	Adam Herman	Rondebosch Boys' High School
7	Nicolene Greeff	Herschel High School
8	Philip van Biljon	Bishops
9	Daniel Mesham	Bishops
10	Michelle van der Merwe	Herschel High School

Grade 8: Pairs

1	Song Min Lee / Simon de Wet	S A College High School
2	Thomas Vidal / Ben Kilpin	Bishops
2	Megan Colman / Somara Sabharwal	St Cyprian's

Grade 9: Individuals

1	Guy Paterson-Jones	Bishops
2	Musa Omar	Livingstone High School
3	James Nevin	Bergvliet High School
4	Samuel Wolski	Bishops
4	Lauren Denny	Rustenburg Girls' High School
6	Gerben Draaijer	Paul Roos Gimnasium
7	Ruth Amoore	Herschel High School

8	Nicholas Cotchobos	Bishops
8	Sylvia van Belle	Fish Hoek High School
8	Megan Higgo	Parel Vallei High School

Grade 9: Pairs

1	Paul Ferrandi / Adam Smith	Bishops
1	Raphael Lee / Martin Page	Fairmont High School
1	Chrisjan Wüst / Nicol Brand	Stellenberg Hoërskool

Grade 10: Individuals

1	Mickey Chew	Elkanah House
2	Robert Spencer	Westerford High School
3	Ihsaan Bassier	Rondebosch Boys' High School
3	Matthew Coulson	Reddam House Constantia
5	Nicholas Cheng	Fairmont High School
6	Ye Jin Hur	Rustenburg Girls' High School
7	Simon Kang	Milnerton High School
7	Allan Moon	St George's Grammar School
9	Uzair Ebrahim	Worcester Gymnasium
9	Aidan Horn	Bishops

Grade 10: Pairs

1	Justin Pierre du Plessis / John Reid	Westerford High School
2	Cian Cullinan / Jason Welz	Rondebosch Boys' High School
3	Michael Brooke / Yaaseen Gallant	Rondebosch Boys' High School

Grade 11: Individuals

1	Sean Wentzel	Westerford High School
2	Ashraf Moolla	Rondebosch Boys' High School
3	Charl du Plessis	Stellenberg High School
4	James Cheon	S A College High School
4	Nina Nicholas	Jan van Riebeeck Hoërskool
6	Choong-Gi Min	Curro Private School
7	Algernon de Vries	Bellville South Secondary School
8	Kira Dsterwald	Springfield Convent
9	Hendrik Wessels	Jan van Riebeeck Hoërskool
10	Chloe Blyth	Herschel High School
10	Tiana du Preez	Stellenbosch Hoërskool

Grade 11: Pairs

1	Gavin le Roux / Matthew Swart	Tygerberg Hoërskool
2	Anja Otto / Emma Stanvliet	Bloemhof Hoër Meisieskool
3	Johann Burger / Jodi Moore	Edgemead High School

Grade 12: Individuals

1	Greg Jackson	Bishops
2	Francois van Niekerk	Bellville Hoërskool
3	Sunghun Jung	Settlers High School
3	Rowan Nicholls	Bishops
5	Yu Sun	Milnerton High School
6	Minkyu Lee	St George's Grammar School
7	Till Bauer	Parel Vallei High School
7	Kieren Davies	International School of Hout Bay
7	Lance Li	S A College High School
10	Ashleigh Avenant	Westerford High School
10	Hankyu Kim	Parel Vallei High School

Grade 12: Pairs

1	Damian Jacobs / Andrew Roginsky	Wynberg Boys' High School
2	Jonathan Antonie / Pepukai Dombo	Parel Vallei High School
2	Reino Barkema / Charles Herbert	D F Malan Hoërskool

On overall performance, the top ten schools were

1. Bishops
2. Rondebosch Boys' High School
3. Herschel High School
4. Parel Vallei High School
5. Rustenburg Girls' High School
6. Westerford High School
7. Reddam College Constantia
8. Wynberg Boys' High School
9. S A College High School
10. Paarl Gimnasium

A free set of competition papers, with answers, may be obtained by sending a stamped self-addressed B5 envelope (220 mm by 160 mm) to

The Director, UCT Mathematics Competition
Department of Mathematics and Applied Mathematics
University of Cape Town
7700 RONDEBOSCH

Books of question papers of previous years, with full solutions, statistics and lists of award winners, are also available — see the order form at the back of the magazine. □

PAMO IN YAMOUSSOUKRO

The Twentieth Pan African Mathematical Olympiad (PAMO) took place in Yamoussoukro, the capital of the Ivory Coast, from 20 to 30 May.

Teams of four from nine countries took part: Benin, Burkina Faso, the Gambia, Ivory Coast, Kenya, Mali, Nigeria, South Africa and Tunisia. South Africa was represented by Kira Düsterwald (Springfield Convent), Thala Msutu (Maritzburg College), Dalian Sunder and Seth Durais (both of Star College, Durban). The Team Leader was Dirk Basson (University of Stellenbosch), with Deputy Leader Koos van Zyl (University of Pretoria).

The teams wrote two $4\frac{1}{2}$-hour papers, each of three questions (see page 24). Each problem was worth just seven points. The papers were set by the Team Leaders and an international panel appointed by the African Mathematical Union. The PAMO structure is very similar to that of the International Mathematical Olympiad. Half the contestants receive medals: Bronze, Silver and Gold, in the ratio 3 : 2 : 1. Those who do not win medals but still solve at least once question completely, win certificates of Honourable Mention.

While the papers were being marked, the teams enjoyed a visit to the spectacular Basilica of Our Lady of Peace, the largest church in the world. It was built by the first President of the Ivory Coast, Felix Houphouët Boigny and opened by Pope John Paul II in 1990.

The Olympiad ended with a gala dinner in Abidjan, the commerical capital and main port of the Ivory Coast. The ceremony was attended by top government officials and the President of the African Mathematical Union, Professor Saliou Toure. When the final scores were announced, the overall winner was the Ivory Coast team, with 127 points out of a maximum of 168, followed by South Africa (113) and Tunisia (97).

There was a tie for individual first place, with 39 points out of a possible 42, between Kouakou Konan (Ivory Coast) and Kira Düsterwald (South Africa). Dalian Sunder and Seth Durais were ranked 8th and 9th respectively, and won Silver Medals. Thala Msutu was ranked 19th and won an Honourable Mention.

Next year's PAMO is scheduled to take place in Abuja, Nigeria. □

PAMO PROBLEMS

Day 1

1. (a) Show that it is possible to pair off the numbers $1, 2, 3, \ldots, 10$ so that the sums of each of the five pairs are five different prime numbers.
(b) Is it possible to pair off the numbers $1, 2, 3, \ldots, 20$ so that the sums of each of the ten pairs are ten different prime numbers?

2. How many ways are there to line up 19 girls (all of different heights) in a row one behind the other so that no girl has a shorter girl both in front of and behind her?

3. In an acute-angled triangle ABC, CF is an altitude, with F on AB, and BM is a median, with M on CA. Given that $BM = CF$ and $\angle MBC = \angle FCA$, prove that triangle ABC is equilateral.

Day 2

4. Seven distinct points are marked on a circle of circumference c. Three of the points form an equilateral triangle and the other four form a square. Prove that at least one of the seven arcs into which the seven points divide the circle has length less than or equal to $c/24$.

5. A sequence $a_0, a_1, a_2, \ldots, a_n, \ldots$ of positive integers is constructed as follows:

- If the last digit of a_n is less than or equal to 5 then this digit is deleted and a_{n+1} is the number consisting of the remaining digits.
(If a_{n+1} contains no digits the process stops.)
- Otherwise $a_{n+1} = 9 \times a_n$.

Can one choose a_0 so that an infinite sequence is obtained?

6. Does there exist a function $f : \mathbf{Z} \to \mathbf{Z}$ such that $f(x + f(y)) = f(x) - y$ for all integers x and y? $\square$

SQUARE CYCLES

Take any number and add up the squares of its digits to get a new number. Repeat this process with the new number, and repeat again. What happens in the long run?

The number 1 is hardly interesting: you just get $1^2 = 1$.

Starting with 2, you get the sequence

$$2 \to 4 \to 16 \to 37 \to 58 \to 89 \to 145 \to 42 \to 20 \to 4$$

after which the sequence cycles through the same numbers.

Starting with 3, you get the sequence

$$3 \to 9 \to 81 \to 65 \to 61 \to 37$$

which appeared in the previous sequence, and again the sequence cycles.

The sequence starting with 4 also cycles, since 4 occurred in the cycle for 2, and the sequence starting with 5 has first step $5 \to 25$, which occurred in the cycle for 3, so that sequence also cycles.

Starting with 6 we have

$$6 \to 36 \to 73 \to 58$$

which is in the cycle for 3, so again we have a cycling sequence.

The cycle for 7 is

$$7 \to 49 \to 97 \to 130 \to 10 \to 1$$

which then just repeats.

What happens for other numbers? Here's something worth investigating, either with pencil, paper and calculator, or with a short computer program. Send your discoveries to the Editor. A Prize Voucher worth R50 will be awarded for the best discussion. □

FERMATA

fermata: a gap of unspecified length in music

SEATING PROBLEM

Mr and Mrs Alberts held a dinner party, and invited four other couples: the Barnes, Charles, Davids and Edwards. They sat down to dinner at a circular table.

No husband and wife sat next to each other. Mr Alberts sat next but one to Mrs Charles. Mrs Edwards sat next to her sister. Mr Davids sat between two ladies, as did Mr Barnes. Mr Edwards sat next but one to Mr Charles, who sat on the left of his father-in-law. Three of the ladies each sat between two men. Mrs Davids sat next but two to her husband, and Mrs Edwards sat next but two to Mrs Barnes.

Can you work out the seating plan? The first correct solution opened on 1 September will win a R50 Prize Voucher. □

COMPUTER LIMERICK

A computer, to print out a fact,
Will divide, multiply, and subtract.
But this output could be
No more than debris
If the input is less than of exact.

□

QUOTE

The man who discredits the supreme certainty of mathematics is feeding on confusion, and can never silence the contradictions of sophistical sciences, which lead to eternal quackery. ... For no human inquiry can be called science unless it pursues its path through mathematical exposition and demonstration.

(Leonard da Vinci, in *Trattato della Pittura*) □

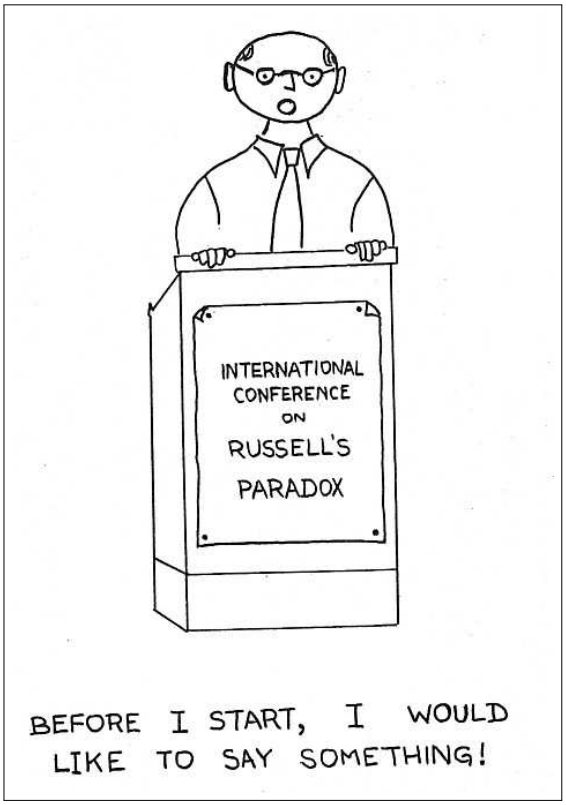
SORT THEM OUT

Here are four facts about four men, their professions and where they live:

1. Bob is younger than the doctor, who lives directly east of John.
2. The doctor lives directly south of Mr Price, who lives exactly 5 km from Harry, who lives exactly 2 km from the soldier.
3. The plumber is older than Mr Conway, and Charlie is older than the accountant.
4. Harry is older than Mr Jones, who lives exactly 3 km from John, who lives directly north of Mr Barnes.

Work out the first names, surnames and professions of the men, and where they live relative to each other.

A R50 Prize Voucher will be awarded for the first correct solution opened on 1 September. □



WISKUNDE

The word "mathematics" comes from ancient Greek, and means "subjects of instruction". It has been taken over into many different languages in more or less that form: *mathematik* in German, *mathematiques* in French, *matematica* in Italian.

In Afrikaans, the subject is known as *wiskunde*. How did that come about?

The story goes back four hundred years to the Flemish mathematician and engineer Simon Stevin (1548-1620). He was active in many areas of science and engineering, both theoretical and practical. Today he is usually remembered for introducing and promoting the use of decimal fractions, in his little book *De Thiende*.

In those days works of science were usually written in Latin, which was the language of scientific discourse. Stevin defied convention by writing in his native tongue, Dutch. He coined a number of Dutch words for scientific concepts. Today, Dutch is one of the few Western European languages that has a lot of mathematical terms that do not stem from Latin, the most notable of which is *wiskunde*, "the art of that which is certain".

Although in the early days of the development of Afrikaans, the word *mathesis* was sometimes used, Stevin's *wiskunde* soon became the standard term. □

LOVE LINE SOLUTION

Label the six (Anne, Bob, Carol, David, Ethel Frank) with the letter M or U, depending on whether they are married or unmarried respectively. We have been told that Ann is married, so is labelled M, and Frank is unmarried and is labelled U.

The list of six labels therefore begins with M and ends with U. Starting with Ann at M, and moving down the line, at some stage there must be a M followed by a U, since Frank at the end is unmarried. In other words, somewhere down the line there is a married person in love with an unmarried person. □

TEN TIPS ON PROBLEM-SOLVING

1. Never *ever* read through a problem carefully. That only takes time and you may get depressed.

2. Mark all the numbers that appear in the problem with a yellow highlighter. Look out for numbers written in words.

3. Look out for key words like *of* (which probably means *multiply*) and highlight them in green.

4. If there are only two numbers in the problem and they are roughly the same size, subtract them.

5. If there are two numbers with one much smaller than the other, divide them if one goes into the other exactly. Otherwise multiply them.

6. If there are three or more numbers, of roughly the same size, add them.

7. If the problem seems to need a formula, use one that has exactly the right number of letters to use all the numbers given.

8. When you have an answer, write it down boldly on a new line, highlight it in blue and put a big confident red tick next to it. Hope that the teacher will see the tick and think she has already marked the answer correct.

9. Never check your working. You may get a different answer, which would be confusing, if not downright depressing.

10. Never spend too much time solving problems. They only wear out your brain. □

ERRATA

Three answers on page 31 of *Mathematical Digest* **158** (April 2010) were incorrect.
The correct answer to *Superbrain* number 3 is 18.
The correct answers to items 6 and 21 of the *Varsity Readiness Test* are 1 and 4 respectively. Thanks to readers David Allison and Khuliso Musweswe for pointing these out. □

THE SEVEN LIBERAL ARTS

In medieval times, the realm of higher learning was classified into two groups of subjects, the *Trivium* and the *Quadrivium*.

The Trivium (three roads) consisted of Grammar (specifically, Latin grammar), Rhetoric (the art of speaking) and Logic (the art of argument). They formed the basis for a level of intellectual development that today we would regard as a Bachelor's degree.

After the three roads of the Trivium came the four roads of the Quadrivium, which consisted of Arithmetic (pure numbers), Music (applied numbers, or number in time), Geometry (matter at rest) and Astronomy (matter in motion, or number in space and time). The Quadrivium was the equivalent of what today we would call a Master's degree.

The Trivium and Quadrivium together formed the Seven Liberal Arts, which all came under the heading of Philosophy, a Greek word which meant "the love of knowledge".

Our modern use of the words "trivia" and "trivial" come from the word "trivium", but not necessarily in the negative sense that would regard Grammar, Rhetoric and Logic as of slight importance. A more likely interpretation is that, at a point where three roads meet, weary travellers in the old days would meet, stop and gossip about minor things. □

LITERARY QUIZ ANSWERS

1. The quatrain is from *The Rubaiyyat of Omar Khayyam*, translated by Edward Fitzgerald.
2. Sherlock Holmes —- his brother's name was Mycroft.
3. Professor Moriarty
4. Major-General Stanley, in the Gilbert and Sullivan operetta *The Pirates of Penzance*.
5. He also claimed to be "very good at integral and differential calculus".
6. Cassio is described by Iago in this way, in *Othello*.
7. In *Romeo and Juliet*, Mercutio is comparing Tybalt's fighting style to the strict rules of mathematics. □

MATHS AT THE BACK

Sixty years ago, every school exercise book had the table below on its back cover.

MEASURES OF TIME

60 seconds	=	1 minute
60 minutes	=	1 hour
24 hours	=	1 day and night
7 days	=	1 week
365 days	=	1 year

MEASURES OF WEIGHTS

16 ounces	=	1 pound (lb.)
100 lbs.	=	1 Cape cwt.
112 lbs.	=	1 Eng. cwt.
2000 lbs.	=	1 Cape ton
2240 lbs.	=	1 Eng. ton

Troy

4 grains	=	1 carat
24 grains	=	1 pennyweight
20 dwts.	=	1 ounce
12 ounces	=	1 pound

CUBIC MEASURE

1728 cubic ins.	=	1 cubic foot
27 cubic feet	=	1 cubic yard

LENGTH OF THE CALENDAR MONTH

Thirty days hath September,
April, June and November;
All the rest have thirty-one,
Excepting February alone,
Which has but twenty-eight days clear,
And twenty-nine in each leap year.

AFSTAND EN OPPERVLAKTE MATE

12 duim	=	1 voet
3 voet	=	1 jaart
1760 jaart	=	1 myl
12 Kaapse dm.	=	1 Kaapse vt.
12 Kaapse vt.	=	1 Kaapse rd.
144 V.K. dm.	=	1 V.K. voet
9 V.K. voet	=	1 V.K. jaart
144 Kaapse V.K. voet	=	1 Kaapse V.K. roede
600 Kaapse V.K. roede	=	1 morgen
1 morgen	=	$2\frac{1}{5}$ Eng. Ak.
1000 Kaapse vt.	=	1033 Eng. vt.
1 Engelse myl	=	426 Kaapse rd
1 V.K. myl	=	$302\frac{1}{3}$ morgen

MEASURES OF LAND AND SURFACE

12 inches	=	1 foot
3 feet	=	1 yard
1760 yards	=	1 mile
12 Cape inches	=	1 Cape foot
12 Cape feet	=	1 Cape rood
144 square ins.	=	1 square foot
9 square feet	=	1 square yard
144 Cape sq. ft.	=	1 Cape sq. rd.
600 Cape sq. rds.	=	1 morgen
1 morgen	=	$2\frac{1}{5}$ Eng. acres
1000 Cape feet	=	1033 Eng. ft.
1 Eng. mile	=	426 Cape rds.
1 square mile	=	$302\frac{1}{3}$ morgen

TYD BEREKENING

60 sekonde	=	1 minuut
60 minute	=	1 uur
24 uur	=	1 dag en nag
7 dae	=	1 week
365 dae	=	1 jaar

MATE VAN GEWIG

16 onse	=	1 pond
100 pond	=	1 Kaapse cwt.
112 pond	=	1 Engelse cwt.
2000 pond	=	1 Kaapse ton
2240 pond	=	1 Engelse ton

Troy

4 grein	=	1 karaat
24 grein	=	1 pennyweight
20 dwts.	=	1 ons
12 onse	=	1 pond

KUBIEKE MATE

1728 kubieke dm.	=	1 kubieke vt
27 kubieke vt.	=	1 kubiekejt.

KALENDER MAANDE VAN DIE JAAR

September, April, Junie en November het dertig dae terwyl die orige maande almal een - en - dertig het, behalwe Februarie, wat ag-en-twintig dae het en nege-en - twintig dae elke skrikkeljaar.

MULTIPLICATION TABLE
VERMENIGVULDIGINGSTAFEL

2 maal times is	3 maal times is	4 maal times is	5 maal times is	6 maal times is	7 maal times is	8 maal times is	9 maal times is	10 maal times is	11 maal times is	12 maal times is
1 are 2	1 are 3	1 are 4	1 are 5	1 are 6	1 are 7	1 are 8	1 are 9	1 are 10	1 are 11	1 are 12
2—4	2—6	2—8	2—10	2—12	2—14	2—16	2—18	2—20	2—22	2—24
3—6	3—9	3—12	3—15	3—18	3—21	3—24	3—27	3—30	3—33	3—36
4—8	4—12	4—16	4—20	4—24	4—28	4—32	4—36	4—40	4—44	4—48
5—10	5—15	5—20	5—25	5—30	5—35	5—40	5—45	5—50	5—55	5—60
6—12	6—18	6—24	6—30	6—36	6—42	6—48	6—54	6—60	6—66	6—72
7—14	7—21	7—28	7—35	7—42	7—49	7—56	7—63	7—70	7—77	7—84
8—16	8—24	8—32	8—40	8—48	8—56	8—64	8—72	8—80	8—88	8—96
9—18	9—27	9—36	9—45	9—54	9—63	9—72	9—81	9—90	9—99	9—108
10—20	10—30	10—40	10—50	10—60	10—70	10—80	10—90	10—100	10—110	10—120
11—22	11—33	11—44	11—55	11—66	11—77	11—88	11—99	11—110	11—121	11—132
12—24	12—36	12—48	12—60	12—72	12—84	12—96	12—108	12—120	12—132	12—144
13—26	13—39	13—52	13—65	13—78	13—91	13—104	13—117	13—130	13—143	13—156
14—28	14—42	14—56	14—70	14—84	14—98	14—112	14—126	14—140	14—154	14—168

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IMO in Kazakhstan

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The eye is the first circle; the horizon which it forms is the second; and throughout nature this primary figure is repeated without end. It is the highest emblem in the cipher of the world.

(Ralph Waldo Emerson, in his essay *Circles*)

Department of Mathematics and Applied Mathematics
University of Cape Town

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Cover: The cover features the Logo of the 51st International Mathematical Olympiad. See page 22 for a report.)

ARCHIMEDES AND THE SOLAR CANNON

Did Archimedes design a solar-powered cannon to repel the Roman invasion of his home city of Syracuse? That is what Cesaro Rossi, a mechanical engineer of the Federico II University of Naples would have us believe.

It all started with the story that Archimedes used huge mirrors to focus the sun's rays on the Roman ships and incinerate them. That story dates from medieval times, and is widely discredited.

The television programme *Mythbusters* attempted to repeat the feat, using authentic Archimedean technology, but was unsuccessful. It was very difficult to focus the sun's rays on a moving target, and even then the small fire set off was easily put out.

Professor Rossi believes that Archimedes attacked the Roman ships with a solar-powered steam cannon instead. Solar reflectors were used to boil water in a sealed chamber at the base of the cannon. When the pressure was high enough, a valve was opened and the superheated steam fired a projectile, a clay ball filled with sulphur and tar designed to burst into flames on impact. The cannon, Professor Rossi calculates, would have had a muzzle velocity of 60 metres per second, and could have fired a 20 cm cannonball a distance of 150 metres.

Other engineers and historians are sceptical. A clay cannonball would break up when fired, it is believed. It is more likely that Archimedes used huge catapults to hurl balls of burning pitch to set fire to the Roman ships. □

GOD'S LANGUAGE

"Do you know calculus? You had better learn it. It's the language God talks."

(Richard Feynman, in conversation with Herman Wouk) □

THE MATHEMATICAL PRESIDENT

Irish history might have taken a different course had a young mathematics teacher Eamon 'Dev' de Valera been appointed professor of mathematics in 1913. That is the thrust of a recent PhD thesis by Cáit Ní Shúilleabháin (University College, Cork) on the mathematical life of the man who later became Prime Minister and President of Ireland.

Eamon de Valera (1882-1975) was born in New York of a Cuban-Spanish father and an Irish mother. At the age of two, his father died and his mother sent him back to Ireland. At school he was a top mathematics student (and a talented rugby player). He got a degree in mathematics at the University College, Cork and became a mathematics teacher. He considered a university career, but after his unsuccessful application for the position at the University of Cork he went into politics.

It was a particularly troubled time in Irish history. De Valera was one of the leaders of the unsuccessful 1916 rebellion against British rule. He and his fellow rebels were captured and sentenced to death by military tribunal, but de Valera's life was saved because of his American birth; Britain was anxious to secure America's support in the Great War and did not want to antagonize Irish Americans. Dev spent one year in prison, and thereafter led the Irish fight for independence.

Mathematics came in useful in his political career. In debates on political structures in the Dail (the Irish parliament) de Valera used Venn diagrams to explain his ideas. Set theory was still a relatively new branch of mathematics, not taught in schools, but it seems that de Valera's logical explanations were sufficiently convincing to persuade the Dail to support his proposals.

As a mathematics student, the young de Valera had a special interest in quaternions, invented by another Irish mathematician, William Rowan Hamilton (1805–1865). Quaternions are a four-dimensional extension of complex numbers which Hamilton believed would revolutionize physics.

In *Inside Europe*, a masterful piece of political journalism of the 1930s, John Gunther devoted a chapter to de Valera, and mentions his mathematical background. According to Gunther,

> "The intellectual pleasure that matters most to him is mathematics. One day going to Rome he asked his secretary what he thought of the quaternary theorem. 'Nothing', the secretary replied, who knew only elementary mathematics. It was a boiling hot day, and the rest of the staff dozed, but Dev spent twelve solid hours teaching the secretary the quaternary theorem. The secretary said that Dev's twelve-hour lecture was the most brilliant intellectual performance he had ever known. When in jail in 1918, incidentally, de Valera spent all his time mastering the Einstein theory."

Although neglected by physicists for many years, quaternions and their generalization octonions (also known as Cayley numbers) are today finding applications in string theory, while computer scientists find quaternions useful in computer graphics, robotics, and signal processing. De Valera would have been pleased. □

THE SYDHARB

The Australian Water Association uses the *sydharb* as a unit for measuring large volumes of water.

A sydharb is approximately 500 gigalitres (500 billion cubic metres), the estimated volume of water in Sydney harbour.

Australian water consumption is about 50 sydharbs a year. □

RATIONAL NUMBERS

Numbers which are ratios of whole numbers, such as $\frac{2}{3}, \frac{7}{5}, \frac{-3}{2}, \frac{4}{-2}$ etc are said to be *rational*. Whole numbers are rational numbers with denominator 1 when reduced to lowest terms.

Every rational number can be expressed as a decimal fraction. Starting with, for example $\frac{3}{7}$, the division process produces the successive digits in the decimal form of the number.

$$\begin{array}{l} 7)\underline{3.^{3}0^{2}0^{6}0^{4}0^{5}0^{1}0^{3}0^{2}0^{6}0^{4}0^{5}0^{1}0^{3}\cdots} \\ \quad 0.\ 4\ 2\ 8\ 5\ 7\ 1\ 4\ 2\ 8\ 5\ 7\ 1\ \cdots \end{array}$$

At each stage in the division by 7 there is a remainder, either 1, 2, 3, 4, 5 or 6. Since there only 6 remainders, after at most six divisions there has to be a repetition in the sequence of remainders, and when that happens the same cycle is repeated.

In the calculation shown, the remainders are 3, 2, 6, 4, 5, 1 and then 3 reappears and the cycle is repeated: we say that the decimal recurs. The *period* of the recurring decimal is the length of the sequence of digits that recur.

Thus $\frac{3}{7} = 0.428571428571428571\ldots$, which can be written succinctly as $0.\dot{4}2857\dot{1}$, where the dots indicate that the sequence of six digits 42857 is repeated ad infinitum. This is a *recurring decimal of period 6.*

Some other examples of rational numbers written out as decimals are

$\frac{1}{3} = 0.333333\ldots = 0.\dot{3}$ (period 1)

$\frac{1}{11} = 0.09090909\ldots = 0.\dot{0}\dot{9}$ (period 2)

$\frac{17}{101} = 0.168316831683\ldots = 0.\dot{1}68\dot{3}$ (period 4)

When the denominator of a rational number contains only powers of 2 and 5, the decimal representation of the number terminates with a string of zeros.

So every rational number has a recurring decimal expansion.

The reverse is also true: every recurring decimal is given by a rational number.

Suppose for example that we have the recurring decimal $0.270270\ldots = 0.\dot{2}7\dot{0}$. Let us denote this number by X.

Then $X = 0.270270270270\ldots.$ (1)
Multiply both sides of the equation by 1000:
$1000X = 270.270.270\ldots.$ (2)

Subtract equation (1) from equation (2):

$$999X = 270.$$

It follows that $X = \dfrac{270}{999} = \dfrac{10}{37}$.

The method works with any recurring decimal, for example $0.\dot{4}6153\dot{8}$.

Set $X = 0.461538461538461538\ldots.$

Multiply by 1 000 000 ($= 10^6$: the decimal recurs with period 6).

$1\ 000\ 000X = 461538.461538461538461538\ldots$

and subtract the two equations to get

$$999\ 999X = 461538,$$

from which it follows that

$$X = \frac{461538}{999999} = \frac{6}{13}.$$

RAMANUJAN AND BEYOND

When the English mathematician G.H. Hardy visited his friend and collaborator Srinivasa Ramanujan in hospital, he must have been a little short of conversation, for he commented that he had travelled in a taxi bearing the number 1729, which he said was "a not very interesting number". The Indian mathematical genius corrected Hardy, by observing that 1729 is the smallest number that can be expressed as the sum of two cubes in two different ways:

$$1729 = 1^3 + 12^3 = 9^3 + 10^3.$$

The smallest number that can be expressed as the sum of *three* cubes in two different ways is 251:

$$251 = 1^3 + 5^3 + 5^3 = 2^3 + 3^3 + 6^3$$

□

PICK A PRIZE PROBLEM

Suppose you take part in a TV game show, and win a prize. The compère shows you a row of four boxes, each of which contains a prize. You have no idea what any of the prizes are. You may open any box and decide whether you like the prize. If you want it, you take it. If you don't like the prize, or think there may be a better prize in the next box, you may reject the prize and open the next box.

If you don't like what is in the second box, you can go on to the third box, and if you still don't like what you see you can open the fourth and last box. The catch is that when you have opened a box and passed on, you have rejected that prize and at no stage can you go back to a box you opened earlier.

What should you do to get the best possible prize? It seems that you may as well simply open one of the boxes at random and accept the prize in it. You would win the top prize and bottom prize with equal probability.

There is, however, a better strategy that increases the chance that you win the top prize, and conversely reduces the chance that you win the bottom prize.

With the boxes in a row in front of you, you open the first box and inspect the prize. Whatever it is, you reject it. Then you open the next box. If the prize it contains is better than the prize in the first box, you accept it. Otherwise you open the third box. If it is better than the prize in the first box, accept it. Otherwise, go to the last box, open it and take the prize.

How will this method work in practice? If we number the prizes 1, 2, 3, 4 (in decreasing value), there are 24 different ways in which they can be arranged in the row of four boxes:

1234 1243 1324 1342 1423 1432
2134 2143 2314 2341 2413 2431
3124 3142 3214 3241 3412 3421
4123 4132 4213 4231 4312 4321

Look at each of the 24 arrangements, and mark the prize you would have won by following the method described above (reject the prize in the first box and accept the prize in any subsequent box if it is better than the prize in the first box).

123$\boxed{4}$	124$\boxed{3}$	132$\boxed{4}$	134$\boxed{2}$	142$\boxed{3}$	143$\boxed{2}$
2$\boxed{1}$34	2$\boxed{1}$43	23$\boxed{1}$4	234$\boxed{1}$	24$\boxed{1}$3	243$\boxed{1}$
3$\boxed{1}$24	3$\boxed{1}$42	3$\boxed{2}$14	3$\boxed{2}$41	34$\boxed{1}$2	34$\boxed{2}$1
4$\boxed{1}$23	4$\boxed{1}$32	4$\boxed{2}$13	4$\boxed{2}$31	4$\boxed{3}$12	4$\boxed{3}$21

The table shows that you would win First Prize 11 times, Second Prize 7 times, Third Prize 4 times and Fourth Prize twice. Not only does the method get you the top prize nearly half the time (as opposed to one-quarter of the time if you simply chose at random), but it also gets you the bottom prize only one-twelfth of the time.

What happens if the game show offers *five* boxes with prizes in them? Now there are 120 different ways in which the boxes can be arranged in order. The rule now is to open the first and second boxes, reject both prizes revealed in them, and carry on opening the boxes until you find a prize better than both the prizes in boxes 1 and 2.

You can check out that this strategy works by writing down all the possible orders of the numbers 1, 2, 3, 4, 5, and noting which prize you get in each case.

In general, when there are n boxes, you reject the first 37% of the boxes and choose the first prize that is better than the prizes rejected. The precise mathematical rule is that you must reject the first n/e of the n boxes, where e is the base of the natural logarithms (with numerical value 2.718281...). The chance that you then get the best prize is $1/e$, or 37%. The solution has become known as "the 37% rule".

The problem was devised in 1949 by Merrill M. Flood, in the form of a young man wanting to select a fiancé from a list of eligible young ladies. It became widely known through Martin Gardner's column in the February 1960 edition of *Scientific American*, subsequently reprinted in his 1966 book *New mathematical diversions from Scientific American.* □

FERMAT IN FOUR

Although it is not possible to find three positive whole numbers x, y and z satisfying the equations $x^4 + y^4 = z^4$ (a particular case of Fermat's Last Theorem), it *is* possible to find four numbers w, x, y and z satisfying $w^4 + x^4 = y^4 + z^4$:

$$133^4 + 134^4 = 158^4 + 59^4.$$

□

FIELDS MEDALS

Every four years mathematicians get together at the International Congress of Mathematicians, held in August this year in Hyderabad, India. One of the highlights of the Congress was the announcement of the award of the Fields Medals.

The Fields Medals are the mathematical equivalent of the Nobel Prizes, but with one important difference: they can only be won by mathematicians under the age of 40. The 2010 winners announced in Hyderabad are

- Ngo Bao Chao, for new algebraic-geometric applications to group theory;
- Stanislav Smirnov, for work in statistical physics;
- Elon Lindenstrauss, for work in ergodic theory and its application to number theory;
- Cedric Villani, for work on the kinetic theory of gases.

Of particular interest to Olympiad enthusiasts is the fact that three of the four Fields medallists were winners of medals in the International Mathematical Olympiad in their schooldays:

- Ngo Bao Chao won Gold Medals for Vietnam in 1988 and 1989.
- Stanislav Smirnov won Gold Medals for the USSR in 1986 and 1987.
- Elon Lindenstrauss won a Bronze Medal for Israel in 1988.

The Fields Medals were proposed in 1924 by the Secretary of the International Mathematical Union, Canadian mathematician John Charles Fields, who donated funds to establish the medals but died before the first medals could be awarded. Despite Fields' insistence that the medals should not be named after any person (particularly himself), today his name is always attached to them.

The Fields Medal are gold, and nine centimetres in diameter. On one side is the head of Archimedes in profile. The Latin inscription "Transire suum pectus mundoque potiri" is a quotation from the Roman poet Manilius which may be translated as "Rise above yourself and grasp the world".

The reverse of the Fields Medal carries the inscription "Congregati ex toto orbe mathematici ob scripta insignia tribuere" (Awarded by mathematicians from all over the world for outstanding work). Behind the inscription is a laurel wreath and Archimedes' diagram of the sphere in the cylinder.

Initially only two Fields Medals were established, but since 1966 up to four have been awarded. A monetary award of 15 000 Canadian dollars accompanies the medal. A Nobel Prize is currently worth a hundred times that amount. Nobel Prizes are often shared, Fields Medals never. □

NEWSPAPER NUMBERS

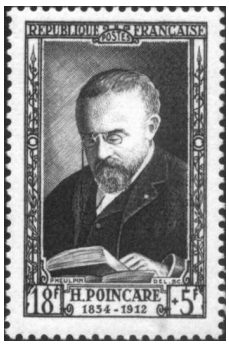
> Elvis purchased the Graceland property in 1957 for a song — about R731 237.
>
> (*The Times*, 17 August 2010)

There are a whole lot of things wrong with that sentence. First of all, it is silly to put "about" in front of a number quoted with six-digit accuracy. Was the price given down to the last cent, and the reporter decided to leave the cents out? There is then the question of where those six digits came from. Obviously the reporter converted the original price of Graceland ($102 500) by looking up the dollar-rand exchange rate of the day and then doing some deft calculator work. That in itself is an absurd thing to do: to convert a 1957 amount in dollars to rands you have to use the dollar-rand exchange rate at the time of the purchase of Graceland (March 1957), which you then find you can't do because the rand did not exist as a unit of currency in 1957 (it was introduced in 1961).

In 1957 the pound was South Africa's unit of currency and the dollar-pound exchange rate was 0.35714. So in 1957 the pound value of Graceland would have been about £36 600. Four years later South Africa changed to the rand, with two rands to the pound, so that the rand value of Graceland was R73 200.

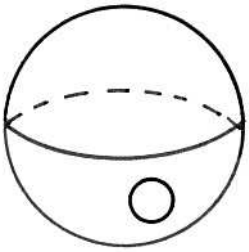
And finally, Graceland was certainly not purchased "for a song". Back in 1957, $102 500 was a huge amount of money, whether in dollars, pounds or not-yet-introduced rands. □

PERELMAN AND THE POINCARÉ PRIZE

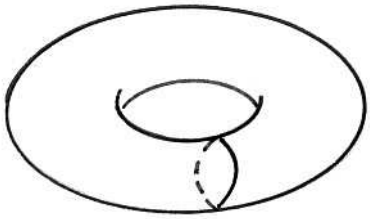


The Poincaré Conjecture was one of the great unsolved mathematical problems of the 20th century. Originally posed by the great French nathematician Henri Poincaré in 1904, it hypothesizes that any three-dimensional body without holes is essentially a sphere.

Another setting of the problem is to note that any simple loop on the surface of a sphere can be contracted to a point. The same is not true for a loop on the surface of a torus, or doughnut. The sphere and the torus are topologically distinct. The Poincaré conjecture asserts that any closed three-dimensional manifold on which any loop can be contracted to a point is topologically a three-dimensional sphere.



A contractible circle on a sphere



A non-contractible circle on a torus

In 2003 a reclusive Russian mathematician Grigory Perelman announced that he had proved the conjecture to be true, and sketched his proof on the internet.

His outline sent mathematicians into a frenzy of work as they tried to fill in the details. When all the gaps were filled, it was recognized that Perelman had indeed solved the problem, and at the International Congress of Mathematicians in Madrid in 2006 he was awarded the mathematician's highest honour, a Fields Medal.

Perelman had meanwhile stopped working on mathematics. He resigned his post at the Steklov Institute in Russia, moved in with his mother in St Petersburg and cut himself off from the outside world. He refused to accept the Fields Medal.

At the dawn of the new millennium the Clay Mathematical Institute, in the United States of America, had made the solution of the Poincaré conjecture one of its seven million-dollar Millennium Mathematics prizes. After yet more careful analysis of Perelman's arguments, the Clay Institute endorsed his proof and on July 1 this year they offered the prize to Perelman. He refused it, saying that he believed his contribution in proving the Poincaré conjecture was no greater than that of U.S. mathematician Richard Hamilton, who first suggested a programme for its solution.

Several charities in Russia appealed to Perelman to accept the prize and donate the money to programmes to alleviate poverty, but to no avail.

Perelman has turned down other awards. In 1982, as a member of the USSR International Mathematical Olympiad team, he won a gold medal, achieving a perfect score. He was 16 years old. It is believed that that was the last time he accepted an award for mathematical achievement. □

AT SIXES AND SEVENS

The phrase "at sixes and sevens" means higgledy-piggledy, or a state of confusion or disagreement.

The phrase is said to have originated in the fifteenth century in the City of London. Different professions had formed themselves into Craft Guilds to set standards of craftsmanship and wages. There were the Goldsmiths, the Drapers, the Ironmongers, the Haberdashers, the Fishmongers and so on. They were ranked in strict order of seniority for ceremonial proceses.

An argument arose between two Guilds, the Merchant Taylors and the Skinners, as to who was sixth and who was seventh in the order of processions in the City, both Guilds having been chartered in 1327. In 1484 they submitted the matter to the Lord Mayor, who ruled that they were to take alternately the positions of six and seven in the formal processions. They have been At Sixes and Sevens ever since. □

THE LEELAVATI AWARD

Simon Singh, author of the best-selling books *Fermat's Last Theorem*, *The Code Book* and *The Big Bang*, was announced the first winner of the Leelavati Prize at the International Congress of Mathematicians in Hyderabad in August 2010.

The Leelavati Prize is for outstanding contributions to the public understanding of mathematics. It is named after the 12th Century mathematical treatise 'Leelavati' by the Indian mathematician Bhaskara. The book consists of a series of problems set in verse, directed to a young lady, Leelavati (who is assumed to have been Bhaskara's daughter).

The Leelavati Prize is worth a million rupees (about R150 000).

Simon Singh was born in 1964 and graduated from Cambridge with a PhD in particle physics. He joined the BBC and in 1996 directed an award-winning documentary on the proof by Andrew Wiles of Fermat's Last Theorem. He followed the television show with a book with the same title, the first popular maths book to become a major best-seller.

His next book *The Code Book* described secret codes from ancient Egypt to quantum cryptography. It was turned into a television series *The Science of Secrecy*, and several television series on mathematics followed.

Branching out into general science writing, Simon Singh co-authored *Alternative Medicine* with Edzard Ernst. In a piece he wrote in *The Guardian* in 2008 he criticized chiropractors for claiming to be able to treat ailments such as childhood colic, asthma or ear infections, when there is no scientific evidence for the effectiveness of chiropractic treatment in such cases. "I can confidently label these treatments as bogus" he wrote. The British Chiropractic Association sued him for libel. Two years of expensive litigation followed, until in April this year Simon Singh was vindicated when the chiropracters dropped the case. □

DID YOU KNOW?

Five out of every three people are bad at fractions. □

PERCENTAGE ERROR

> "[A] contract for information technology services was illegally extended over a period of 49 months, increasing the original R180 million contract value by 587 percent to a total of R1.056 billion."
>
> (From a report of the Auditor-General of financial irregularities in the Department of Water Affairs, quoted in the *Cape Times*, 15 July 2010).

This unfortunately common type of error was spotted by David Allison. The writer had apparently divided 1056 by 180 to arrive at the figure of 587%.

What is the mistake?

The difference between the original contract price and the increased price is R1056 million — R180 million = R876 million. So the ratio of the increase to the original price is $876/180 = 4.87$, or 487%.

Think of it this way.

Increasing a number by 100% means that you add 100% of A, namely A, to A itself to get $2A$.

Increasing a number by 200% means that you add 200% of A (namely $2A$) to A to get $3A$.

If you increase a number A by 100% you get $2 \times A$.
If you increase a number A by 200% you get $3 \times A$.
If you increase a number A by 300% you get $4 \times A$.
In general, if you increase a number A by $100n\%$ you get $\dfrac{n-100}{100} \times A$. □

FERMAT'S LAST EQUATION

Pakistan's long, long wait for a Test victory over Australia finally came to an end amid scenes of excruciating tension on the fourth morning at Headingley, as a simple equation of 40 runs for victory with seven wickets still standing was made to look as complex as the proof to Fermat's Last Theorem.

(Andrew Miller on `www.cricinfo.com`, 24 July 2010)

(Submitted by Dirk Laurie) □

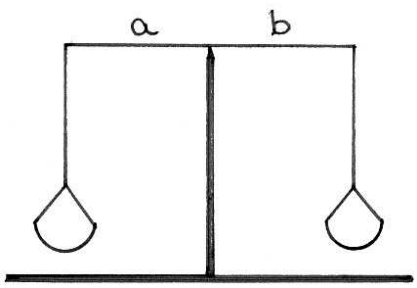
THE POTATO PUZZLE

The local township spaza weighs out vegetables on an ancient homemade scale: a beam with a pan hanging at each end, with a set of standard brass weights.

Customers have noticed that they get seem to get different amounts of vegetables, depending on which side of the scale the vegetables are put. The shopkeeper agrees that the arms of the scale are perhaps not equal in length, and suggests a compromise: If a customer wants two kilograms of potatoes, he will weigh out one kilogram with the weight in the left pan and potatoes in the right hand pan, and the second kilogram with the weight in the right hand pan and potatoes on the left.

Is this a fair way of weighing out the potatoes? Is the customer getting the full two kilograms?

Suppose the lengths of the two arms of the balance are a and b. With the weight on the left and potatoes on the right, the "law of moments" says that $1 \times a = W_1 \times b$, where W_1 is the weight of the first bag of potatoes.



Now put the weight on the right and potatoes on the left. This time we have $W_2 \times a = 1 \times b$, where W_2 is the weight of the second bag of potatoes.

The total weight of potatoes is then

$$W_1 + W_2 = \frac{a}{b} + \frac{b}{a} = \frac{a^2 + b^2}{ab}.$$

This is where some basic algebra becomes useful.

Since for all values of a and b, $(a-b)^2 \geq 0$, it follows that $a^2 - 2ab + b^2 \geq 0$. Therefore $a^2 + b^2 \geq 2ab$, from which it follows (remembering that a and b are both positive numbers, so that we can divide through by ab) that

$$\frac{a^2 + b^2}{ab} \geq 2.$$

So when potatoes are weighed out in this way, the customer always gets more than two kilograms. The spaza owner should invest in an accurate scale. $\square$

BAT DROP

According to the rules of baseball, a baseball bat must be no more that 2.75 inches in diameter at its thickest part and no more than 42 inches in length. It typically weighs no more than 33 ounces.

In American baseball jargon the *drop* of a baseball bat is the difference $Weight - Length$. If a bat is 33 inches long and weighs 30 ounces, then the drop is -3.

Bat drops are usually negative, and in baseball arithmetic a bat drop of -4 is regarded as greater than a bat drop of -3. Baseball players believe that a larger bat drop helps to increase swing speed. Bats with smaller drops create more power.

The "drop zone" of a bat is regulated in American baseball, depending on the level of play. Little League has no restrictions. Senior League baseball (ages 10–14) has a drop zone regulation of no lighter than -8. High school and college baseball has a drop zone regulation of no lighter than -3.

There are several points about the definition of "bat drop" that causes one to pause and scratch one's head. First of all, is there any physical meaning or real-life interpretation of the subtraction of two different units, weight and length? Would different units (grams and centimetres, for example) make a difference? Furthermore, what is meant by "power"? Perhaps it means the speed (or kinetic energy) of the ball as it leaves the bat. The swing speed is best interpreted as angular velocity (degrees per second), or perhaps the linear velocity of a specific point on the bat, for example the "sweet spot".

Is there a similar concept for a cricket bat? □

ECONOMICS QUOTE

Population, when unchecked, increases in a geometrical ratio. Subsistence increases only in an arithmetical ratio.

(English political economist Thomas Malthus, 1766 – 1834) □

SHARP CALCULATOR COMPETITION

The Sharp Calculator Competition is sponsored by Sharp Electronics (S.A.) (Pty) Limited, who have donated three calculators as prizes.

These prizes are offered for the three best solutions of the problems below. Solutions may be in English or Afrikaans. Partial solutions will be judged on their merits. The Editor reserves the right to make no award if no satisfactory entries are received, and his judgement will be final. The winner of an nth prize in a previous competition will only be eligible for an mth prize if $m < n$.

Only high school students may compete. Entrants must give their full name and home address, age, school and grade. Solutions must be certified by a mathematics teacher at the school as the entrant's own unaided work.

The closing date for receipt of entries is 1 December 2010, and the results will be announced in our January 2011 edition.

SHARP

1. The vertex A of equilateral triangle ABC is fixed, and the vertex B moves along a fixed straight line. Find the path traced out by the vertex C.

2. Find the sum of the series
$$\sum_{r=1}^{n} \sqrt{1 + \frac{1}{r^2} + \frac{1}{(r+1)^2}}.$$

3. If $\dfrac{(a-b)(c-d)}{(b-c)(d-a)} = k$, determine the value of $\dfrac{(a-c)(b-d)}{(a-b)(c-d)}$. □

QUOTE

Two and two the mathematician continues to make four, in spite of the whine of the amateur for three, or the cry of the critic for five.

(American artist James McNeill Whistler) □

NUMBER PATTERN

Rita Hendrikse

$$
\begin{aligned}
1 \times 9 &+ 10 = 11 \\
12 \times 9 &+ 10 = 111 \\
123 \times 9 &+ 10 = 1111 \\
1234 \times 9 &+ 10 = 11111 \\
12345 \times 9 &+ 10 = 111111 \\
123456 \times 9 &+ 10 = 1111111 \\
1234567 \times 9 &+ 10 = 11111111 \\
12345678 \times 9 &+ 10 = 111111111 \\
123456789 \times 9 &+ 10 = 1111111111
\end{aligned}
$$

$$
\begin{aligned}
9 \times 9 &+ 7 = 88 \\
98 \times 9 &+ 6 = 888 \\
987 \times 9 &+ 5 = 8888 \\
9876 \times 9 &+ 4 = 88888 \\
98765 \times 9 &+ 3 = 888888 \\
987654 \times 9 &+ 2 = 8888888 \\
9876543 \times 9 &+ 1 = 88888888 \\
98765432 \times 9 &+ 0 = 888888888
\end{aligned}
$$

$$
\begin{aligned}
1 \times 1 &= 1 \\
11 \times 11 &= 121 \\
111 \times 111 &= 12321 \\
1111 \times 1111 &= 1234321 \\
11111 \times 11111 &= 12345321 \\
111111 \times 111111 &= 12345654321 \\
1111111 \times 1111111 &= 1234567654321 \\
11111111 \times 11111111 &= 123456787654321 \\
111111111 \times 111111111 &= 12345678987654321
\end{aligned}
$$

Rita wins a R50 Prize Voucher for these patterns.

Readers are invited to send their favourite number patterns to the Editor. The best contributions will win R50 Prize Vouchers. □

SHARP SOLUTIONS

1. *Solve the equations* $x^2 - yz = 1$, $y^2 - zx = 2$ *and* $z^2 - xy = 3$.

Subtracting the first equation from the second, we get

$$y^2 - x^2 + yz - zx = (y - x)(x + y + z) = 1.$$

Subtracting the second equation from the third, we get

$$z^2 - y^2 + zx - xy = (z - y)(x + y + z) = 1.$$

So

$$y - x = \frac{1}{x + y + z} = z - y.$$

It follows that x, y and z form an arithmetic progression, with common difference $d = \dfrac{1}{x + y + z}$.

The second equation now gives

$y^2 - (y - d)(y + d) = y^2 - (y^2 - d^2) = d^2 = 2$. So $d = \pm\sqrt{2}$.

Since x, y, z are in arithmetic progression, $y = \dfrac{x + y + z}{3}$.

From $\dfrac{1}{x + y + z} = d$ we $y = \dfrac{1}{3d} = \pm\dfrac{1}{3\sqrt{2}}$.

So we have two solutions:

$$x = \frac{1}{3\sqrt{2}} - \sqrt{2} = -\frac{5}{3\sqrt{2}}, \quad y = \frac{1}{3\sqrt{2}} \quad \text{and} \quad z = \frac{1}{3\sqrt{2}} + \sqrt{2} = \frac{7}{3\sqrt{2}}$$

$$x = -\frac{1}{3\sqrt{2}} + \sqrt{2} = \frac{5}{3\sqrt{2}}, \quad y = -\frac{1}{3\sqrt{2}} \quad \text{and} \quad z = -\frac{1}{3\sqrt{2}} - \sqrt{2} = -\frac{7}{3\sqrt{2}}.$$

From the Irish university competition *Superbrain*, 1994.)

2. *Find all pairs* (a, b) *of natural numbers satisfying the equations*

$$a^{6a} = b^b.$$

Since $a^{6a} = b^b$, it follows that $a \leq b$ and $b = ka$, where k is a natural number. Then $a^{6a} = (ka)^{ka} = k^{ka}.a^{ka}$.

It follows that $a^{6a-ka} = k^{ka}$ and hence $a^{6-k} = k^k$. So $k < 6$.

We now try the different possible values of k.

$k = 1$. Then $a^5 = 1$ and it follows that $a = 1$ and $b = 1$.

$k = 2$: $a^4 = 2^2 = 4$. There are no solutions in this case.

$k = 3$: $a^3 = 3^3 = 27$. Then $a = 3$ and $b = 9$.

$k = 4$: $a^2 = 4^4 = 256$. Then $a = 16$ and $b = 64$.

$k = 5$: $a = 5^5$ and $b = 5^6$.

(From the Swiss Olympiad training programme, January 2009.)

3. *The squares of a rectangular grid are coloured alternatively black and white in a chess board fashion. Each square contains an integer. The sum of the numbers in each row and the sum of the numbers in each column is even. Prove that the sum of all the numbers in the black squares is even.*

   Without loss of generality we assume that the top left square is white. Since the sum of all the columns is even, the sum of all the squares is even. Therefore, if the sum of the black squares is even, so is the sum of the white squares.

   The sum of every second row starting from the top plus the sum of every second column starting on the left will include all the black squares on the board. Since each row and column has an even sum, this sum is even. The white squares in this array are counted twice, so the sum of the numbers in those squares, counted twice, is even. So the sum of the numbers in all the black squares is even.

No prizeworthy entries were received. □

NUMBER WORDS

tithe: Derived from the old English word *tegotha* for one tenth, it was once a tax levied by the Church, which required every parishioner to contribute one tenth of his annual product of land or labour for the upkeep of the church and provision of social services.

noon: Today it means midday, or 12 o'clock. The word, however, derives from the Latin words for nine (*novem*) and ninth (*nonus*), in a rather unusual way. In the old Roman Catholic Church there were seven offices, or times for prayer, set down for a specific time each day. Nones was to be observed during the ninth hour after sunrise, i.e. 3 o'clock in the afternoon. The office was later moved to midday, but the name was retained. Later, *nones* morphed into *noon*.

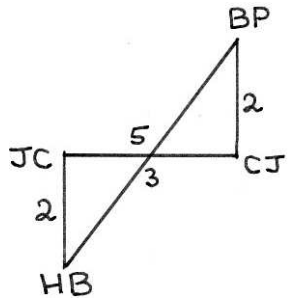
decimate: Today the word is often taken to mean reducing something to one tenth of its original size or number. The term originated in the Roman army, where *decimatio* was a punishment meted out to cowardly or mutinous soldiers. The disgraced soldiers were divided into groups of ten, and each group drew lots to choose one of its members. The loser was then bludgeoned to death by the other nine. □

PRIZEWINNERS AND SOLUTIONS

Seating Problem (July 2010, p. 26):
Clockwise round the table, the diners are: Mr Albert, Mr Charles, Mrs Barnes, Mr Edwards, Mrs Davids, Mr Barnes, Mrs Alberts, Mr Davids, Mrs Charles, and Mrs Edwards. Kayla Arnold (Sweet Valley Primary School) wins a R50 Prize Voucher for her solution.

Sort Them Out (July 2010, p. 27):
Abdullah Valley (Harding Secondary School) wins a R50 Prize Voucher for working out who the names of the men, their professions and where they live relative to each other.

1. Bob Price (accountant)
2. Charlie Jones (doctor)
3. John Conway (soldier)
4. Harry Barnes (plumber).



All entries that did not win a Prize went into the Editor's Hat, and one lucky name was drawn. The winner of the Editor's Hat Prize Voucher worth R50 is Jonathan Crowther (Sweet Valley Primary School).

Competitions are open to all, but only readers who are at school are eligible for prizes. Entries to separate competitions must be submitted on separate sheets of A4 paper, each of which must give the sender's name, age, grade, school and home address. To save postage, entries to different competition may all be sent in the same envelope. Solutions may also be sent in by email or fax. □

A NEW LOW

> Now 1.7cm striker Peter Crouch has taken matters to a new low after being accused of cheating on his fiancée.
>
> (*Independent On Line*, 9 August 2010)

The next day the IOL article was corrected to read 1.7m, but it's still wrong; Peter Crouch's actual height is 1.98m. And the relevance of a striker's height to his capacity for cheating was not explained. □

BOOK REVIEW

Handbook of Cubik Math, by Alexander H. Frey, Jr. and David Singmaster. Lutterworth Press, 2010. ISBN: 978 0 7188 9209 8.

Erno Rubik (or Rubik Ernö, in strict Hungarian style) was a teacher of interior design and architecture in Budapest when he invented what the world today knows as Rubik's Cube. According to legend, he took a month to solve his own puzzle.

Rubik's Cube became a worldwide craze in the early 1980s. Rubik became an instant millionaire, and went on to invent further puzzles, Rubik's Clock, Rubik's Snake and Rubik's Revenge (a $4 \times 4 \times 4$ Cube).

There was an immediate demand for books on how to solve the Cube, and one of the first was *Notes on Rubik's Magic Cube* by David Singmaster, published in 1979 and described in *Scientific American* as "the definitive treatise".

Singmaster realised from the start that Rubik's Cube was a wonderful way of demonstrating the power of group theory, an abstract branch of mathematics of importance in many fields, notably theoretical physics. The basic ideas of group theory (isomorphism, normal subgroups, commutators and conjugates) all arise naturally in understanding how to solve the Cube.

Now revised and rewritten by Singmaster and Alexander H. Frey, Jr, the new *Handbook of Cubik Math* offers a comprehensive and elegant mathematical analysis of the Cube. The solution is intuitive, and does not require memorising long sequences of moves. In showing how to solve the Cube, the authors explain the background theory, and how to apply it to similar puzzles. □

QUOTE

Inspiration is needed in geometry, just as much as in poetry.

(Russian poet Alexander Pushkin) □

IMO IN KAZAKHSTAN

The 51st International Mathematical Olympiad (IMO) took place from 2 to 14 July in Kazakhstan.

It was attended by 517 students, of whom fewer than 50 girls were from 97 countries.

The South African team was
Steve Bailey (St David's Marist School)
Greg Jackson (Bishops)
Ashraf Moolla (Rondebosch Boys' High School)
Dylan Nelson (Benoni High School),
Dessi Nikolov (Eunice High School)
Sean Wentzel (Westerford High School).

The Team Leader was David Hatton (UCT), with Deputy Team Leader Maciej Stankiewicz (UCT). Professor John Webb (UCT) attended in his capacity as Secretary of the Advisory Board of the IMO.

The Team Leaders arrived ahead of the teams and met in Almaty, the former capital of Kazakhstan, to set the question papers. They were then flown to Astana, the new capital, for the Opening Ceremony. This was set for July 6th, a public holiday to celebrate the 70th birthday of the President of Kazakhstan, Nursultan Nazarbayev.

The teams stayed in Baldauren, a scenic resort on the shores of Lake Shchuchye, where the contest took place. On two successive days they wrote a gruelling $4\frac{1}{2}$ hour paper consisting of just three problems. Then it was time for relaxation and excursions in the most picturesque part of Kazakhstan. Back in Astana the Team Leaders were hard at work marking the papers and fighting for points for their students.

When the results were finally announced, the top country was once again China, followed by Russia, the USA and Korea. Fifth place was a tie between Thailand and the host country, Kazakhstan (their best performance ever in an IMO). Japan, Turkey, Germany and Serbia filled up the remaining top ten places.

The presence (and performance) of African teams at the IMO is a constant source of interest. This year the number of African teams was, disappointingly, down to five. Benin, Mauritania, Senegal and Zimbabwe were present at the IMO in Germany in 2009, but did not send teams this year. However, the performance of the five African countries that sent teams was

noteworthy: for the first time, every African team present at an IMO won a medal.

South Africa led the African field with two Bronze Medals (Greg Jackson and Sean Wentzel) and four Honourable Mentions, followed by Morocco (one Bronze Medal and three Honourable Mentions). The Ivory Coast, in their first outing to an IMO, won a Silver Medal and two Honourable Mentions. Tunisia won one Silver Medal and one Honourable Mention) and Nigeria won their first IMO medal (a Bronze).

For the question papers and full IMO results, visit the IMO website `www.imo-official.org`.

The 2011 IMO will be held in the Netherlands. □

HERON AND THE STEAM TURBINE

While the story of Archimedes and the steam cannon (page 1) is highly speculative, there is good evidence for the design of a steam turbine by another Greek mathematician, Heron of Alexandria, some three hundred years later.

In Heron's design, a hollow sphere on a spindle was filled with water and heated over a fire. Steam escaped through angled nozzles, causing the sphere to rotate on the spindle. The *aelopile*, as it was called, was no more than a toy, and the world had to wait for the Industrial Revolution, about 1600 years later, for the power of steam to be harnessed. The first steam engines were piston driven, and only in 1884 was Heron's idea revived of using jets of steam to generate power. □

STEVE BAILEY: A TRIBUTE

Dessi Nikolov
Eunice High School

On the 19th of August this year, the world lost a great son, brother, friend and a multi-talented teenager with a zest for life like no other. I was fortunate enough to have Steve in my IMO team this year and would like to share some of my fondest memories of him.

My first impression of Steve was a truly overwhelming one. One of the evening activities at the 2009 Stellenbosch Camp was a speed challenge in arithmetic. Two at a time, we had to multiply two digit numbers and the person to answer correctly first, would advance. Needless to say, Steve won that evening hands down. He could produce an accurate answer in a matter of seconds, every time. Later I asked him how he managed to do that, and he simply shrugged and replied that he had practised when he was younger in his spare time. This introduced me to the immense determination and perseverance he showed when it came to any challenge he was presented with.

Six months later, on a flight from Frankfurt to the IMO in Astana, Kazakhstan, I woke up to find the only light switched on in the entire plane, and Steve firmly tackling geometry problems in the early hours of the morning. His hard work and dedication was rewarded with his 11 points and honourable mention 2 weeks later. He was particularly proud to have made the team, and having proven himself worthy of the green blazer, meant a great deal to him.

Once the contest was over, we had time to relax and explore our exciting location. This was when I discovered Steves passion for playing and watching sport. He had a vast range of interests. From being a true Liverpool supporter and a keen football player himself, to the endless descriptions I had to endure of Johannesburgs finest golf courses and his plans to run the Comrades in the next few years. In the afternoons, he would often go for a jog around the complex, and complain that it was way too small to get any decent exercise; while I promptly told him that he was crazy.

Another admirable quality that he possessed, and will always be remembered for, would most certainly be his warmth and friendliness as a person.

At the airports and when we arrived, he was usually the first to walk up to other teams and befriend them. Soon after his passing, I received e-mails and messages from many of the IMO contestants expressing their condolences. This shows that even despite the little time he had spent with them, he was able to touch their lives and make a difference.

Personally, I will always remember Steve for his optimism. All through the time I knew him, I never saw him angry or sad. Always with a smile and a joke to tell, he brought a sense of calm and contentment to the team.

E. Kubler-Ross once said: "Death is simply a shedding of the physical body like the butterfly shedding its cocoon. It is a transition to a higher state of consciousness where you continue to perceive, to understand, to laugh, and to be able to grow."

Steve's spirit will continue to live in the memories of those fortunate to have known him. □

CONVERSION ERROR

> What it shows is a complete scan of the heavens ... from the Planck satellite orbiting the sun almost 1 609 344km from Earth.
>
> (*Sunday Times*, 18 July 2010)

Where did that figure 1 609 344 come from? Seven-digit accuracy is not appropriate for something moving relative to the Earth.

It's not too difficult to work out what happened. The original article appeared in the London newspaper *The Times*, in which the distance of the Planck satellite from Earth was given as "almost a million miles". The local *Sunday Times* sub-editor thought that South African readers would not know what a "mile" is, so hauled out his (her?) calculator and did a metric conversion.

A mile is 1760 yards, a yard is 3 feet, a foot is 12 inches and an inch is 2.54 cm. Those figures are all exact. So *exactly* one million miles is exactly

$$1000000 \times 1760 \times 3 \times 12 \times 2.54 = 1.609344 \times 10^{11} \text{ cm} = 1\ 609\ 344 \text{ km}.$$

So the descriptive and approximate "million miles" became the ridiculously accurate and therefore meaningless 1 609 344 km. If conversion of the measurement to metric units was really felt to be necessary, it should have been "almost 1.6 million kilometres". □

IOI IN WATERLOO

The 2010 International Olympiad in Informatics (IOI) was held at the University of Waterloo, Canada, in August. A total of 304 students from 80 countries took part.

All four South African team members won Bronze Medals. They were:
Francois Conradie (De Kuilen High School/University of Stellenbosch)
Graham Manuell (De La Salle Holy Cross College)
Kosie van der Merwe (Brackenfell High School/University of Cape Town)
Sean Wentzel (Westerford High School, Cape Town)

The Team Leader was Peter Waker, Manager of the Standard Bank/CSSA Computer Olympiad. The Deputy Leader was Keegan Carruthers-Smith, a postgraduate student at the University of Cape Town and former IOI medallist.

The team was chosen from the more than 33 000 candidates who took part in the 2009 Standard Bank/CSSA Computer Olympiad. Additional training was given at three Training Camps run by the South African Computer Olympiad Trust.

The IOI participants spent two days – 5 hours per day – writing computer programs to solve a range of problems.

This year, as an experiment, the organizers had a live scoreboard available on the internet. Spectators worldwide could log in and follow the contest as it developed.

One of the aspects that became clear early on in the contest was the tremendous difference in IT education between countries. Some participants wrote a program to solve the first (easy) problem within 6 minutes; a few could not do so in 5 hours.

Many participating countries report that the number of students graduating in informatics has decreased this decade. Especially noticeable at the IOI was the small number of female participants. Although there were more than 300 participants, only nine were girls; three from the same country - Libya.

The overall winner was Gennady Kosotkevich of Belarus. Although he is only 15, Gennady was competing in his 5th IOI and is close to beating all time record holder, South African Bruce Merry, who competed in six IOIs and won a medal at each.

Second was Bulgarian Rumen Hristow who was competing in his third IOI. Third was Adrian Jaskolka, a first time competitor from Poland. USA team members Wenyu Cao and Michael Cohen came fourth and fifth respectively. Canadian Yu Cheng came sixth.

It was noticeable that the Chinese team, which usually dominates the top places, came in at 10th, 11th, 27th and 32nd. Commented South African team leader Peter Waker: "The Chinese policy of dropping gold medal winners from the next year's team to get a wider distribution of medals is unusual and probably prevented them from dominating the first few places this year."

South Africa ranked 28th of the 80 countries, just below Hong Kong and Brazil, and just above Serbia and Israel. South Africa was also the top ranked African country; well above Egypt (35th), Madagascar (70th), Ghana (72nd), Libya (78th) and Nigeria (79th).

Each team had a Canadian student as guide – usually one who spoke their home language. This was possible because the University of Waterloo attracts such a cosmopolitan student body.

There were organized outings to a Theme Park and to the Niagara Falls. In addition many participants used the opportunity to attend appropriate lectures at the university and explore the town of Waterloo. The South African team went ice skating and go-karting with their guide. Many decided they want to return to the University of Waterloo for postgraduate studies.

Enquiries

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SUBTRACTION PROBLEM

Teacher: "How many times can you subtract 6 from 30"

Tom: "Five time, because five times six is 30"

Dick: "Only once. After that it's 24."

Harry: " As often as you like. Every time you get 24" □

IPMO 2010 RESULTS

The Interprovincial Mathematics Olympiad (IPMO) has been a regular annual event since 1990. Teams of ten from different provinces (the term corresponds more to sporting divisions than political) meet in various centres in friendly competition. The IPMO has two divisions, Junior (up to grade 9) and Senior (grades 10 to 12).

This year's IPMO took place on the afternoon of Saturday 11 September, with teams representing Boland, Eastern Cape, Free State/Northern Cape, Gauteng North, Gauteng South, KwaZulu-Natal Coast, KwaZulu-Natal Midlands, Kwazulu-Natal North Coast and Western Province. Many provinces entered more than one team in each section, making a total of 21 Junior and 16 Senior teams.

The Olympiad started at 2 pm, with the team members writing an individual one-hour test, a conventional multiple-choice paper of 15 questions. After a break for refreshments and discussions of team tactics, a second paper of ten difficult problems was tackled by the team as a whole.

The papers were quickly marked and the scores sent to the IPMO organizer, Professor Peter Dankelman, at the University of KwaZulu-Natal in Durban. By 5.30 pm the results came back and were announced to the eagerly waiting teams.

In the Junior section, the Free State/Northern Cape team came out top with 1566 points out of a possible 2000, followed by Boland (1485) and Western Province (1386).

In the Senior section (grades 10 to 12), Western Province headed the rankings with 1267 points, followed by the Western Province B team (1159) and Gauteng North (1150). □

MORE NEWSPAPER NUMBERS

Developments to provide sustainable water supplies for the town (Mossel Bay) and PetroSA are well under way, including the building of the largest desalination plant in South Africa.

When fully commissioned, the plant will produce 15ml of water per day, 10 of which will go to the town and five to the refinery.

(*Cape Times*, 25 August 2010) □

UCT MATHEMATICS COMPETITION 2010 REPORT

In April this year over 6500 participants from Western Cape high schools thronged the University of Cape Town campus to take part in the annual UCT Mathematics Competition. The results were announced in May and published in the July 2010 edition of *Mathematical Digest*.

The full report on the 2010 Competition has now been published. It contains includes the question papers for all five high school grades, with full solutions, a list of all the individual winners and the top schools, and a detailed statistical analysis of the individual problems and the papers as a whole.

The book is a valuable resource for the classroom teacher. Statistics of performance on individual problems indicate weaknesses which need to be addressed. While the first ten problems in each paper are relatively easy, the problems get increasingly trickier as the paper progresses. The last five problems in each paper are deliberately made very difficult, and full marks is a rare achievement.

The papers can be used by pupils preparing for participation in maths competitions, and their scores may then be evaluated by comparison with the histograms of the 2010 contestants.

All the problems can be understood in the context of school mathematics. Some focus on essential skills, others offer useful problem-solving experience and require lateral thinking. The more challenging problems provide insights into the type of original thinking required in university mathematics courses. Working through the grades 11 and 12 papers, in particular, is recommended as excellent preparation for university entrance exams and placement tests.

Copies of *The 2010 UCT Mathematics Competition* cost R40 each (including VAT, packing and postage). Reports on earlier years are still available. Use the form at the back of the magazine to order copies. □

WIMBLEDON BUG

At the 2010 Wimbledon Championships, in the Gentlemen's Singles tournament, the American 23rd seed John Isner beat the French qualifier Nicolas Mahut in the longest match in tennis history, measured both by time and number of games. After 11 hours and 5 minutes of play over three days, the match finished 6-4, 3-6, 6-7(7), 7-6(3), 70-68.

Now for the geek angle:

On the second day of play, the courtside scoreboard stood still at 47-47 and later went dark. IBM programmers said it was only programmed to go to 47-47 but would be fixed by the next day. The on-line scoreboard at the official website lasted slightly longer: At 50-50 it was reset to 0-0. Users were asked to "please add 50 to the Isner/Mahut game score". An IBM programmer worked on the computerised scoring system until 11:45 pm to accommodate the match's scores for the next day. There was no way, he said to display a score above 99–99. The reason is that panels which display the numbers are limited to two digits for each player. □

MATHS DATES FOR 2011

UCT Mathematics Competition:

Thursday 14 April Website `www.mth.uct.ac.za/competition`

South African Mathematics Olympiad:
Closing date for entries: 4 February
1st round: 16 March
2nd round: 17 May
3rd round: 7 September Website: `www.samf.ac.za/samo`

Pan African Mathematics Olympiad:
Tunisia, 26 March to 4 April Website: `www.pamo-official.org`

International Mathematical Olympiad:
The Netherlands, 12 to 24 July Website: `www.imo-official.org`

International Olympiad of Informatics:
Thailand, 22 to 29 July Website: `www.ioinformatics.org` □

PUZZLE PAGE

1. Put the eight digits 2, 3, 4, 5, 6, 7, 8 and 9 in the eight circles (one in each circle) below to make the addition correct.

$$\frac{1}{\bigcirc \times \bigcirc} + \frac{\bigcirc}{\bigcirc \times \bigcirc} + \frac{\bigcirc}{\bigcirc \times \bigcirc} = 1$$

2. $$\frac{1}{2n+1} = \frac{1}{2n+1}$$
$$\frac{1}{2n+1} + \frac{1}{(2n+1)(4n+1)} = \frac{2}{4n+1}$$
$$\frac{1}{2n+1} + \frac{1}{(2n+1)(4n+1)} + \frac{1}{(4n+1)(6n+1)} = \frac{3}{6n+1}$$

   How does the pattern continue?

3. Find three odd numbers a, b and c such that

$$\frac{3}{2011} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}.$$

4. Anne, her brother Bert, her daughter Ceri and her son Dave play tennis. The best player's twin sibling and the worst player are of opposite sex. The best player and the worst player are of the same age. Who is the best player?

5. Four dice are thrown. What is the probability that the sum shown is divisible by 3?

6. The following riddle, in verse, was composed by Jonathan Swift, the author of *Gulliver's Travels*.

   I'm up and down and round about,
   Yet all the world can't find me out;
   Though hundreds have employed their leisure,
   They never yet could find my measure.

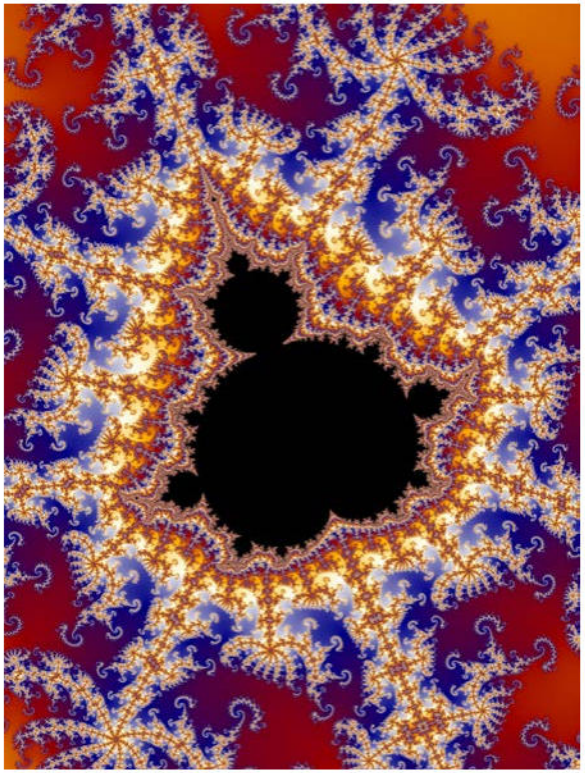
   What is the answer to the riddle?

7. What type of racing is done "in Archimedean mode"?

Send your solutions to the Editor. BACKPACKS (full sets of all available back numbers of *Mathematical Digest*) will be sent to the ten readers who submit the best entries to two or more of these puzzles. □

mathematical digest

No. 162 January 2011



The Mandelbrot Set

mathematical digest

Number 162 **January 2011**

The computer is rigour's only true friend. The notion is of the essence for anyone who has been motivated to write a computer program—even a short one—from scratch... In most cases a non-rigorous program fails completely, and the slightest departure from absolute rigour makes it scream "error" at the programmer... The computer programmer soon learns that a program that works on one computer, with its operating system, will not work on another. He will swear at the discrepancies, but I cannot imagine a better illustration of the changeability and arbitrariness of axiomatic systems.

(Benoit Mandelbrot, in *Fractals, Computers and Mathematics Education*, ICME7, Quebec, 1992)

Department of Mathematics and Applied Mathematics
University of Cape Town

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Cover: The Mandelbrot Set (see page 1).

BENOIT MANDELBROT 1924–2010

Benoit Mandelbrot, the creator of fractals, died on 14 October 2010. He was born in Poland in 1924, and studied in France and the USA. Most of his career was spent at IBM's Thomas J. Watson Research Center, where he was an IBM Fellow.

Mandelbrot's work covered pure mathematics and a broad range of applied fields such as information theory, economics, financial mathematics, cosmology and fluid dynamics. He became convinced that the theme of self-similarity was common to a number of problems in the real world. In 1975, he coined the term *fractal* to describe certain geometric structures, and published his ideas in *Fractals: Form, Chance and Dimension* in 1977.

Mandelbrot believed that fractals provided realistic models of many irregular phenomena in the real world. In many ways they are more intuitive and natural than the artificially smooth circles and lines of conventional Euclidean geometry. Fractals describe the shapes of mountains, coastlines and rivers. In biology, fractals model the structure of blood vessels and lungs and even the head of a cauliflower. In cosmology, fractals can explain the clustering of galaxies. Fractals are found in music, painting, architecture and the unpredictable movement of stock market prices.

While on secondment as Visiting Professor of Mathematics at Harvard University in 1979, Mandelbrot began to study structures called Julia sets that arose through considering iterations of the simple mapping $z \to z^2 + c$ in the complex plane. These had been investigated by Gaston Julia and Pierre Fatou half a century before. Mandelbrot's contribution was to use a computer to study the sets, and what is now named the Mandelbrot Set (see cover) was the spectacular result.

In his introduction to *The Fractal Geometry of Nature*, published in 1982, he wrote "Clouds are not spheres, mountains are not cones, coastlines are not circles, and bark is not smooth, nor does lightning travel in a straight line." His informal style of writing made the book accessible to a wide readership.

Mandelbrot received many awards, prizes and honorary degrees, from the naming of a small asteroid after him to the French Legion of Honour. Paying tribute, French President Nicolas Sarkozy said "Mandelbrot had a powerful, original mind that never shied away from innovating and shattering preconceived notions." □

P VERSUS NP

The P versus NP problem ranks as the biggest unsolved problem in theoretical computer science. Introduced in 1971 by Stephen Cook in his paper "The complexity of theorem proving procedures", its importance was recognized by the Clay Mathematics Institute when it was ranked as one of the seven Millennium Prize Problems, with a prize of US$1 000 000 for the first correct solution.

In simple terms, the question is whether every problem whose solution can be easily *checked* by a computer can also be easily *solved* by a computer.

The term "easily" needs to be made more precise. An algorithm to solve a problem is measured in terms of the length of time a computer would take to run it. If the size of the problem is measured by an integer n, and the running time for the algorithm is given by a polynomial $p(n)$, then the algorithm is said to run "in polynomial time", and the problem is "in class P", or simply "P".

If the solution of the problem can be checked in polynomial time, the problem is "in class NP", or simply "NP".

In general, any algorithm that will run in polynomial time is regarded as acceptable, but if "exponential time" is needed, you had better find another algorithm.

For example, consider the "subset-sum" problem:

> Given the set of integers
>
> $$\{6, 14, 23, 26, 38, 45, 50, 70, 84, 93\}$$
>
> find a subset of these numbers which adds up to 211.

The solution is $\{6, 14, 38, 69, 84\}$, and checking that the solution is correct amounts to no more than calculating $6 + 14 + 38 + 69 + 84$. But finding that solution may not have been quick at all.

There are 10 numbers on the list, and the number of subsets of a set of 10 elements is 2^{10} (including the empty subset and the whole set). So all you had to do was program a computer to check every subset-sum until it made a hit. Since 10^n is an exponential function, not a polynomial, and any exponential function goes off to infinity faster than any polynomial, this algorithm cannot be executed in polynomial time.

There may be a polynomial-time algorithm for solving the subset-sum problem, but the fact is that none is known. The subset-sum problem is therefore in NP (quickly checkable), but not necessarily in P (quickly solvable).

In general, P implies NP (any quickly solvable problem is quickly checkable). The big issue is whether there are problems in NP which are not in P. If it turned out that P does not equal NP, it would mean that some NP problems are harder to compute than to verify: they could not be solved in polynomial time, but the answer could be checked in polynomial time.

Round the world, many computer scientists are working on the problem. There was a surge of excitement in August last year when Vinay Deolalikar of Hewlett-Packard Labs in Palo Alto, California, released a draft paper with the simple title " $P \neq NP$". Deolalikar is a respected computer scientist, so his paper was taken seriously. But, given the speed of the internet, the paper came immediately under the scrutiny of the top experts in the field. Regrettably, within a couple of days some serious flaws had been found. The problem remains unsolved, and the million dollar prize is still unclaimed.

Reference

New Scientist 14 August and 21 August 2010 □

PROBLEMS IN PROPORTION

Henry VIII had six wives. How many wives did Henry IV have?

If the genuine Mona Lisa is worth a billion Euros, how much would ten genuine Mona Lisas be worth?

A 105-piece orchestra will take 54 minutes to play Beethoven's Fifth Symphony.

How long will an 80-piece orchestra take to play Beethoven's Tenth Symphony?

Ten pairs of socks take two hours to dry on a washing line. How long will 20 pairs of socks take?

A musical octave consists of eight notes. How many notes are there in two octaves? □

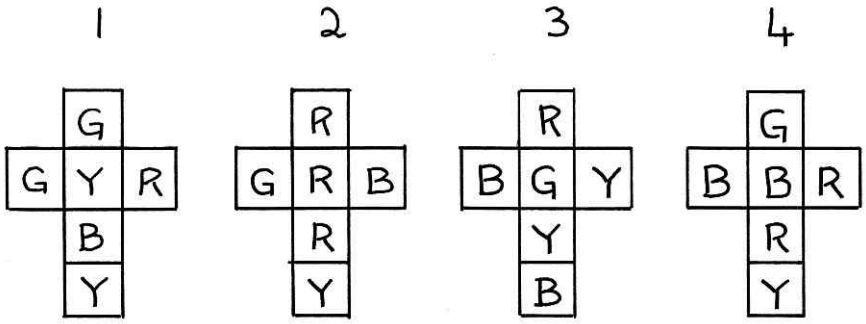
INSTANT INSANITY

You have four cubes, with their sides coloured with four different colours. The object is to arrange the cubes in a $4 \times 1 \times 1$ rectangular block so that each of the four long sides of the block show four different colours.

The puzzle is over a hundred years old, and was originally known as *The Great Tantalizer*. It was marketed as *Instant Insanity* in the 1960s, and is often known by that name today.

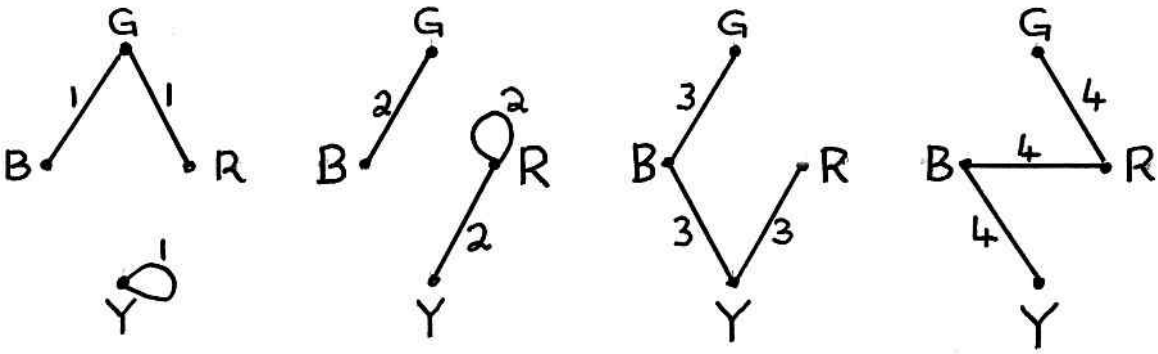
How should the puzzle be tackled? Trial and error is not a good idea, as there are too many combinations to work through. The graphical method below is quite easy to implement, and works for all puzzles of this type.

Suppose the four cubes are those shown by their nets below, with their faces coloured Blue (B), Green (G), Red (R) and Yellow (Y). Number the cubes 1, 2, 3, 4.

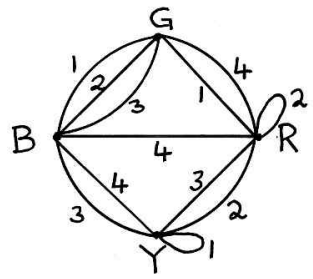


For each cube, draw a little graph with four vertices, labelled (for the four colours) B, G, R, Y.

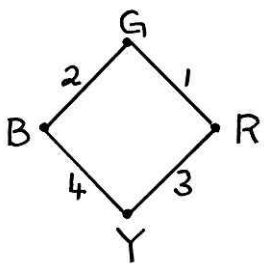
Each cube has three pairs of opposite faces. For each pair of opposite faces, draw a line connecting the points representing the colours of those two faces and label the line with the number of that cube. If two opposite faces are the same colour, draw a little loop from that point back to itself.

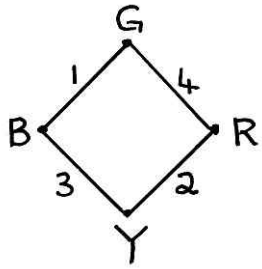


Now combine the four graphs into one graph.



Now find two loops in the combined graph, such that each loop consists of four lines, with different numbers, linking the four different colours. The two loops must use different lines.





These two loops are the key to solving the puzzle.

Use the first loop G-2-B-4-Y-3-R-1-G to arrange the cubes in a line so that the front and back faces have two different colours.

Start with Cube 2, and put a Green face at the back and a Blue face in the front. Next to it you set Cube 4 with a Blue face at the back and a Yellow face in the front. Next to it you place Cube 3 with a Yellow face at the back and a Red Face in the front, and at the end of the row you put Cube 1 with a Red face at the back and a Green face at the front.

The four cubes are now positioned so that the front and back faces show all four different colours.

To sort out the colours on the top and bottom faces, you have to rotate each of the four cubes about axes through the centres of the front and back faces. This keeps the colours on the front and back faces unchanged.

The key to rotating each of the cubes is given by the second loop G-1-B-3-Y-2-R-4-G.

Rotate Cube 1 so that a Green face is on the top and a Blue face is on the bottom. Then rotate Cube 3 so that a Blue face is on the top and a Yellow face is on the bottom. Then rotate Cube 2 so that a Yellow face is

on the top and a Red face is on the bottom. Finally rotate Cube 4 so that a Red face is on the top and a Green face is on the bottom.

It may happen that you can separate out a different pair of loops from the combined graph. That would mean that the puzzle has a different solutions. On the other hand, if you cannot separate out a pair of loops as required, then the puzzle has no solution. □

THE TEN COMMANDMENTS FOR MATHS SUCCESS

1. Thou shalt read the problem unto the uttermost end thereof.
2. Whatsoever thou doest to one side of an equation, do ye also to the other.
3. Thou shalt use thy Common Sense in solving word problems else thou wilt have flagpoles 9000 feet in height, yea ... even fathers younger than sons.
4. Thou shalt ignore the teachings of false prophets that thou mayest ignore tables and cast thy lot in with calculators, for the truth is not in them.
5. When thou knowest not, thou shalt look it up, and if thy search still elude thee, then thou shalt ask thy all-knowing teacher.
6. Thy correct answer does not prove that thou hast solved thy problem correctly. This argument convincest none, nay, least of all, thy teacher.
7. Thou shalt first see that thou hast copied thy problem correctly before bearing false witness that the answer book lieth.
8. Thou shalt look back even unto thy youth and remember thy arithmetic.
9. Thou shalt not remove brackets without having regard to the sign.
10. Thou shalt read, mark, learn and inwardly digest the language of mathematics, and verily A's and B's shall follow thee even unto Thy graduation. □

THE LEGEND OF THE APPLE

There are many legends in science, and the most famous is certainly the story of the falling apple that inspired Newton to discover the law of gravity. Legends are often untrue, but the legend of the apple is well-founded.

One of the early biographers of Newton, William Stukeley, related how he visited Newton in his old age.

> After dinner, the weather being warm, we went into the garden and drank thea under the shade of some appletrees; only he, and my self. Amidst other discourse, he told me, he was just in the same situation, as when formerly, the notion of gravitation came into his mind. Why should the apple always fall perpendicularly to the ground, thought he to himself, occasioned by the fall of an apple, as he sat in a contemplative mood.

This is the first written account of the story. The original manuscript of Stukeley's *Memoirs of Sir Isaac Newton's Life* has been published digitally by the Royal Society as part of its 350th anniversary. It may be viewed at `www.royalsociety.org/turning-the-pages`.

In further celebration of its 350th anniversary, in May last year the Royal Society sent a piece of wood from the famous apple tree into orbit in the space shuttle, in the care of British astronaut Dr Piers Sellers.

"Sir Isaac absolutely loved it, I've got to tell you," Sellers joked. "We had him in the window and he got to watch his little wood chip float by and ponder the laws of gravitation and everything. I think it was a treat for him." □

RUBIK IN TWENTY

Rubik's Cube was invented by Hungarian sculptor and professor of architecture Erno Rubik. Since the very earliest days of the Cube it has been known that it can be twisted into 43 252 003 274 489 856 000 different configurations. Many different solution algorithms were quickly found, but the big problem was to work out the minimum number of moves needed to unscramble any cube, where a "move" is any rotation of one of the faces, through any angle.

At the end of 2010 a team of four cube freaks, Morley Davidson, (a mathematician from Ohio's Kent State University), John Dethridge (an engineer at Google in Mountain View, California), Herbert Kociemba (a mathematician from Germany) and Tomas Rokicki (a programmer from Palo Alto, California) gave the answer: any configuration can be unscrambled in at most 20 moves.

In 2008 Rokicki had shown that 22 moves were sufficient. But were 22 moves always necessary? It was known that there were configurations that needed at least 20 moves. Was there a cube that needed 21 or 22 moves?

The team classified the 43×10^{18} configurations into 2.21 billion sets of 19.5 billion positions, and then used symmetry arguments (from a branch of mathematics known as group theory) to reduce the number of configurations before they were checked by computer.

Previous computer algorithms started with the computer implementing a sequence of moves, then checking whether the resulting position was closer to the solution. If not, that sequence of moves was discarded. About 4000 cubes per second could be tackled in this way. Rokicki realised that these dead-end sequences were actually solutions to a different starting position. That insight led him to devise an algorithm that could check one billion cubes per second.

On a standard PC the algorithm would have needed around 35 years to check every configuration. Then Google stepped in and offered the use of their supercomputers. Even then, it took a couple of weeks to run the program.

The conclusion: all cubes can be unscrambled in 20 moves. Most can be solved in 15-19 moves, but a small fraction of configurations (about 300 million) require the full 20 moves. □

POLICE PERCENTAGES

Case Rijsdijk
The Wilderness

It all stems from the government's decision not to release crime statistics.

What this means is that our local police during neighbourhood watch meetings are not allowed to give us numbers, but are allowed to give comparative percentages.

Our local captain gives percentages and then sometimes explains them with some numbers – especially when the percentages appear to rocket! (No-one will report his transgressions!) Now fortunately we live in a relatively crime free area, the most common crime being common assault (drunken brawls!) and theft from motor vehicles (laptops left on seats when you go to the beach!!?? – insurance fraud most likely).

So when the officer reads out the crime "statistics" for the month, below is a typical session:

"Rape is up by 100%" says the officer.

"What?" shouts a member of the public.

"Yes" replies the officer, "there were none last month and 1 this month".

He continues, "Housebreaking is up by 300% – in others words, there were none last month and 3 this month and theft from cars 75%".

"Is that an increase or decrease?" asks someone.

"No, it's a 75% increase from 3 last month to 4 this month" replies the officer. □

SIR JOSEPH'S WILL

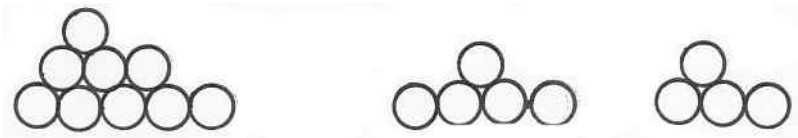
The Last Will and Testament of Sir Joseph Porter K.C.B. directed that £2 000 000 be divided among his 100 female relatives. Each sister was to receive £200 000, each cousin £10 000 and each aunt £2500.

When the lawyers tracked down all 100 ladies they were surprised that the money divided up exactly. How many sisters, cousins and aunts did Sir Joseph have?

The answer is on page 15. □

COIN STACKS

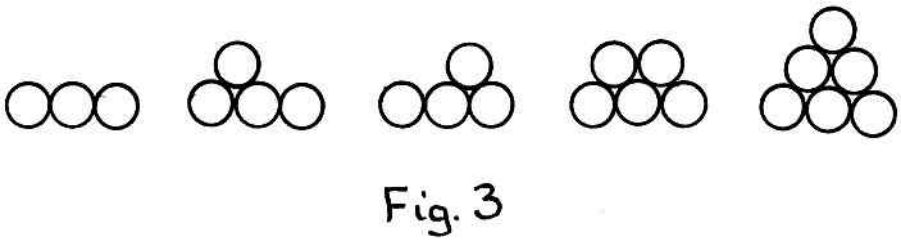
A coin stack is an arrangement of identical coins in rows such that the coins in the bottom row form an unbroken line, and each row of coins above the first forms an unbroken line, with each coin touching two coins in the row below. Some examples are shown.



How many different coin stacks are there, with a given number of coins in the bottom row?

With one coin in the bottom row there is only one coin stack (fig. 1).
With two coins in the bottom row, there are two possible coin stacks (fig. 2).

With three coins in the bottom row, there are 5 possible coin stacks (fig. 3).



How many coins are there in a coin stack with n coins in the bottom row?

Suppose we use c_n to denote this number. We know that $c_1 = 1$, $c_2 = 2$ and $c_3 = 5$, and it is not difficult to work out that $c_4 = 13$ and $c_5 = 34$.

After that, sketching and counting coin stacks gets more tiring. Is there a formula for c_n?

A good idea is to try to work out c_n in terms of the values of its predecessors $c_{n-1}, c_{n-2}, \ldots, c_2, c_1$.

We start with the base row of n coins. It is a coin stack with just one row.

On that base can be placed any coin stack based on $n - 1$ coins.

Or we could put on the base row any coin stack based on $n-2$ coins, in two positions.

Or we could put on the base row any coin stack based on $n-3$ coins, in three ways.

So we see that

$$c_n = 1 + 1 \times c_{n-1} + 2 \times c_{n-2} + 3 \times c_{n-3} + \ldots + (n-1)c_1. \tag{1}$$

Replacing n by $n+1$ throughout gives

$$c_{n+1} = 1 + 1 \times c_n + 2 \times c_{n-1} + 3 \times c_{n-2} + \ldots nc_1. \tag{2}$$

If we subtract these two equations we get

$$c_{n+1} - c_n = c_n + c_{n-1} + c_{n-2} + \ldots c_2 + c_1. \tag{3}$$

Again we replace n by $n+1$ in equation (3) to get

$$c_{n+2} - c_{n+1} = c_{n+1} + c_n + c_{n-1} + \ldots + c_2 + c_1. \tag{4}$$

We finally subtract equation (3) from equation (4) to obtain

$$c_{n+2} - 2c_{n+1} + c_n = c_{n+1}$$

which simplifies to

$$c_{n+2} = 3c_{n+1} - c_n.$$

This formula allows us to calculate the successive values of c_n: 1, 2, 5, 13, 34, 89, 233, ...

These numbers should look familiar: they are all numbers that occur in the famous Fibonacci sequence $1, 1, 2, 3, 5, 8, 13, 34, 55, 89 \ldots$, in which each term after the first two is the sum of the two preceding terms. The coin stack numbers found so far are the first, third, fifth, seventh etc Fibonacci numbers.

That gives rise to an intelligent guess: if F_n denotes the nth Fibonacci number, then

$$c_n = F_{2n-1}.$$

To establish that the pattern continues, all we need do is check that the alternate Fibonacci numbers satisfy the same recurrence relation as the coin stack numbers. We must prove that

$$F_{2n+3} = 3F_{2n+1} - F_{2n-1}.$$

This is not difficult:

$F_{2n+3} = F_{2n+2} + F_{2n+1} = (F_{2n+1} + F_{2n}) + F_{2n+1} = 2F_{2n+1} + F_{2n}$
$= 2F_{2n+1} + (F_{2n+1} - F_{2n-1}) = 3F_{2n+1} - F_{2n-1}$.

The two sequences $\{c_n\}$ and $\{F_{2n-1}\}$ start off with the same values $c_1 = F_1 = 1$, $c_2 = F_3 = 2$ and after that satisfy the same recurrence relation. So they must be identical. □

POWER PATTERNS

Henry Thackeray
University of Pretoria

$$\begin{aligned}
153 &= 1^3 + 5^3 + 3^3 \\
370 &= 3^3 + 7^3 + 0^3 \\
371 &= 3^3 + 7^3 + 1^3 \\
4150 &= 4^5 + 1^5 + 5^5 + 0^5 \\
4151 &= 4^5 + 1^5 + 5^5 + 1^5 \\
54748 &= 5^5 + 4^5 + 7^5 + 4^5 + 8^5 \\
92727 &= 9^5 + 2^5 + 7^5 + 2^5 + 7^5 \\
93084 &= 9^5 + 3^5 + 0^5 + 8^5 + 4^5 \\
194979 &= 1^5 + 9^5 + 4^5 + 9^5 + 7^5 + 9^5 \\
548834 &= 5^6 + 4^6 + 8^6 + 8^6 + 3^6 + 4^6
\end{aligned}$$

□

HEISENBERG'S HOLIDAY

When Heisenberg went on vacation
His colleagues were in consternation.
  For while studies had shown
  That his speed was well known,
His position was pure speculation.

Those colleagues then carried out testing
And found a result interesting.
  They found his position
  With perfect precision
But couldn't be sure he was resting.

□

OCTOBER 823

Last year, October had five Fridays, five Saturdays and five Sundays. A story started going around the internet that this was a highly unusual event, happening only once every 823 years.

Newspaper columnists took up the intriguing story and radio chat show hosts were asked for their comments on the phenomenon. Gullible readers of astrology columns wondered about the planetary significance.

Unfortunately, the story was just plain wrong. A year can start on any one of seven days, and could either be an ordinary year or a leap year. So there are only $7 \times 2 = 14$ different calendars. And you do not have to go back 823 years to find a year with an October with five Fridays, five Saturdays and five Sundays. A mere 11 years turns out to be enough: the calendar page for October 1999 looks just the same as the page for October 2010.

In fact, for the two-hundred-year period from 1901 to 2100 every calendar repeats after exactly 28 years. So 2038, 2066 and 2094 all have Octobers with five Fridays, five Saturdays and five Sundays.

Why does the calendar repeat after 28 years? Because 365 days of a normal year are one day more than 52 weeks, the calendar moves on one day every year. So if your birthday was on a Saturday last year, then this year it will be on a Sunday. Except that, every fourth year, the 29th of February makes the calendar "leap" forward by an extra day. In a period of 28 years there are exactly 7 leap years, so in 28 years the calendar moves on $28+7 = 35$ days, which is a whole number of weeks. Therefore the calendar repeats.

So the special only-once-in-823-years October story was a hoax. Whoever first thought it up must be very proud of the amount of attention, albeit short-lived, that it won. But why did the perpetrator of this successful little fable choose 823? It seems to have no connection with special numbers that come up in calendar calculations, such as 7, or 52 or 365. It's a prime number, to be sure. But does it have any other interesting properties? □

QUOTE

Eternity is very long, especially towards the end.

(Woody Allen) □

THE LEAP YEAR

Thousands of years ago the priests and mathematicians of ancient Egypt calculated that the year is $365\frac{1}{4}$ days long, and constructed their calendar on that basis.

In 63 B.C. an ambitious young politician named Julius Caesar was elected *pontifex maximus* of the Roman Republic. One of his responsibilities was the calendar. At that time the Roman calendar was a mess.

After his military successes, Caesar had the time and power to implement drastic reforms, and decreed in 44 B.C. that the Roman calendar should follow the Egyptian model. To cater for the extra quarter-day per year, Caesar laid down that one day should be added to February every fourth year: the sixth day before the beginning of March, namely 24 February, was to be repeated. This strange arrangement meant that what we know call a *leap year* was originally called a *bissextile year*.

This sensible reform became known as the Julian calendar and was adopted throughout the Roman Empire. It had, however, a basic flaw: it assumed that the year is $365\frac{1}{4}$ days long, a figure that represented the limits of astronomical accuracy at that time. Today we have a far more accurate measurement: 365 days 5 hours 48 minutes 46 seconds, or 365.242199 days. So the Julian year of 365.25 days is .007801 days (about 11 minutes) too long.

Now $\dfrac{1}{.007801} = 128.02$, so that means that an extra day creeps into the Julian calendar every 128 years. This discrepancy could not be detected at first, but as the centuries rolled on, astronomers noticed that the equinoxes were occurring rather earlier than expected. Since the date of Easter was determined in terms of the Spring Equinox, this problem was of great concern to the Church.

It was not until 1582 that the necessary adjustment to the calendar was made, at the decree of Pope Gregory XIII. The key idea behind the Gregorian reform is the arithmetical observation that

$$365 + \frac{1}{4} - \frac{3}{400} = 365.2425$$

which is very close to the true length of the year, 365.2422 days.

This meant that in order to keep the calendar in step with the seasons, three leap years had to be dropped every 400 years. The easiest way to

do this was to declare that years divisible by 100 should not be leap years, except for years divisible by 400. So the year 2000 was a leap year, but 2100 will not be a leap year.

The Gregorian calendar was quickly adopted by Catholic countries in Europe, but Protestant countries refused to accept the decree of the Pope. The infamous massacre by Catholics of thousands of Huguenots in France in 1572 was too fresh in Protestant memories. But gradually they saw the sense of the reform. The United Kingdom, for example, fell into line with Europe in 1752.

The Eastern Orthodox churches held out much longer. Only in 1923 did they agree on a "Revised Julian Calendar", coming up with a slightly different way of determining when a leap year takes place. There should be a leap year every fourth year, as Julius Caesar had laid down, except that centennial years (years divisible by 100) should only be leap years when they leave a remainder of 200 or 600 when divided by 900.

So while the Gregorian calendar drops three leap years every 400 years, making a year of average length $365 + \frac{1}{4} - \frac{3}{400} = 365.2425$ days, the Revised Julian Calendar drops seven leap years every 900, making a year of average length $365 + \frac{1}{4} - \frac{7}{900} = 365.24222$ days. That is much closer to the astronomical year of 365.24199 days.

The year 2000 was a Leap Year in both calendars and 2400 will be the same. But 2800 will be a Leap Year in the Gregorian but not the Revised Julian calendar. What will happen then? Will the Western and Eastern calendars again diverge? We shall have to wait and see. □

PRIZEWINNERS AND SOLUTIONS

Gwyneth Allwright of Uniondale wins a BACKPACK (a complete set of all available back numbers of *Mathematical Digest*) for her solutions of four of the six puzzles on page 31 of the October 2010 edition of the magazine. More BACKPACKS can be won, so the solutions are being held back until the April 2011 edition. □

SIR JOSEPH'S WILL

See page 9: Sir Joseph had 7 sisters, 49 cousins and 44 aunts. □

SHARP CALCULATOR COMPETITION

The Sharp Calculator Competition is sponsored by Sharp Electronics (S.A.) (Pty) Limited, who have donated three calculators as prizes.

These prizes are offered for the three best solutions of the problems below. Solutions may be in English or Afrikaans. Partial solutions will be judged on their merits. The Editor reserves the right to make no award if no satisfactory entries are received, and his judgement will be final. The winner of an nth prize in a previous competition will only be eligible for an mth prize if $m < n$.

Only high school students may take part. Entrants must give their full name and home address, age, school and grade.

The closing date for receipt of entries is 1 March 2011, and the results will be announced in our April 2100 edition.

SHARP

1. Prove that the equation $m^2 - 5n^2 = 1$ has infinitely many solutions in natural numbers.
2. Let $ABCD$ be a quadrilateral with $AD = BC$ and $\angle A + \angle B = 120°$. Let P be a point exterior to the quadrilateral such that P and A lie on opposite sides of the line CD and triangle DPC is equilateral. Prove that triangle APB is also equilateral.
3. A sequence $\{a_n\}$ is defined by

$$a_{n+1} = \frac{a_n + c}{1 - ca_n}.$$

Prove that it is possible to choose values for a_0 and c so that $a_n > 0$ for $n = 0, 1, \ldots, 2010$ but $a_{2011} < 0$. □

X QUOTE

I'm sorry to say that the subject I most disliked was mathematics. I have thought about it. I think the reason was that mathematics leaves no room for argument. If you made a mistake, that was all there was to it.

(Malcolm X) □

FERMAT'S CHRISTMAS THEOREM

All prime numbers leaving a remainder of 1 on division by 4 (i.e. all prime numbers of the form $4n+1$) can be expressed as the sum of two squares.

The first few examples are

$$5 = 1^2+2^2, 13 = 2^2+3^2, 17 = 1^2+4^2, 29 = 2^2+5^2, 37 = 1^2+6^2, 41 = 5^2+5^2$$

It is easy to check that the other prime numbers (those leaving a remainder of 3 on division by 4: 3, 7, 11, 19, 23, 31 etc) cannot be expressed as the sum of two squares.

The French mathematician Albert Girard made this observation in 1632, but gave no proof. His more famous contemporary, Pierre de Fermat claimed in a letter to Marin Mersenne that he had proved the result. Fermat's letter was dated 25 December 1640, and today the result is referred to as Fermat's Christmas Theorem.

Fermat

Mersenne

The result can be extended to composite numbers, all of whose prime divisors are of the form $4n+1$. The reason is a simple algebraic identity that shows that any number which is the product of two numbers, each of which is the sum of two squares, is itself a sum of two squares:

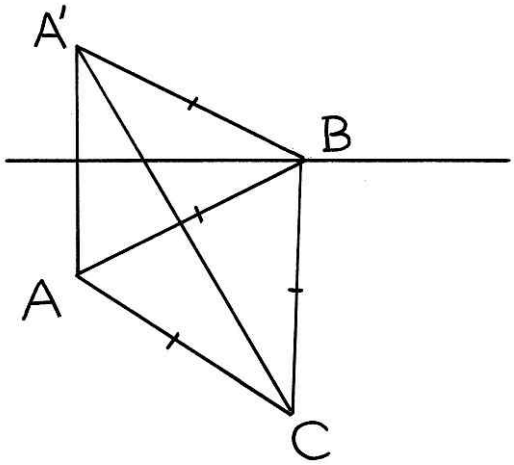
$$(a^2+b^2)(c^2+d^2) = (ab-cd)^2 + (ac+bd)^2.$$

So any number, all of whose prime factors are of the form $4n+1$, is expressible as the sum of two squares.

What about numbers that contain prime factors of the form $4n+3$? $\square$

SHARP SOLUTIONS AND RESULTS

1. *The vertex A of equilateral triangle ABC is fixed, and the vertex B moves along a fixed straight line. Find the path traced out by the vertex C.*



Let A' be the reflection of the point A in the fixed straight line. Suppose that B is to the right of $A'A$. Then $AB = A'B$.

Now consider the circle with centre B through A', A and C. Then $\angle AA'C = \frac{1}{2}\angle ABC = 30^\circ$ (the angle at the centre is twice the angle at the circumference).

So the path traced out by C is a straight line through A' at an angle of 30° to $A'A$.

If B is to the left of $A'A$, the path of C is the reflection of the line above in the line $A'A$.

2. *Find the sum of the series*

$$\sum_{r=1}^{n} \sqrt{1 + \frac{1}{r^2} + \frac{1}{(r+1)^2}}.$$

$$\sum_{r=1}^{n} \sqrt{1 + \frac{1}{r^2} + \frac{1}{(r+1)^2}} = \sum_{r=1}^{n} \sqrt{\frac{r^2(r+1)^2 + (r+1)^2 + r^2}{r^2(r+1)^2}}$$

$$= \sum_{r=1}^{n} \sqrt{\frac{r^4 + 2r^3 + 3r^2 + 2r + 1}{r^2(r+1)^2}} = \sum_{r=1}^{n} \sqrt{\frac{(r^2 + r + 1)^2}{r^2(r+1)^2}}$$

$$= \sum_{r=1}^{n} \left(\frac{r^2 + r + 1}{r(r+1)}\right) = \sum_{r=1}^{n} \left(1 + \frac{1}{r} - \frac{1}{r+1}\right) = n + 1 - \frac{1}{n+1}.$$

3. *If* $\dfrac{(a-b)(c-d)}{(b-c)(d-a)} = k$, *determine the value of* $\dfrac{(a-c)(b-d)}{(a-b)(c-d)}$.

$$\frac{(a-b)(c-d)}{(b-c)(d-a)} = k$$
$$\Rightarrow (a-b)c - d(a-b) = k(b-c)d - k(b-c)a$$
$$\Rightarrow d(k(b-c) + (a-b)) = (a-b)c + k(b-c)a$$
$$\Rightarrow d = \frac{(a-b)c + k(b-c)a}{(a-b) + k(b-c)}.$$

So $b - d = b - \dfrac{(a-b)c + k(b-c)a}{(a-b) + k(b-c)}$

$$= \frac{b(a-b) + kb(b-c) - (a-b)c - k(b-c)a}{(a-b) + k(b-c)}$$
$$= \frac{(a-b)(b-c) + k(b-c)(b-a)}{(a-b) + k(b-c)}$$
$$= \frac{(a-b)(b-c)(1-k)}{(a-b) + k(b-c)}. \tag{1}$$

Similarly $c - d = c - \dfrac{(a-b)c + k(b-c)a}{(a-b) + k(b-c)}$

$$= \frac{c(a-b) + kc(b-c) - (a-b)c - k(b-c)a}{(a-b) + k(b-c)}$$
$$= \frac{k(b-c)(c-a)}{(a-b) + k(b-c)}. \tag{2}$$

Using (1) and (2) we have $\dfrac{(a-c)(b-d)}{(a-b)(c-d)}$

$$= \frac{a-c}{a-b} \cdot \left(\frac{(a-b)(b-c)(1-k)}{(a-b) + k(b-c)}\right) \cdot \left(\frac{(a-b) + k(b-c)}{k(b-c)(c-a)}\right) = \frac{k-1}{k}.$$

SHARP

No prizeworthy entries were received. □

DID YOU KNOW?

On a scale in which the radius of the earth's orbit around is sun is represented by one inch, a light year is represented by one mile. □

2010 COMPUTER OLYMPIAD RESULTS

The 2010 Computer Olympiad was dominated by Western Cape students. For the first time in its 27-year history a single province took all the medals in the final round.

The Standard Bank/CSSA Computer Olympiad is an annual competition to identify, encourage and reward programming skills among high school pupils. The competition attracted 38 000 entries, of whom 3500 went on to a second round. The top sixteen were then invited to the final round, which took place at the University of Cape Town during the weekend of 2 and 3 October. The winners were announced at a gala function at the Kelvin Grove club on Monday 4 October.

The runaway winner of the Olympiad Gold Medal was Sean Wentzel (grade 11, Westerford High School). Sean won R35 000 in prize money for himself and R4 000 for his school. A regular participant in Maths and Science Olympiads, Sean represented South Africa at the International Mathematics Olympiad in Kazakhstan in July 2010 and the International Olympiad in Informatics in Canada in August 2010. He came home from both Olympiads with a Bronze Medal.

Computer Olympiad Silver Medals went to Kieren Davies (grade 12, International School of Hout Bay) and Vaughan Newton (grade 11, Bishops). Bronze Medals went to Ashraf Moolla (grade 11, Rondebosch Boys' High School), Robert Spencer (grade 10, Westerford High School) and Bennie Swart (grade 12, Bellville High School).

The dates for the 2011 Computer Olympiad are:
1st Round: Friday 11 March 2011 (at local school)
2nd Round: Friday 13 May 2011 (at local school)
3rd Round: Saturday 3 and Sunday 4 September 2011 (at UCT)
Awards Function: Monday 5 September 2011

The next International Olympiad in Informatics will take place in Thailand, July 22–29 2011.

Enquiries:
Peter Waker, Manager: S.A. Computer Olympiad
P.O. Box 13013
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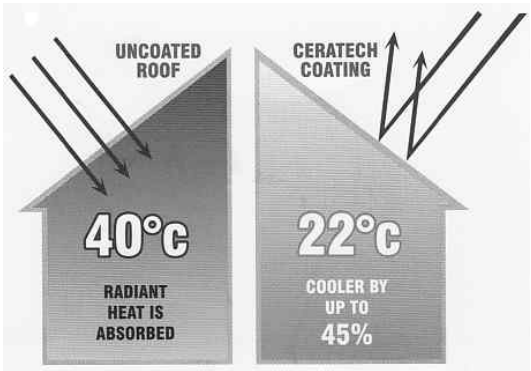
Email: `info@olympiad.org.za` □

PERCENTAGE ERROR

A flier arrived in the Editor's mailbox recently, promoting a roof coating that will help to keep the house cool.



In the house with an uncoated roof the radiant heat is absorbed and the temperature rises to 40°C. With the coating, heat is reflected and the temperature is 22°C.

That, according to the advertisement, means that the house is "cooler by up to 45%".

Any use of a percentage in an advertisement is worth checking. The drop in temperature from 40 to 22 is 18, and $\frac{18}{40} = 45\%$. So the numbers are correct.

But does it make sense to measure a drop in temperature by a percentage?

The problem with the measurement of temperature is that there are different scales. In this case the temperatures were given in Celsius degrees. Suppose that the temperatures are given in Fahrenheit degrees instead.

Since 0°C = 32°F (freezing point of water) and 100°C = 212°F (boiling point of water) a temperature of x°C translates to $(9x/5 + 32)$°F. So 40°C = 104°F and 22°C = 71.6°F. The drop in temperature (in degrees Fahrenheit) is then 32.4 and the percentage drop is $\frac{32.4}{104} = 31\%$. That is not as impressive as the 45% drop on the Celsius scale.

The drop is even less on the Kelvin scale.

To convert from Celsius to Kelvin, just add 273. The two temperatures are now 313°K and 295°K. With a drop of 18, the percentage drop is a mere $\frac{18}{313} = 5.75\%$.

Conclusion: it is meaningless to use percentages to describe temperature changes. □

PUZZLE PAGE

Blockout

Divide the 7×7 grid into 14 blocks. Each block must be either a rectangle or a square, and must contain just one of the numbers shown, which gives the number of squares in the block.

						5
	4		2	3		
2				3		
			3		4	
		5			2	
	3			6		
4			3			

Six Sums

Fill in numbers into the blank squares so that the six calculations shown are correct.

9	+		−		= 3
+	■	×	■	+	
	−		+		= 2
÷	■	−	■	÷	
	+	5	÷		= 3
= 4		= 7		= 3	

Crossnumber

ACROSS
1. A Fibonacci number
2. A square
3. A perfect number

DOWN
1. A square
2. A cube
3. A multiple of 7

1	2	3
2		
3		

Six Squares

The integers 1, 3, 8 and N have the property that the product of any two of them, increased by 1, is a perfect square. What is the value of N?

Square Plot

A point P lies inside a square $ABCD$ If $AP = 1$, $BP = 2$ and $CP = 3$, determine DP.

Answers are on page 28. □

OLYMPIAD PROBLEMS

For those hoping to make it into the third round of the South African Mathematics Olympiad, here are some warm-up problems. Each solution is quite simple (when you see it, of course) and also contains an important idea which could be used in other problems.

1. In triangle ABC, the bisector of $\angle A$ meets BC in D. Prove that
$$\frac{AB}{AC} = \frac{BD}{DC}.$$
2. Prove that the equation
$$x^4 - 5x^3 - 4x^2 - 7x + 4 = 0$$
has no negative solutions.
3. Simplify
$$\sqrt[3]{2+\sqrt{5}} + \sqrt[3]{2-\sqrt{5}}.$$
4. The sides a, b and c of a triangle satisfy the equation
$$a^2 + b^2 + c^2 = ab + bc + ca.$$
Prove that the triangle is equilateral.
5. Which is larger: $\sqrt[8]{8!}$ or $\sqrt[9]{9!}$?

Solutions are on page 29. □

SHAMPOO SHAM

Advertisers in the press, radio and television are prohibited by law from making false claims. But they are not prohibited from making empty claims.

Everybody must have seen the television advertisement for a shampoo that is guaranteed to give you hair that is "up to 100% dandruff-free". But have you thought what that numerical claim could mean?

Think of it this way. Suppose you use the shampoo diligently for some months and after that your dandruff has been reduced by a mere 10%. Do you have cause for a complaint of misleading advertising? Regrettably, none at all, since any percent less than one hundred is covered by those clever little words "up to".

The same shampoo, in the same television commercial, is said to be "recommended by 86% of dermatologists". Now there's a definite figure, not qualified by a sneaky little "up to". There's a certain authenticity attached to that figure of 86%. It's not the same as the classic advertisement of some years ago for a soap that was claimed to be chosen by "nine of ten of film stars". Nobody would believe that sort of naive stuff today. But 86% sounds more scientific, more exact, than "nine out of ten", and on top of that dermatologists are surely more credible than film stars even when it comes to shampoo (which is only a liquid soap after all).

But when you think that the figure of 86% could just mean that six out of seven dermatologists recommended the shampoo (after all, $6/7 = 0.86$ to two decimal places) the claim starts losing its impact. And why, one asks, did that seventh dermatologist *not* recommend the shampoo? □

PRIMES AND POWERS

Can every odd number greater than one be expressed as the sum of a prime number and a power of 2?

It's quite easy to check that the result is true for small numbers:

$3 = 2 + 2^0$, $5 = 3 + 2^1$, $7 = 3 + 2^2 = 5 + 2^1$, $9 = 5 + 2^2 = 7 + 2^1$, $11 = 3 + 2^3 = 7 + 2^2$, $13 = 5 + 2^3 = 11 + 2^1$, etc.

However, the result is not always true. What is the first odd number that *cannot* be expressed as the sum of a prime number and a power of 2? □

THE GUNFIGHTERS' PARADOX

Three gunfighters Tom, Dick and Harry agree that Tombstone is not big enough for more than one of them, so they agree to fight a three-way duel. They stand some distance apart and will fire at each other in turn, drawing lots to decide in which order they will shoot.

Tom and Dick never miss, but Harry only hits his target half the time.

What strategies should each of them adopt? Who has the best chance of surviving?

If Tom gets the first shot, he should shoot Dick. For if he shoots Harry, then Dick shoots him with the next shot. Similarly, if Dick has the first shot, he should shoot Tom.

The next shot goes to Harry, and the survivor of the first shot then has a 50% chance of surviving Harry's shot.

What should Harry do? He should not try to kill Tom or Dick, because even if he succeeds he would then be killed by the survivor. And if he fails, he is killed anyway. So Harry's best option is to fire into the air.

If the three gunfighters adopt this strategy, then Harry has the best chance of surviving (50%) while Tom and Dick have equal chances of surviving: 25%. □

THE GREENHOUSE EFFECT

Everybody knows about "the greenhouse effect. What is not commonly known is that the term was coined by a mathematician nearly two hundred years ago.

A pioneer in the mathematical analysis of heat flow, the great French mathematical physicist Jean Baptiste Joseph Fourier (1768–1830) used the term in a paper published in 1824, entitled *Remarques générales sur les températures du globe terrestre et des espaces planétaires*, in the journal Annales de Chemie et de Physique.

□

FIBONACCI MEETS PYTHAGORAS

The Fibonacci numbers

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89 \cdots$$

start with the first two terms equal to 1, and thereafter each term is the sum of its two predecessors.

If the nth Fibonacci number is denoted by F_n, then

$$F_1 = F_2 = 1 \quad \text{and} \quad F_{n+2} = F_{n+1} + F_n \quad \text{for} \quad n \geq 1.$$

The Fibonacci numbers can be used to generate Pythagorean triangles: right-angled triangles with sides of integer lengths a, b and c such that $a^2 + b^2 = c^2$, since the numbers

$$F_n F_{n+3}, \quad 2F_{n+1}F_{n+2}, \quad F_{n+1}^2 + F_{n+2}^2$$

satisfy the Pythagorean relationship $a^2 + b^2 = c^2$.

For example:

with $n = 1$ we get the triple 3, 4, 5;

with $n = 2$ we get the triple 5, 12, 13;

with $n = 3$ we get 16, 30, 34 (which is just double the 8, 15, 17 triangle);

with $n = 4$ we get the triple 39, 80, 89.

To check that this process always works, we need to check that

$$(F_n F_{n+3})^2 + (2F_{n+1}F_{n+2})^2 = (F_{n+1}^2 + F_{n+2}^2)^2.$$

This is not difficult:

$$\begin{aligned}
&(F_{n+1}^2 + F_{n+2}^2)^2 - (2F_{n+1}F_{n+2})^2 \\
&= (F_{n+1}^2 + F_{n+2}^2 - 2F_{n+1}F_{n+2})(F_{n+1}^2 + F_{n+2}^2 + 2F_{n+1}F_{n+2}) \\
&= (F_{n+1} - F_{n+2})^2 (F_{n+1} + F_{n+2})^2 \\
&= (F_n F_{n+3})^2
\end{aligned}$$

where the basic relation between three successive Fibonacci numbers is used in the last line. $\square$

ROOT PATTERN

$\sqrt{11-2} = 3$

$\sqrt{1111-22} = 33$

$\sqrt{111111-222} = 333$

$\sqrt{11111111-2222} = 3333$

$\sqrt{1111111111-22222} = 33333$

$\sqrt{111111111111-222222} = 333333$

$\sqrt{11111111111111-2222222} = 3333333$

Explanation:

$$\sqrt{\frac{10^{2n}-1}{9} - 2\Big(\frac{10^n-1}{9}\Big)}$$
$$= \tfrac{1}{3}\sqrt{10^{2n}-1-2(10^n-1)}$$
$$= \tfrac{1}{3}\sqrt{10^{2n}-2\times 10^n+1}$$
$$= \tfrac{1}{3}(10^n-1)$$

□

ERRATA

There were several mistakes in *Mathematical Digest* (October 2010).

Rational Numbers, page 4: At the right hand end of a line a bit more than half way down the page one reads about "six digits 42857". That should be 428571.

Percentage Error, page 13: The expression at the end of the last line should be $(n+1)A$.

Number Patterns, page 17: The 10's in the first set of equations should be 2, 3, 4, etc. And in the third set of equations, there's a 4 missing on the right hand side of the fifth equation.

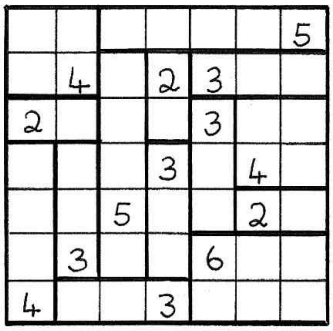
The Sydharb, page 3: A *sydharb* is approximately 500 million (not billion) cubic metres.

Our thanks to readers Henry Thackeray, David Allison and Marlene Steenkamp's grade 9 class at Hoërskool President for pointing out these errors.

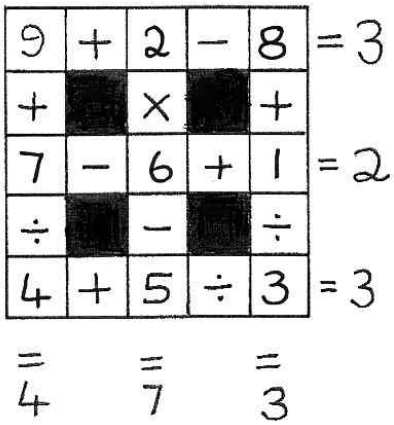
□

PUZZLE ANSWERS

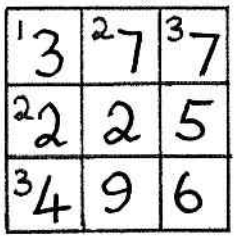
Blockout



Six Sums

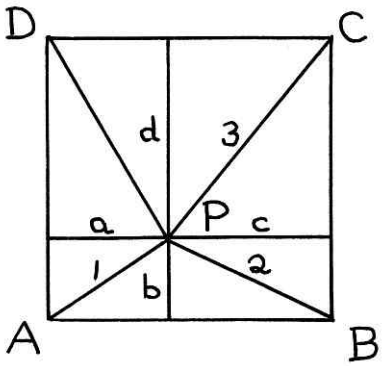


Crossnumber



Six Squares $N = 120$

Square Plot



$AP^2 = a^2 + b^2 = 1$
$BP^2 = b^2 + c^2 = 4$
$CP^2 = c^2 + d^2 = 9$
$DP^2 = d^2 + a^2 = (a^2 + b^2) - (b^2 + c^2) + (c^2 + d^2) = 1 - 4 + 9 = 6.$
So $DP = \sqrt{6}$. $\square$

SOLUTIONS TO OLYMPIAD PROBLEMS

1. Consider the areas of triangles ABD and ADC with bases BD and DC (figure 1). Since they have the same altitude, it follows that

$$\frac{\text{Area}\, BDA}{\text{Area}\, DCA} = \frac{BD}{DC}.$$

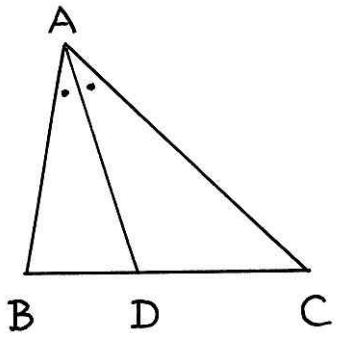


Figure 1

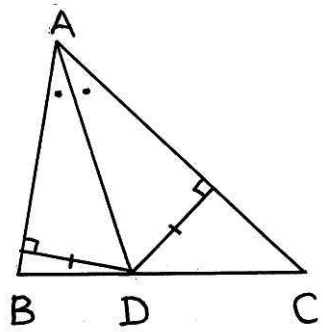


Figure 2

Now drop perpendiculars from B and C onto AC and AB respectively (figure 2). Since AD bisects $\angle BAC$, the two perpendiculars have the same length. Putting it another way, the two triangles ABD and ADC have the same perpendicular heights from AB and BC, respectively. It follows that

$$\frac{\text{Area}\, BDA}{\text{Area}\, DCA} = \frac{AB}{AC}.$$

So

$$\frac{BD}{DC} = \frac{AB}{AC}.$$

2. $x^4 - 5x^3 - 4x^2 - 7x + 4 = (x^2 - 2)^2 - 5x^3 - 7 > 0$ for all negative values of x. So there are no negative solutions.

3. Let $X = \sqrt[3]{2+\sqrt{5}} + \sqrt[3]{2-\sqrt{5}}$.
Cube both sides, using the result that $(a+b)^3 = a^3 + b^3 + 3ab(a+b)$:
$X^3 = (2+\sqrt{5}) + (2-\sqrt{5}) + 3\sqrt[3]{2+\sqrt{5}}\sqrt[3]{2-\sqrt{5}}(\sqrt[3]{2+\sqrt{5}} + \sqrt[3]{2-\sqrt{5}})$
$= 4 + 3\sqrt[3]{-1}X = 4 - 3X$.
So we have the equation $X^3 + 3X - 4 = 0$, which factorizes to $(X-1)(X^2+X+4) = 0$, from which it follows that $X = 1$.

4. The equation is equivalent to $\frac{1}{2}(a-b)^2 + \frac{1}{2}(b-c)^2 + \frac{1}{2}(c-a)^2 = 0$, from which it follows that $a = b = c$.

5. Raise both to the power of 72: $(\sqrt[8]{8!})^{72} = (8!)^9$ and $(\sqrt[9]{9!})^{72} = (9!)^8$. Then $(9!)^8 - (8!)^9 = (8!)^8(9^8 - 8!) > 0$. So $\sqrt[8]{8!} < \sqrt[9]{9!}$. $\square$

MATHS PUNS

The fattest knight at King Arthur's round table was Sir Cumference. He acquired his size from eating too much pi.

A piece of writing paper is an ink-lined plane
An inclined plane is a slope up.
A slow pup is a stupid dog.
That proves a piece of writing paper is a stupid dog.

A psychic quadrilateral is one in which each angle knows the size of the angle opposite it.

Rulers, compasses and protractors will be banned in schools because they are weapons of maths instruction.

When Noah sent two snakes out from the Ark with the instruction to "Go forth and multiply" they said they couldn't because they were adders. So Noah gave them hammer and nails and a saw and told them to construct log tables.

In an Indian reservation an anthropologist observed a woman sitting outside her wigwam on a hippopotamus skin. She was watching over two little boys whose mothers had sat them down on two buffalo skins nearby. The woman weighed 50 kg and the two boys each weighed 25 kg. The anthropologist wrote in his journal: "The squaw on the hippopotamus is equal to the sons of the squaws on the other two hides."

In an eternal triangle all angels are equal.

There was the person who sent ten puns to friends, with the hope that at least one of the puns would make them laugh. No pun in ten did.

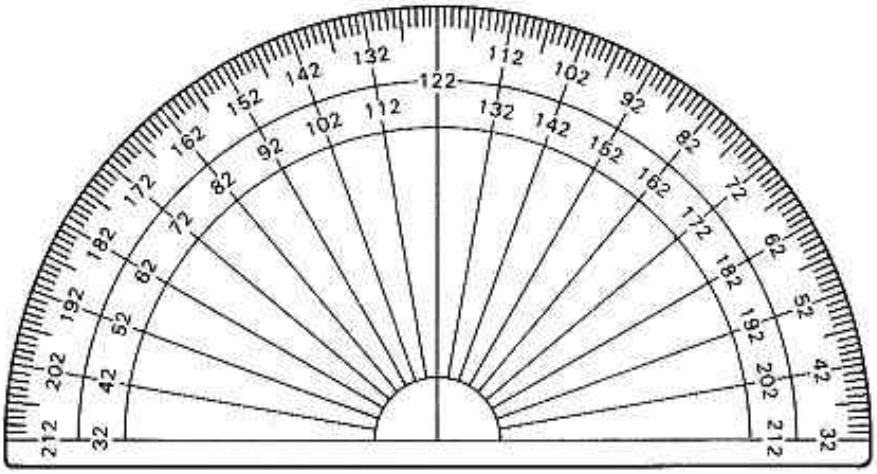
Send your mathematical puns to the Editor.
The worst examples will win a BACKPACK: a set of all available back numbers of *Mathematical Digest*. □

THE TETRAHEDRON

The Tetrahedron: a new member of the percussion family which brings an extra dimension into musical interpretation.

(With apologies to Gerard Hoffnung.) □

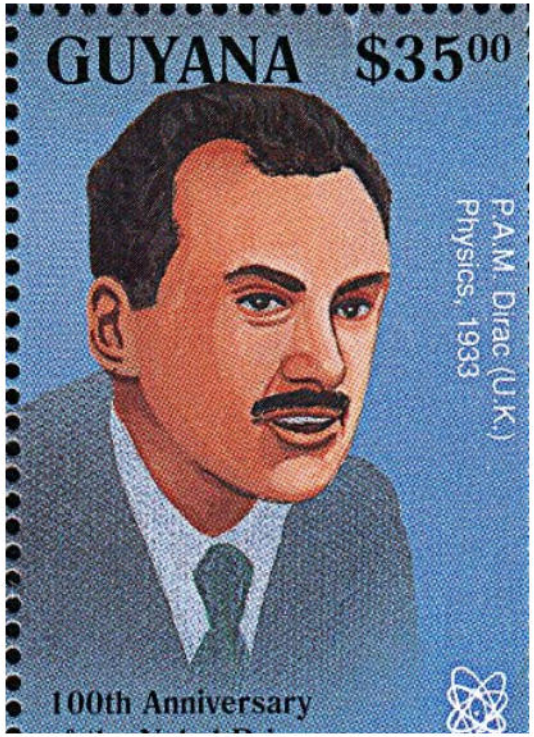
THE FAHRENHEIT PROTRACTOR



(W. Wynne Wilson) □

mathematical digest

No. 163 April 2011



Paul Dirac (1902–1984)

mathematical digest

Number 163 **April 2011**

It seems that if one is working from the point of view of getting beauty in one's equations, and if one has really a sound insight, one is on a sure line of progress. If there is not complete agreement between the results of one's work and experiment, one should not allow oneself to be too discouraged, because the discrepancy may well be due to minor features that are not properly taken into account and that will get cleared up with further development of the theory.

(Paul Dirac)

Department of Mathematics and Applied Mathematics
University of Cape Town

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Cover: Paul Dirac (see page 2)

GOOGLE AND MATHEMATICS

Simon Hampton
Director of Public Policy, Google

Maths is very important to Google. It's the basis of everything we do: from the algorithms that deliver answers to your search queries, to the way in which your Gmails are grouped in conversations, to the technology advances which are enabling us to develop driverless cars. It's so important to us, that our founders named the company after a very large number — a Googol.

Maths is given a lot of importance by policy makers too — its contribution to creativity and innovation was celebrated in some of the events organised by the European Union back in 2009. And in December last year, the latest OECD PISA report and league tables highlighted how well (or not) Europe's various national education systems are performing in the core educational fields of reading, mathematics and science.

Over the last few months, our attention's been drawn to another mathematical league table — the one that tracks the winners of the International Mathematical Olympiad, first held in Romania in 1959. The IMO is the world championship of secondary school mathematics, designed to test ingenuity and insight and tax the sharpest minds in the world.

Google has always encouraged and supported excellence in the sciences, whether in its own staff, or in its work with academics around the world. And so yesterday* together with the Advisory Board of the International Mathematical Olympiad, we were proud to announce that we are making a gift of one million euros to the organisation to help cover the costs of the next five global events.

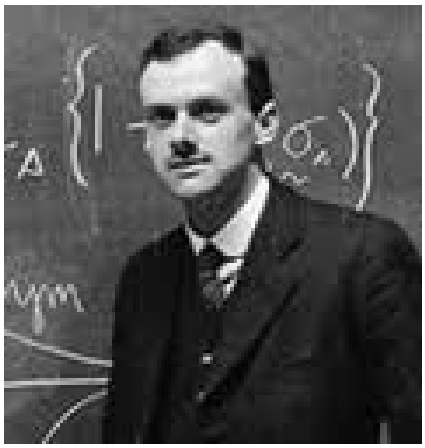
We're delighted that we can help the IMO bring young mathematicians from around the world together to celebrate a shared passion, to push themselves and to compete to be the best in the world. This year's event takes place in Amsterdam in July and we'll be watching out for the results with particular interest.

And I don't doubt that quite a few Googlers will also spend some time in coming months trying solve the sort of mathematical challenges that will be put to the world?s young Maths Olympians.

(*Blogpost, 21 January 2011) □

THE STRANGEST MAN

The Strangest Man: the hidden life of Paul Dirac, quantum genius. By Graham Farmelo (Faber and Faber, 2009)



Paul André Maurice Dirac (1902-1984) was born in Bristol and graduated from Bristol University as an electrical engineer. He then won a scholarship at St John's College, Cambridge where he earned a PhD in 1928. He was the Lucasian Professor of Mathematics at Cambridge (the Chair held by Isaac Newton) from 1932 to 1969.

Paul Dirac was one of the greatest theoretical physicists of the 20th century, second only to Albert Einstein. His towering achievement was the discovery in 1928 of an equation that provided a description of elementary particles, such as electrons, for the first time accounting for relativity in the context of quantum mechanics. Dirac's quantum theory explained the hydrogen spectrum perfectly and predicted the existence of antimatter (the positron) by quantum theory. The positron was first observed in 1932 and in the next year Dirac was awarded the Nobel Prize for Physics for his role in its discovery. Today, Positron Emission Tomography (PET) scanning is a valuable tool in medicine.

Quantum theory underpins physics and chemistry. It explains the behaviour of fundamental particles, why electricity flows through copper but not wood and why the laser works. It is the foundation of electronics.

Dirac's landmark textbook, *Principles of Quantum Mechanics*, published in 1930, is still used today. In that book, Dirac showed how to marry the matrix mechanics of Werner Heisenberg and the wave mechanics of Erwin Schrödinger into a single theory based on the notion of an infinite-dimensional Hilbert space.

Graham Farmelo's magnificent biography of Dirac describes the blossoming of theoretical physics in the first half of the 20th century, with Dirac at the centre. It was a troubled period in world history: two World Wars bracketing a catastrophic economic depression and the rise of Hitler and Stalin, followed by the Cold War. Against this dramatic background Farmelo unfolds the life story of "the strangest man", Paul Dirac.

Dirac did not fit easily into society. His tyrannical father, a Swiss emigré, forced him to speak only French at mealtimes, and Dirac's taciturnity in adulthood became legendary. His colleagues in Cambridge invented a unit to measure conversational participation: a rate of one word per hour was defined to be *one dirac*. He was naive in society and awkward with strangers, who may have thought him dismissive, even rude. One of the most famous stories about Dirac relates how, at the end of a public lecture, a member of the audience commented that he did not understand a certain part of the discussion. A long and awkward pause followed. When the Chairman asked whether he was going to answer the question, Dirac replied "It was not a question." His response (literal, and not intended to be rude), would today be seen as a symptom of autism.

Given his unemotional and withdrawn nature (also characteristic of autism), it is surprising that Dirac married, and given his childhood in an unhappy home, it is more surprising that the marriage was happy and lifelong. What undoubtedly helped was the fact that his wife, Manci, the sister of the Hungarian physicist Eugene Wigner, was an ebullient woman and a loving wife.

During World War II, Dirac returned to his engineering roots and conducted important research on uranium enrichment by gas centrifuge.

After WWII he travelled extensively, and after his retirement at age 67 from the Lucasian Professorship he moved to the USA, where he was offered the position of Visiting Eminent Professor at the State University of Florda. He died in Tallahassee in 1984, where he is buried.

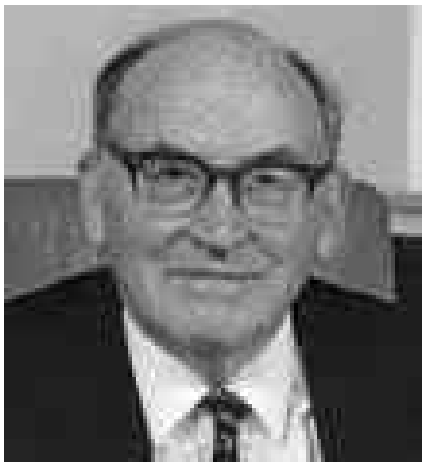
Dirac was a modest man, and refused the offer of a knighthood. In 1973 he was given the highest honour that the Queen may bestow: the Order of Merit. Only 24 people may hold this award at any given time.

In 1995 a stone commemorating Dirac and featuring his groundbreakinq equation was unveiled in Westminster Abbey, next to Newton's gravestone and close to Darwin's.

Graham Farmelo's biography of Dirac is compelling reading, and is highly recommended. □

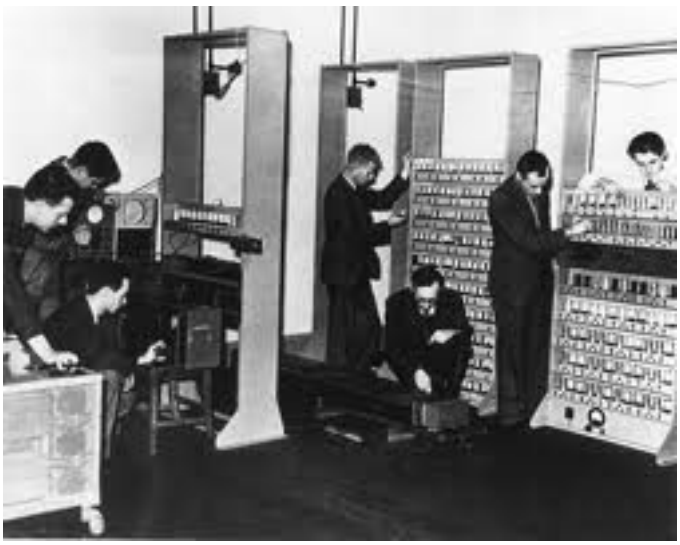
MAURICE WILKES AND THE EDSAC

Sir Maurice Wilkes, the "father of computing" in the United Kingdom, died in November 2010 at the grand old age of 97.



After graduating with a First in Mathematics at Cambridge, he earned a PhD in 1938 in Physics. His research on the propagation of radio waves meant that when war broke out the following year he was drafted to work with the radar pioneer Robert Watson-Watt in developing British air defences. After the war he returned to Cambridge and became the head of the Cambridge Mathematical Laboratory, a post he held till 1970.

Visiting the USA in 1946 Wilkes met the computer pioneers Presper Eckert and John Mauchly. There he learned about ENIAC, (the "Electronic Numerical Integrator and Computer"), which had been designed to calculate shell trajectories for the US army. On the return voyage on the Queen Mary Wilkes started designing what was to become the first stored-program digital computer, EDSAC (the "Electronic Delay Storage Automatic Computer".



EDSAC

In an era before the invention of transistors, EDSAC used 3000 thermionic valves for computation. Memory units were mercury-filled tubes through which beams of sound were directed. Input was via punched tape and

output was by means of a teleprinter. At 650 operations per second, it was amazingly fast compared with electro-mechanical calculators of the day.

EDSAC was completed in early 1949. It covered a floor area of 20 square metres and stood 2 metres high. Wilkes and a student assistant wrote a simple program to print a table of squares and on 6 May 1949 the program was loaded into the computer on punched tape. Half a minute later the teleprinter printed out $1, 4, 9, 16, 25, \ldots$. The world's first functioning stored-program computer was up and running.

The world's first scientific publication using computer calculations (a paper on genetics by RA Fisher) was completed with the help of EDSAC. EDSAC was also used by South African Nobel prizewinners Aaron Klug and Sidney Brenner in their work on molecular biology.

EDSAC remained in operation until 1958 when it was replaced by EDSAC II, with the mercury delay lines replaced by magnetic storage. EDSAC II was the first computer to have a micro-programmed control unit.

Wilkes also pioneered software development, and built one of the first high-speed distributed computing networks, the Cambridge Ring, long before networking became fashionable. The network was originally designed to avoid the need for a printer at every computer.

Maurice Wilkes was the first president of the British Computer Society, a fellow of the Royal Society, and a fellow of the Royal Academy of Engineering. His many awards included the Faraday medal from the Institution of Electrical Engineers in London and a knighthood. He died on 29 November 2010.

In memory of Maurice Wilkes and other computer pioneers at the University of Cambridge, a working replica of EDSAC is to be built by the Computer Conservation Society at The National Museum of Computing at Bletchley Park, the British WWII code-breaking centre.

Professor Andrew Hopper, Head of the Computer Laboratory at Cambridge University, said: "EDSAC set computing standards for academia and commerce. It was so successful that in the nine years following 1949 it was used by Cambridge University researchers in studies such as genetics, meteorology and X-ray crystallography and even helped two researchers win Nobel prizes. EDSAC also led directly to the first commercially applied computer, the LEO, that broke new ground by enabling the catering company J Lyons & Co. Ltd. to perform payroll calculations in 1953." □

AN INTERESTING CYCLIC QUADRILATERAL RESULT

Michael de Villiers
University of KwaZulu-Natal

A few months ago I experimentally discovered the following interesting result as shown in Figure 1 using the dynamic geometry software *Sketchpad:* "The respective intersections E, F, G and H of the angle bisectors of angles A, B, C and D of a cyclic quadrilateral $ABCD$, with the circumcirle, form a rectangle."

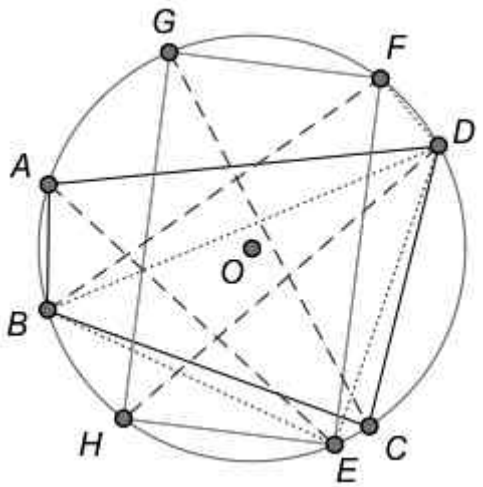


Figure 1

The result follows directly from a useful Lemma proved on p. 190 of my book *Some Adventures in Euclidean Geometry* in the process of proving another result. The Lemma basically states that the angle bisector of $\angle BAD$ intersects the circumcircle at E, the midpoint of arc $BHCD$. Let us now prove the Lemma as well as the desired result.

Proof
In Figure 1, connect E with B and D, and B with D. Since $\angle BAE = \angle DAE$, it follows that chord BE = chord DE (equal angles subtend equal chords). Hence, E is the midpoint of arc $BHCD$. Similarly, G is the midpoint of arc $BAFD$. Hence, E and G are diametrically opposite each other and lie on the diameter through O. Similarly, points F and H lie on the same diameter through O. Thus, since its diagonals are equal and bisect each other, $EFGH$ is a rectangle.

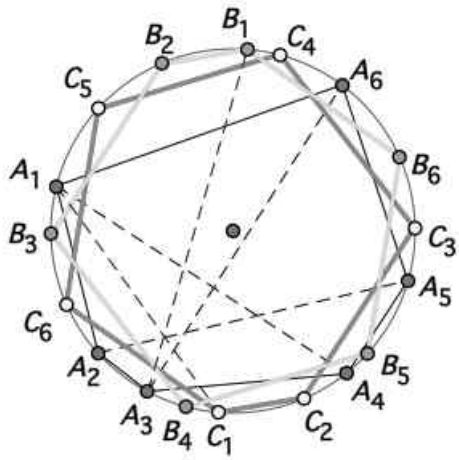


Figure 2

Further generalization

The result can be further generalized and applied to a cyclic hexagon by subdividing a cyclic hexagon $A_1A_2A_3A_4A_5A_6$ into six cyclic quadrilaterals $A_1A_2A_3A_4$, $A_2A_3A_4A_5$, $A_3A_4A_5A_6$, $A_4A_5A_6A_1$, $A_5A_6A_1A_2$ and $A_6A_1A_2A_3$. In quadrilateral $A_1A_2A_3A_4$, construct the respective angle bisectors of $\angle A_2A_1A_4$ and $\angle A_4A_3A_2$ to intersect the circumcircle respectively at C_1 and B_1 as shown in Figure 2.

Then from the aforementioned result, C_1 and B_1 are diametrically opposite points lying on the diameter through the centre. Therefore, repeating the same constructions for the other five cyclic quadrilaterals in a cyclic fashion, we obtain two hexagons $B_1B_2B_3B_4B_5B_6$ and $C_1C_2C_3C_4C_5C_6$ with corresponding vertices diametrically opposite each other. In other words, the one hexagon can be given a half-turn (a rotation by 180°) to map onto the other.

Hence, if we connect any set of 12 corresponding points from these two hexagons in a cyclic fashion, we obtain a 12-gon with opposite sides equal and parallel, since the configuration has half-turn symmetry. For example, $B_1B_2C_5B_3C_6B_4C_1C_2B_5C_3B_6C_4$, and other similar sets, all have half-turn symmetry, and therefore opposite sides are equal and parallel.

In a similar fashion, the result can be further generalized to cyclic octagons, cyclic decagons, etc., by respectively subdividing them into hexagons, octagons, etc., but that's left as an exercise to the reader.

Reference

De Villiers, M. (2009). *Some Adventures in Euclidean Geometry.* Lulu Publishers. (Available as PDF download or in printed book form at http://www.lulu.com/content/7622884). □

ST IVES AND THE EGYPTIAN CONNECTION

As I was going to St Ives
I met a man with seven wives.
Each wife had seven sacks,
Each sacks had seven cats,
Each cat had seven kits:
Kits, cats, sacks and wives,
How many were going to St Ives?

This popular riddle was composed over 200 years ago. What is the answer? You seem to be asked to add up the number of kits, cats, sacks and wives. Without thinking too hard, you start doing some arithmetic:

There were 7 wives, $7 \times 7 = 49$ sacks, $7 \times 7 \times 7 = 343$ cats and $7 \times 7 \times 7 \times 7 = 2401$ kits. So we add up kits, cats, sacks and wives to get $7 + 49 + 343 + 2401 = 2800$ items going to St Ives.

But that's only one possible answer to the problem. After, the polygamous man in charge of the party was also going to St Ives, making a total of 2801 in the party.

Of course, if you know something about geometric series, you will know the formula which gives the answer quickly:

$$1 + 7 + 7^2 + 7^3 + 7^4 = \frac{7^5 - 1}{7 - 1} = 2801.$$

But everybody knows that this is a catch question, and that the man and his wives and their sacks stuffed with moggies were not travelling to St Ives, since they met somebody who was.

So there was only one person going to St Ives, the narrator.

On the other hand, the riddle seems to ask how many "kits, cats, sacks and wives" were going to St Ives, and if they were travelling away from St Ives, then the answer to the riddle is "none".

However, that is not the only interpretation of "met". It could be that the lone traveller overtook the man with his seven wives and their luggage on the road to St Ives. Or maybe they were travelling along two different roads to St Ives and met where the roads joined. In either case, the number going to St Ives was 2800, if you just count kits, cats, sacks and wives, or 2801 if you count the ladies' husband, or 2802 if you include the narrator.

Finally, the nursery rhyme does not specify that the narrator was travelling alone to St Ives. Without clarification on that point, the riddle can't be answered.

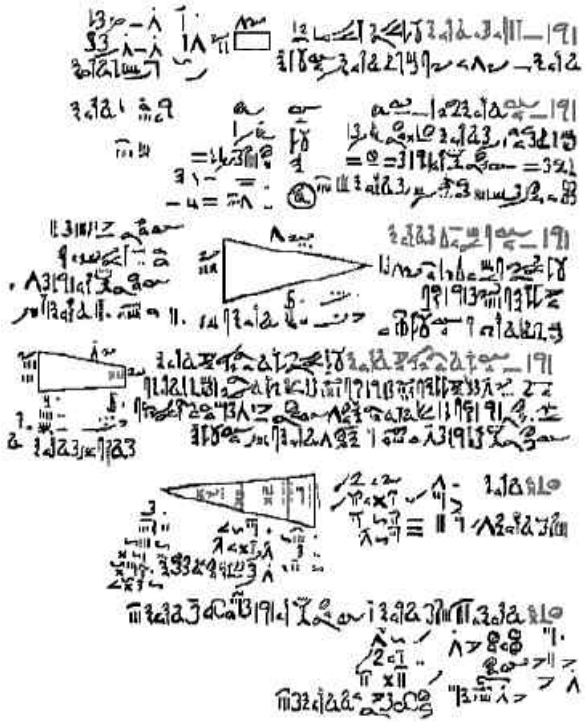
This shows how an amusing little nursery rhyme riddle can be turned into an exercise in tedious nitpicking.

What is far more interesting, however, is that the St Ives problem (without the riddle twist) can be traced back to ancient Egypt. The Rhind Mathematical Papyrus, which dates from 1650 B.C., contains a puzzle of sevens that is remarkably like the St. Ives riddle.

In the Rhind Papyrus (named after the 19th Century Scottish antiquarian who acquired it) the scribe Ahmes poses a problem about seven houses each with seven cats that each eat seven mice and each mouse could have eaten seven grains of barley and each barley grain would have produced seven hekats of grain. (A hekat was a unit of volume, equal to about 5 litres.) The problem was to add up the items listed. The solution given by Ahmes

$$7 + 7^2 + 7^3 + 7^4 + 7^5 = 19607$$

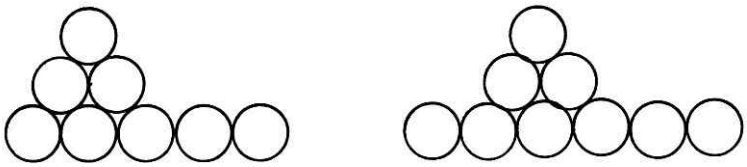
shows that the mathematicians of Egypt knew something about geometric series over 3650 years ago.



Above: A portion of the Rhind papyrus. The papyrus roll, about 30 cm wide and 5 metres long, can be seen in the British Museum in London. □

SYMMETRIC COIN STACKS

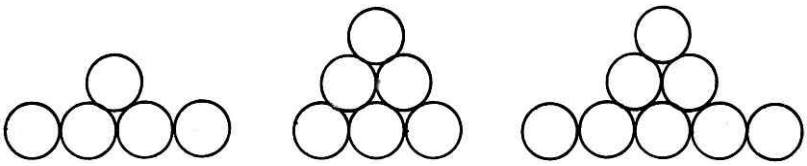
A coin stack is an arrangement of identical coins in rows such that the coins in the bottom row form an unbroken line, and each row of coins above the first forms an unbroken line, with each coin touching two coins in the row below.



How many coin stacks with a base of n coins can be formed?

In *Mathematical Digest* **162** (January 2010) the answer was shown to be F_{2n-1}, where F_n is the nth Fibonacci number.

Here is a variation of the problem: how many *symmetric* coin stacks can be formed?



Let s_n denote the number of symmetric coin stacks with n coins in the bottom row.

It is easy to see that $s_1 = 1$, $s_2 = 2$, $s_3 = 3$ and $s_4 = 5$. These numbers leads to the guess that Fibonacci numbers are once again involved.

Suppose now we have n coins in the bottom row. On the base row we can place symmetric coin stacks with $n - 1$ coins: there are s_{n-1} options.

We can't put on the base any coin stacks with $n - 2$ coins, because the stack must be symmetric. But we can place a stack with $n - 3$ coins: there are s_{n-3} ways of doing so.

The next possibility is to build a coin stack with n coins in the base row and $n - 5$ coins in the second row: there are there s_{n-5} ways of doing so.

Continuing in this way, we see that

$$s_n = s_{n-1} + s_{n-3} + s_{n-5} + \cdots + s_3 + s_1 + 1$$

if n is even. Here the last two terms are $s_1 = 1$, (when just one coin is placed centrally in the second row) and 1 (the stack with no coins in the second row).

When n is odd, we have

$$s_n = s_{n-1} + s_{n-3} + s_{n-5} + \cdots + s_4 + s_2 + 1.$$

Here the last two terms are $s_2 = 2$ (with the second row having just two coins, centrally placed) and 1 (the stack with with no coins in the second row).

Replace n by $n + 2$. We then have:

$s_{n+2} = s_{n+1} + s_{n-1} + s_{n-3} + s_{n-5} + \cdots + s_3 + s_1 + 1$ if n is even, and

$s_{n+2} = s_{n+1} + s_{n-1} + s_{n-3} + s_{n-5} + \cdots + s_4 + s_2 + 1$, if n is odd.

It is now clear that

$$s_{n+2} = s_{n+1} + s_n$$

which is exactly the Fibonacci relation.

Summing up, the sequence $\{s_n\}$ starts off with $s_1 = 1 = F_2$ and $s_2 = 2 = F_3$, and thereafter satsifies the same recurrence rule as the Fibonacci sequence. It follows that $s_n = F_{n+1}$ for all positive integers n. □

POLYNOMIAL PROBLEM

My computer has been programmed with a polynomial p with positive integer coefficients. It will answer the question "What is the value of $p(n)$ for any integer n?"

How many questions must I ask in order to work out what polynomial has been programmed into the computer?

The answer may be found on page 27. □

QUOTE

God used beautiful mathematics in creating the world.

(Paul Dirac) □

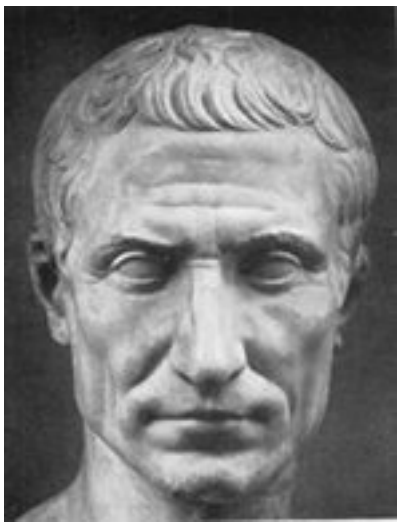
THE BINARY LEAP YEAR

The year is 365.2422 days long. To make this inconvenient number fit into calendars with whole numbers of days in a year, a system of Leap Years is needed.

Julius Caesar decreed a calendar with years of 365 days, with an extra day every four years. That was equivalent to using $365 + \frac{1}{4} = 365.25$ as the number of days in a year. This number was a little too large, so that more Leap Years were added than needed.

Over the years, the error built up to ten days, until in 1582 Pope Gregory XIII decreed that there should be a Leap Year in every year divisible by 4, except in years that were divisible by 100 but not by 400. That amounted to using $365 + \frac{1}{4} - \frac{3}{400} = 365.2425$ as the length of the year, which is a very good approximation to the true year of 365.2422 days. Today's Western Calendar follows Pope Gregory's rule.

Pope Gregory XIII



Julius Ceasar

However, the Gregorian year is too long by 0.0003 days, and after some 3000 years it will get out of step with the Sun.

A much more accurate method is to note that the fraction $\frac{31}{128} = 0.2421875 = 0.2422$ to four decimal places. That is a very good approximation. In practical terms, it means that in order for a calendar to keep in step with the Sun, it needs to have 365 days, with 31 Leap Years every 128 years.

Since the denominator of this fraction is $128 = 2^7$, and the numerator is $31 = 2^5 - 1$, an obvious solution is to Think Binary. It has been suggested

that Pope Gregory's Leap Year Rule should be replaced by a Binary Leap Year Rule:

> A year is a Leap Year when its binary representation ends in at least two zeros, but no more than six.

Putting the next ten years in binary shows that how the system works.

$$2011 = 11111011011$$

$$2012 = 11111011100\text{: a Leap Year}$$

$$2013 = 11111011101$$

$$2014 = 11111011110$$

$$2015 = 11111011111$$

$$2016 = 11111100000\text{: a Leap Year}$$

$$2017 = 11111100001$$

$$2018 = 11111100010$$

$$2019 = 11111100011$$

$$2020 = 11111100100\text{: a Leap Year}$$

Continuing this table will show that Binary Leap Years and Gregorian Leap Years coincide until $2048 = 2^{11} = 100000000000$ which is a Gregorian Leap Year but not a Binary Leap Year. The last time that happened was $1984 = 11111000000$. Going further ahead, $2100 = 100000110100$ will be a Binary Leap Year but not a Gregorian Leap Year.

The Binary Leap Year is a neat idea. It has mathematical appeal, and is very accurate. But it will take a bit more than mathematical elegance and precision to displace a 400-year-old Papal edict. □

BRAHMAGUPTA QUOTE

As the sun eclipses the stars by its brilliance, so the man of knowledge will eclipse the fame of others in assemblies of the people if he proposes algebraic problems — and still more if he solves them.

(Indian mathematician Brahmagupta 598-760 A.D.) □

THE CAUCHY-SCHWARZ-BUNYAKOVSKY INEQUALITY

If $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ are real numbers, then

$$(\sum_{i=1}^{n} a_i b_i)^2 \leq (\sum_{i=1}^{n} a_i^2)(\sum_{i=1}^{n} b_i^2).$$

The inequality was published in 1821 by the French mathematician Augustin-Louis Cauchy. The German mathematician Hermann Schwarz rediscovered it in 1888. Meanwhile, a Russian mathematician Viktor Bunyakovsky had shown in 1859 that the inequality works for integrals as well:

$$(\int_a^b f(x)g(x)dx)^2 \leq (\int_a^b (f(x))^2 dx)(\int_a^b (g(x))^2 dx)$$

The inequality, known under different combinations of these three names, has important applications in many areas of mathematics (and is incidentally an essential item in an Olympiad toolbox).

There are a number of different proofs of the result; the proof below is a clever little application of the theory of quadratics.

The proof starts with the statement that, for any real number x

$$\sum_{i=1}^{n} (a_i - x b_i)^2 \geq 0.$$

Square out the general term and split the sum into three sums:

$$\sum_{i=1}^{n} a_i^2 - 2x \sum_{i=1}^{n} a_i b_i + x^2 \sum_{i=1}^{n} b_i^2 \geq 0 \quad \text{for all } x.$$

This quadratic expression in x does not take negative values, so its discriminant is less than or equal to zero:

$$(2\sum_{i=1}^{n} a_i b_i)^2 - 4(\sum_{i=1}^{n} a_i^2)(\sum_{i=1}^{n} b_i^2) \leq 0.$$

and cancelling out the factor 4 on each side gives the result.

The proof of Bunyakovsky's version of the inequality follows the same lines: just replace the sums by integrals.

It is often useful to know under which circumstances an inequality becomes an equality. That happens here when the discriminant is zero, in which case the quadratic has just one root. The only way this can happen is when there is a number x such that $a_i = b_i x$ for all i (or if $b_i = 0$ for all i, when the quadratic collapses to linear).

For more information on the Cauchy-Schwarz Inequality and its application to a range of problems, see Graeme West's *Inequalities for the Olympiad Enthusiast*, one of a set of Olympiad Training Notes which may be ordered using the form at the back of the magazine. □

POLITICS PUTS PAID TO PAMO

The Pan African Mathematics Olympiad is an annual event that takes place in different African countries every year, and South Africa has been a regular participant since 2000. This year the Olympiad was scheduled to be held in March in Tunisia, but the political unrest in that country forced a postponement. The Tunisian PAMO organization has expressed the hope that they will be able to host the event in September. □

ALPHAMETRIC

$$\begin{array}{cccccc} T & W & E & N & T & Y \\ T & W & E & N & T & Y \\ T & H & I & R & T & Y \\ \hline S & E & V & E & N & T & Y \end{array}$$

Replace the letters in the above sum by digits, in a one-to-one fashion, to make the calculation correct.

The first three readers to send in a correct answer will win a BACKPACK: a set of all available back numbers of *Mathematical Digest*. □

SHARP CALCULATOR COMPETITION

The Sharp Calculator Competition is sponsored by Sharp Electronics (S.A.) (Pty) Limited, who have donated three calculators as prizes.

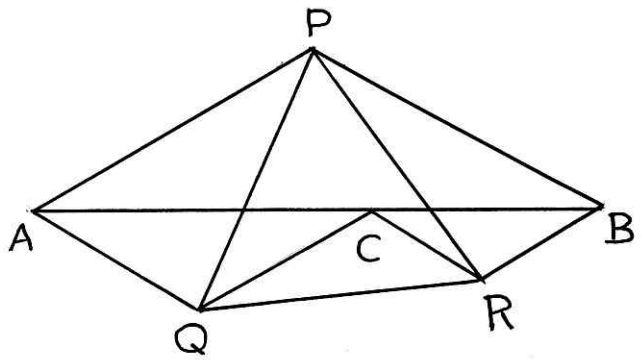
These prizes are offered for the three best solutions of the problems below. Solutions may be in English or Afrikaans. Partial solutions will be judged on their merits. The Editor reserves the right to make no award if no satisfactory entries are received, and his judgement will be final. The winner of an nth prize in a previous competition will only be eligible for an mth prize if $m < n$.

Only high school students may take part. Entrants must give their full name and home address, age, school and grade. Solutions must be certified by a mathematics teachers at the school as the entrant's own unaided work.

The closing date for receipt of entries is 1 June 2011, and the results will be announced in our July 2011 edition.

SHARP

1. Triangle APB is isosceles with $\angle PAB = \angle PBA = 30°$. The point C lies between A and B. Two isosceles triangles AQC and CRB are constructed, with Q and R on the opposite side of AB from P, with $\angle CAQ = \angle ACQ = \angle RCB = \angle CBR = 30°$.



   Prove that triangle PQR is equilateral.

2. Solve the equation
$$m^3 - n^3 = 2m^2 - 3m + 1,$$
   where m and n are integers.

3. For which value(s) of k is there at least one common tangent to the circle $x^2 + y^2 = 1$ and the parabola $y = x^2 + k$? □

PYTHAGORAS AND THE JUBILEE

In Jewish tradition, a "jubilee" occurs every 50 years. The Book of Leviticus lays down that after seven "sabbatical years" (meaning seven cycles of seven years) the next year (the 50th) is be celebrated as a year of rejoicing, when debts are waived and indentured labourers are freed from their obligations.

The word "jubilee" comes from the Hebrew word "yobhel" (ram). A ram's horn, or *shofar*, is blown to announce the beginning of festivals.

The first century Jewish scholar Philo of Alexandria wrote that 50 was a holy number, "the principle of the generation of the universe".

Some scholars believe that the Jubilee Number 50 has its origin in the 3-4-5 triangle of Pythagoras. The sum of the first two numbers, 7, is the number of days of the week, the sum of all three numbers, 12, is the number of months of the year, and the sum of the squares of the three numbers, 50, gives the Jubilee.

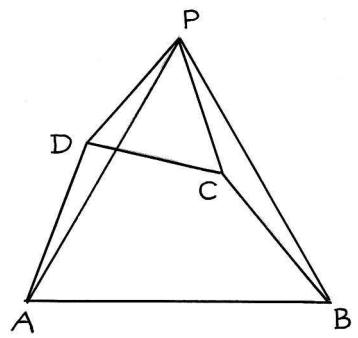
Other even more adventurous writers have speculated that Pythagoras was tutored by the Old Testament prophet Ezekiel, and that it was via this route that the Jubilee found its way into Jewish tradition.

Ezekiel was born in 622 B.C. and was in Babylon during the Captivity, which ended in 538 B.C. Very little is known about Pythagoras, and stories about his life date from hundreds of years after his death. He is believed to have been born some time around 570 B.C. (when Ezekiel would have been 52 years old). Pythagoras is said to have travelled widely in his youth, at least as far as Egypt, in pursuit of knowledge. Babylon was renowned in those days as a centre of mathematical learning, and would have been a natural destination for a bright young mathematician. Pythagoras was also interested in the religions of the time, so would have been keen on meeting the ageing Ezekiel while in Babylon.

But it's all speculation—there is no hard evidence that Pythagoras and Ezekiel met. The story is, at best, flaky. □

SHARP SOLUTIONS AND RESULTS

1. *Let $ABCD$ be a quadrilateral with $AD = BC$ and $\angle A + \angle B = 120°$. Let P be a point exterior to the quadrilateral such that P and A lie on opposite sides of the line CD and triangle DPC is equilateral. Prove that triangle APB is also equilateral.*



We shall prove that triangles ADP and BCP are congruent. Since $AD = BC$ and $DP = CP$ (given), all we have to prove is that $\angle ADP = \angle BCP$.

$\angle ADP = 60° + \angle ADC$

$= 60° + (360° - \angle A - \angle B - \angle DCB)$

$= 60° + 360° - 120° - \angle DCB$ (since $\angle A + \angle B = 120°$)

$= 300° - (360° - 60° - \angle PCB)$

$= \angle PCB.$

Since triangles ADP and BCP are congruent, it follows that $AP = BP$. Moreover, since $\angle DPA = \angle CPB$, it follows that $\angle APB = \angle DPC = 60°$. So triangle APB is equilateral.

2. *Prove that the equation $m^2 - 5n^2 = 1$ has infinitely many solutions in natural numbers.*

This is a *Diophantine equation* (a single equation with more than one unknowns, all of which are whole numbers). It is in fact a special type of Diophantine equation, known as *Pell's equation*, and a great deal has been written about solving Pell's equation.

It is not difficult, by trial and error, to find one solution of the equation: $m = 9$ and $n = 4$. (Check: $9^2 - 5 \times 4^2 = 81 - 80 = 1$.)

For any solution (m, n) of the equation we have $m^2 - 5n^2 = 1$. Now $1 = (m^2 - 5n^2)^2 = (m^2 + 5n^2)^2 - 4(m^2)(5n^2) = (m^2 + 5n^2)^2 - 5(2mn)^2$. This shows that (M, N) is also a solution, where $M = m^2 + 5n^2$ and $N = 2mn$. And it is a larger solution, since $M > m$ and $N > n$.

So, starting with our original solution $m = 9$ and $n = 4$ we use the formula above to find another solution: $m = 9^2 + 5 \times 4^2 = 161$ and $n = 2 \times 9 \times 4 = 72$. And from this larger solution we generate another one: $m = 161^2 + 5 \times 72^2 = 51841$ and $n = 2 \times 161 \times 72 = 23184$. And so on.

Another way of showing that there are infintely many solutions (pointed out by reader David Allison) is to note that, if m and n satisfy the equations, so do $M = 9m + 20n$ and $N = 4m + 9n$.

For $M^2 - 5N^2$

$= (9m + 20n)^2 - 5(4m + 9n)^2$

$= 81m^2 + 360mn + 400n^2 - 5(16m^2 + 72mn + 81n^2)$

$= m^2 - 5n^2 = 1$

However, the formulas for M and N are not easy to spot.

SHARP

3. *A sequence $\{a_n\}$ is defined by*

$$a_{n+1} = \frac{a_n + c}{1 - ca_n}.$$

Prove that it is possible to choose values for a_0 and c so that $a_n > 0$ for $n = 0, 1, \ldots, 2010$, but $a_{2011} < 0$.

The structure of the term

$$\frac{a_n + c}{1 - ca_n}$$

suggests the right hand side of the trig identity

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}.$$

We now observe that we can find acute angles α and β such that

$$\tan\alpha > 0, \tan(\alpha + \beta) > 0, \ldots, \tan(\alpha + 2010\beta) > 0$$

but $\tan(\alpha + 2011\beta) < 0$.

So we set $a_0 = \tan\alpha$ and $c = \tan\beta$. Then $a_n = \tan(\alpha + n\beta)$.

(These problems appeared in the Baltic Way Mathematics Competition, 1990) $\square$

VARSITY READINESS TEST

How ready are you for varsity maths? Try the problems below (no calculators!) and then check out the correct answers on page 31. Any mistake you make will flag a weakness in your mathematical background requiring fixing before signing up for first-year university mathematics.

1. What is the remainder when $3x^3 + x^2 + 8x - 7$ is divided by $x^2 + x + 2$?

   (1) $4x - 3$ (2) $2x + 1$ (3) $3x - 2$ (4) $x - 4$ (5) 5

2. When $\frac{1}{41}$ is written as a recurring decimal, what is the hundredth digit after the decimal point?

   (1) 0 (2) 2 (3) 3 (4) 4 (5) 9

3. $2^n - 2^{n-1}$ is equal to

   (1) 2^{n-1} (2) 2 (3) $2^{n-1} - 1$ (4) 2^{2n} (5) $\frac{1}{2}$

4. $\frac{1}{2n+1} + \frac{1}{(2n+1)(4n+1)} + \frac{1}{(4n+1)(6n+1)}$ is equal to

   (1) $\frac{5}{(2n+1)(6n+1)}$ (2) $\frac{2}{4n+1}$
   (3) $\frac{4}{(2n+1)(4n+1)}$ (4) $\frac{3}{6n+1}$
   (5) $\frac{3}{(2n+1)(4n+1)(6n+1)}$

5. If $\frac{1}{x+3} = 4$, then $\frac{3}{x+2}$ is equal to

   (1) 1 (2) -2 (3) 3 (4) -4 (5) 5

6. $(ab)^{-1}(a^{-1} + b^{-1})^{-1}$ is equal to

   (1) $a^{-1} + b^{-1}$ (2) $ab(a^{-1} + b^{-1})$ (3) $(a+b)^{-1}$
   (4) $ab(a+b)^{-1}$ (5) $(ab)^{-1}(a+b)$

7. What is the coefficient of x in the expansion of $(2x-1)(3x+2)(4x-3)$?

   (1) 7 (2) 8 (3) -9 (4) 10 (5) -11

8. $\cos^2 x - \sin^2 x$ is equal to

(1) $(\cos x - \sin x)^2$ (2) 1 (3) -1 (4) $\cos 2x$ (5) $\sin 2x$

9. $\sqrt[3]{4} \times \sqrt[4]{8}$ is equal to

(1) $\sqrt{2}$ (2) $2\sqrt[7]{12}$ (3) $\sqrt[7]{32}$ (4) $\sqrt[12]{32}$ (5) $2\sqrt[12]{32}$

10. $\log_2 4 \times \log_9 3$ is equal to

(1) $\log_{11} 7$ (2) $\log_{11} 11$ (3) $\log_{11} 12$ (4) $\log_{11} 9$ (5) $\log_{11} 8$

11. The sum of n terms of an arithmetic series is $\frac{1}{2}n(3n+7)$. The tenth term of the series is

(1) 32 (2) 47 (3) 29 (4) 38 (5) 51

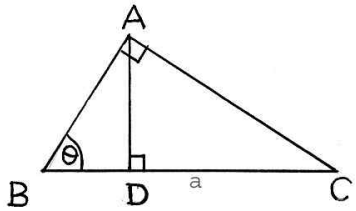
12. Find the sum of all integers between 100 and 200 which leave a remainder of 3 when divided by 10.

(1) 1684 (2) 1522 (3) 1376 (4) 1480 (5) 1296

13. The sides of a right-angled triangle are in a geometric progression and the shortest side has length 2. What is the length of the hypotenuse?

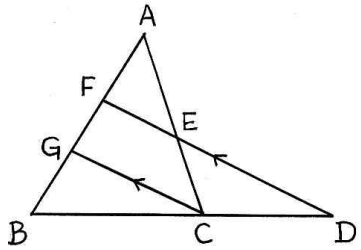
(1) $\sqrt{11}$ (2) $1+\sqrt{5}$ (3) $2\sqrt{3}$ (4) $\sqrt{10}$ (5) $3\sqrt{2}-1$

14. In triangle ABC, $\angle A = 90°$ and D is the foot of the altitude from A. If $\angle B = \theta$ and $BC = a$, then AD is equal to



(1) $a\sin^2\theta$ (2) $a\cos^2\theta$ (3) $a\tan\theta$ (4) $a\sin 2\theta$ (5) $a\sin\theta\cos\theta$

15. In the figure, $AF = 6$, $AE = 9$, $FB = 16$, $EC = 6$ and $DF \parallel CG$. Calculate $\dfrac{BC}{CD}$.



(1) $\dfrac{8}{3}$ (2) $\dfrac{3}{2}$ (3) 1 (4) 3 (5) $\dfrac{16}{9}$

16. The maximum value of $\dfrac{1}{x^2+5x+7}$ is

(1) $\dfrac{3}{4}$ (2) $\dfrac{1}{7}$ (3) $\dfrac{4}{3}$ (4) $\dfrac{1}{2}$ (5) $\dfrac{1}{4}$

17. The roots of the quadratic equation $x^2 - 5x - 3 = 0$ are a and b. Which of the following equations has roots $a - 1$ and $b - 1$?

(1) $x^2 + 4x - 9 = 0$ (2) $x^2 + 5x - 1 = 0$ (3) $x^2 + 7x - 3 = 0$
(4) $x^2 - 3x - 7 = 0$ (5) $x^2 + x - 5 = 0$

18. $\cos 65^\circ \sin 40^\circ + \cos 25^\circ \cos 40^\circ$ is equal to

(1) $\cos 15^\circ$ (2) $\cos 25^\circ$ (3) $\cos 35^\circ$ (4) $\cos 55^\circ$ (5) $\cos 65^\circ$

19. If $\tan A = \dfrac{1}{2}$ and $\tan(A + B) = \dfrac{1}{3}$, then $\tan(2A + B)$ is equal to

(1) $\dfrac{1}{4}$ (2) $\dfrac{5}{6}$ (3) $\dfrac{2}{5}$ (4) 1 (5) $\dfrac{7}{6}$

20. If $a = 0.432432432432\ldots$ and $b = 0.45454545\ldots$ are recurring decimals as indicated, then $a + b$ is equal to

(1) $\dfrac{359}{405}$ (2) $\dfrac{361}{407}$ (3) $\dfrac{363}{409}$ (4) $\dfrac{365}{411}$ (5) $\dfrac{367}{413}$

Answers are on page 27. □

NUMBER PATTERNS

$$1 \times 1 = 1$$
$$11 \times 11 = 121$$
$$111 \times 111 = 12321$$
$$1111 \times 1111 = 1234321$$
$$11111 \times 11111 = 123454321$$
$$111111 \times 111111 = 12345654321$$
$$1111111 \times 1111111 = 1234567654321$$
$$11111111 \times 11111111 = 123456787654321$$
$$111111111 \times 111111111 = 12345678987654321$$

What happens next? □

FERMAT MADE SIMPLE

Three-and-a-half centuries ago, the French lawyer and mathematician Pierre de Fermat stated that he had found a simple proof that it is not possible to find positive integers x, y, z and n, with $n > 2$, such that $x^n + y^n = z^n$.

Fermat never got round to writing his proof down for posterity, with the result that for the next 350 years mathematicians great and small racked their brains to find the elusive simple proof. Finally, in the 1990s, British mathematician Andrew Wiles found a proof of what had by then become known as Fermat's Last Theorem. But Wiles' proof was not, in any sense, simple.

Convinced that a simple statement should have a simple proof, a Cape Town computer programmer, Peter van Vliet, slaved away at Fermat's Last Theorem until he was convinced he had found an argument that the world's greatest mathematicians had missed for centuries. So pleased was he with his achievement that he placed advertisements in the Saturday and Sunday editions (26/27 February 2011) of the *Weekend Argus* announcing his discovery to the world. The advertisements cost him R10 000.

When asked for their comments by local journalists, several Cape Town mathematicians expressed scepticism, and wondered why Mr van Vliet had paid so much money to have his proof published when any reputable maths journal would have published it for free (as long as it was right).

Sad to say, Mr van Vliet's proof was wrong. His argument, not by any means easy to follow on first reading, seemed to rely on the irrelevant observation that, if x, y and z are positive integers such that $x^2+y^2 = z^2$, then $x^n + y^n \neq z^n$. Actually, he did not prove this little bit of algebra, relying rather on having tested "billions" of Pythagorean triples on a computer.

The Editor invites readers to submit one-page proofs of Fermat's Last Theorem for publication in *Mathematical Digest*. Incorrect proofs will be published free of charge, if they are interesting. Correct one-page proofs will win a prize of R10 000. $\square$

GOLDEN OLDIES

1. A clock strikes six in five seconds. How long will it take to strike 12?
2. There are ten men at a meeting. Each shakes hand once with each of the others. How many handshakes take place?
3. How many different solutions can be found of the equation $3^m + 7 = 5^n$, where m and n are positive integers?
4. A student bought a book for R50, sold it for R60, then bought it back for R80 and finally sold it for R90. How much money did he make our lose in this series of transactions?
5. A taxi driver travels from Langa to Khayelitsha at an average speed of 50 km/hour. How immediately returns to Langa at 70 km/h. What is his average speed for the trip there and back?
6. If the notation $n!$ is used to denote the product $1 \times 2 \times 3 \times \cdots \times (n-1) \times n$, how many zeros are at the end of 50!?
7. Find three positive integers whose sum is equal to their product.
8. If a hen-and-a-half can lay an egg-and-a-half in a day-and-half, how long will it take ten hens to lay ten egg?
9. How much soil is there in a trench 1 metre deep, two metres wide and 3 metres long?
10. A ship is anchored in a harbour. A ladder is hanging over the side, with its bottom rung just touching the water. The ladder has 40 rungs, 20 cm apart. The tide comes in and rises at a rate of 50 cm per hour. How many rungs will lie above the water after three hours?

    Answers may be found on page 31.

QUOTE

At school I had a passion for maths. There is always an answer with mathematics and I like that. I like the neatness.

(Natalie Portman, Oscar-winning star of *The Black Swan*, in an interview in the *Sunday Times*, 20 February 2011) □

DIRAC AND THE FOUR TWOS PROBLEM

Paul Dirac, the pioneer of Quantum Theory, was one of the towering figures of Twentieth Century mathematical physics. The anecdote below is told by Graham Farmelo in his biography of Dirac, *The Strangest Man* (Faber and Faber, 2009).

> [Dirac had an] appetite for mathematical games and puzzles that served no purpose at all beyond entertainment. Once, he gave a devastating performance in a game that had been introduced at Göttingen in 1929. The challenge was to express any whole number using the number 2 precisely four times, and using only well-known mathematical symbols. The first few numbers are easy:
>
> $$\begin{aligned} 1 &= (2+2)/(2+2) \\ 2 &= (2/2)+(2/2) \\ 3 &= (2\times 2)-(2/2) \\ 4 &= 2+2+2-2. \end{aligned}$$
>
> Soon the game becomes much more difficult, even for Göttingen's finest mathematical minds. They spent hundreds of hours playing the game with ever-higher numbers — until Dirac found a simple and general formula enabling *any* number to be expressed using four 2s, entirely within the rules. He had rendered the game pointless.

Dirac's formula is given on page 31. □

COMPUTER PREDICTION

How would you feel if you had exceeded the speed limit on a deserted road in the dead of night, and a few days later received a demand for a fine that had been automatically printed by a computer coupled to a radar system and vehicle identification device? It might not be a demand at all, but simply a statement that your bank account had been debited automatically.

(Computer pioneer Maurice Wilkes, in an essay written in 1964) □

PARTITION PROBLEM

In how many ways can a positive integer n ($n \geq 3$) be split into a sum of three positive integers?

For each of $3 = 1+1+1$ and $4 = 1+1+2$ there is just one such partition.

There are two partitions of 5 ($1+1+3$ and $1+2+2$) and three partitions of 6: ($1+1+4$, $1+2+3$ and $2+2+2$.

How many partitions can you find for 7, 8, 9, etc?

From your observations, can you guess a formula for the number of partitions of an integer n? Using a computer to count the number of partitions could be a good idea.

Send your discoveries to the Editor. The best discussions will win a BACKPACK: a set of all available back numbers of *Mathematical Digest*. □

BIRTHDAY PROBLEM

Seven friends Alf, Ben, Cal, Dan, Eve, Fan and Guy have their birthdays on successive days, but not necessarily in that order.

Ben's birthday is first and Cal's birthday is on a Wednesday.
Eve's birthday is as many days before Alf's as Dan's is after Alf's.
Guy's birthday is as many days before Fan's as it is after Alf's.

On what days of the week do their birthday's fall?

The first three readers to send in the answer will win a BACKPACK: a set of all available back numbers of *Mathematical Digest*. □

POLYNOMIAL SOLUTION

Since the degree of the polynomial is not specified, it may seem that there could be no limit to the number of questions needed. One obvious question is to ask the computer to calculate $p(0)$. That would give the value of the constant term of p. After that each question would give an equation relating all the other unknown coefficients of the polynomial.

The very surprising answer is that only two questions need be asked, no matter how large the degree of the polynomial is.

The first question is

"What is the value of $p(1)$?

On receiving the answer N, the second question is:

"What is the value of $p(10^N)$?"

The answer will be a string of digits, separated by blocks of zeros. The coefficients of the polynomial appear between the blocks of zeros.

For example, suppose that the polynomial $p(x)$ hidden in the computer is

$$x^3 + 2x^2 + x + 3.$$

The answer to the first question "What is the value of $p(1)$?" is 7.

When asked for the value of $p(10^7)$ the computer will give the answer

$$1 \times 10^{21} + 2 \times 10^{14} + 1 \times 10^7 + 3 = 1\ 000\ 000\ 200\ 000\ 010\ 000\ 003,$$

revealing the coefficients of the polynomial: 1, 2, 1, 3, separated by blocks of six zeros.

The key condition of the problem is that all coefficients are positive, so that $p(1) = N$ is bigger than all the coefficients, and the evaluation of $p(10^N)$ involves only addition of positive numbers. □

VARSITY READINESS ANSWERS

The sequence of correct answers is:

15144 35452 14254 34142

□

BOOK REVIEWS

Andrew Schaug
University of Cape Town

How to Read and Do Proofs by Daniel Solow. Fifth Edition, John Wiley & Sons, Inc, 2010.

As the author remarks in the foreword, there are many books on all sorts of topics in mathematics at high school and early university level, but there are very few that explicitly devote themselves to the most important conceptual leap maths students face: the idea of proof. This book is a conscientious attempt to address this lack, and considers several kinds of elementary methods, building up a system for students to follow when confronted with any of several sorts of elementary problems.

However, this book fails in this attempt for a few major reasons. Unfortunately, the author takes a very long time to develop most of his ideas, and seems to credit the reader with little capability of independent thought, never getting past the very basics, but rather repeating them over and over in different ways.

Instead of providing a real discussion of the matter, acknowledging its complexity, and imparting helpful suggestions, the author seems to have invented his own pet procedure of solving 'generic' problems, expecting the reader to follow this to the letter.

Worse, those routines of his own invention are quite confusing. The author makes an effort to avoid formality, but instead of being accessible, the writing style is verbose, clumsy and repetitive, and points which many books may have communicated informally but concisely are either laboured or unclear.

He introduces several of his own sometimes ill-chosen names (like the 'backward uniqueness method', or 'choose method'), making it very difficult to read sections of the book independently or navigate the very uniform text, and the effort required to reading the whole book through would be better spent reading others with a similar focus. There is a table of proof methods on the inside cover which summarizes the majority of the book's contents, and could easily have replaced it, were it not also badly worded. He spends much of the latter portion of the book on rather dry applications to problems involving functions and vectors, clearly realizing that

many introductory textbooks do not clear up the difficulties enough - but here he spends far too long on them.

There are simple problems at the end of each section, and despite the vast amount of choice, the author seems to have made no effort to make them interesting or original.

This book will only disenchant interested readers by further instilling the idea that mathematics is a matter of formulaic routine, despite the fact that it was meant to address the very aspect of mathematics which makes it exactly the opposite.

All of that said, there are no obvious inaccuracies, and students having fundamental difficulties with first or second year university mathematics may find a great deal of the advice very helpful. However, it is certainly not the first book I would recommend for this purpose.

Foundations of Mathematics by Thomas Q. Sibley. John Wiley & Sons, Inc 2009.

This wonderful book is, as it aims to be, a fine bridge between standard calculation and proof-based mathematics, presenting initially difficult concepts in a gentle but very thorough way. In fact, the section introducing mathematical logic and the general ideas and methods of proof is more far-reaching, more concise and better organised than the entirety of the book discussed above.

Several of the later sections provide very good introductions to more specific branches of mathematics that might be encountered later on, like group theory, order theory, graph theory, and introductory analysis. As the title indicates, the purpose of the book is to consider some more interesting aspects of higher mathematics which the reader may not have encountered before, and not formulaic practicality - a fair amount is dedicated to such abstract notions as the nature of infinity and the axiom of choice, for example. Of course, the book is general and has to limit its scope; some major theorems and most of the technicalities one would find in books devoted specifically to these subjects are left out. The focus, as it should be, is always on the main concepts, their motivations, and what proofs in these areas are like. Even so, having read this book will leave the interested reader with quite a good idea of the essentials of several branches of mathematics, the basis of the subject as a whole, and some understanding of their context and history. The author includes well thought-out, informal explanations of the more difficult concepts when they are intro-

duced, and tends to end each section with very interesting historical notes and some tantalising indication of more advanced topics, which will hopefully encourage the reader to pursue the subject further. Each section can be read on its own, so even readers familiar with much of the material are more than likely to find a great deal that is completely new.

The book ends off with a brief comparison of general philosophical viewpoints on the nature of mathematics itself, with their relative strengths and weaknesses, and allows the reader to draw his or her own conclusions.

Overall, this book is an excellent introduction to interesting mathematics beyond the school curriculum, and to more abstract ways of thinking. □

STOREY SUMS

- Ahmed, Bongani, Charlie and Dirk live one on each floor of a four-storey block of flats.
- Their ages are 6, 9, 10 and 11, but not necessarily in that order.
- Ahmed lives directly above the 10-year-old and directly below the 9-year-old.
- Bongani lives above the 6-year-old.
- Bongani is more than one floor away from Dirk, who is more than one year younger than Bongani.

Find the ages of each of the boys, and the floors on which they live.

The first three correct entries received will each win a BACKPACK (a set of all available numbers of *Mathematical Digest*). □

SINGIN' THE BLUES

Why do blues singers start every song by counting "One-two-and-a-one-two-three"? It's because that reminds them of school maths and that makes them depressed and that is what you need to sing the blues. □

READER'S PROBLEM

Submitted by Wiggert Perfrangitaurus

Let $a_1,\ a_2,\ \cdots,\ a_n$ be an arbitrary sequence of positive real numbers. In the Euclidean plane, consider a circle C of radius 1, as well as n circles $C_1,\ C_2,\ \cdots,\ C_n$ of respective radii $a_1,\ a_2,\ \cdots,\ a_n$. It is desired to arrange $C_1,\ C_2,\ \cdots,\ C_n$ in a necklace around C such that:

$$C_i \text{ is tangent to } C \text{ for } i = 1, 2, \cdots, n,$$

$$C_i \text{ is tangent to } C_{i+1} \text{ for } i = 1, 2, \cdots, n-1, \text{ and}$$

$$C_n \text{ is tangent to } C_1.$$

State and prove a necessary and sufficient condition on $a_1, \cdots,\ a_n$ for this to be possible. □

GOLDEN OLDIES SOLUTIONS

(1.) 11 seconds (2.) 45

(3.) None: one side of the equation is even, the other side odd.

(4.) R20 (5.) $58\frac{1}{3}$ km/h (6.) 12

(7.) $1 + 2 + 3 = 1 \times 2 \times 3$ (8.) A day-and-a-half. (9.) None.

(10.) 40 As the tide comes in, the ship rises, and so does the ladder □

DIRAC'S FORMULA

Dirac's formula for expressing any number n using exactly four 2s and conventional mathematical symbols is

$$n = -\log_2 \log_2 \left(\sqrt[2]{\sqrt{\cdots\sqrt{2}}} \right)$$

where there are n square root signs inside the brackets.

Since the figure 2 in front of the first root sign is unnecessary, Dirac's solution solves the same problem, using only three 2s. □

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The mathematics are distinguished by a particular privilege, that is, in the course of ages, they may always advance and can never recede.

(Edward Gibbon, in *Decline and Fall of the Roman Empire*)

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Cover: The Swedish artist Oscar Reutersvärd (1915–2002) is renowned as the creator of impossible figures: images that appear to be solid objects but cannot be constructed in the real world. In 1982 three postage stamps featuring Reutersvärd's impossible figures were produced in Sweden.

THE CHINESE REMAINDER THEOREM

The oldest reference to a result in number theory, known today as the "Chinese Remainder Theorem", occurs in the *Sun Tzu Suan-Ching* (the "Mathematical Classic of Sun Tzu"). The exact date of this work is uncertain, but it's somewhere between 280 A.D. and 473 A.D.

One of the problems in this treatise is stated as follows:

> We have things of which we do not know the number;
> if we count them by threes, the remainder is 2;
> if we count them by fives, the remainder is 3;
> if we count them by sevens, the remainder is 2.
> How many things are there?

Put another way, the problem is to find a number N such that

$$\begin{array}{l} N-2 \text{ is divisible by } 3, \\ N-3 \text{ is divisible by } 5, \text{ and} \\ N-2 \text{ is divisible by } 7. \end{array}$$

From yet another point of view, what we want is a number which occurs in each of the following arithmetic sequence:

$$\begin{array}{rrrrl} 2, & 5, & 8, & 11, & \ldots \\ 3, & 8, & 13, & 18, & \ldots \\ 2, & 9, & 16, & 23, & \ldots \end{array}$$

It's not difficult to find an answer to this particular problem in the last form. If you write down a few more terms of each sequence you soon hit on a common number: 23. Sun Tzu described how to solve *all* problems of this type.

In general terms, the problem is to find a number x, which leaves

$$\begin{array}{rll} & \text{a remainder of} & b_1 \text{ when divided by } m_1, \\ & \text{a remainder of} & b_2 \text{ when divided by } m_2, \\ & \vdots & \vdots \\ \text{and} & \text{a remainder of} & b_n \text{ when divided by } m_n, \end{array}$$

where no two of the divisors $m_1,\ m_2,\ \ldots,\ m_n$ have any factor in common.

The first point to note is that there must be infinitely many solutions, for if x is a solution, so are $x \pm M$, $x \pm 2M$, $x \pm 3M$, where $M = m_1 m_2 m_3\ \ldots\ m_n$. Conversely, any two solutions differ by a multiple of M. Sun Tzu set out to find the smallest positive solution.

The Chinese method starts by finding numbers a_k $(1 \leqslant k \leqslant n)$ such that $a_k y_k$ leaves a remainder of 1 when divided by m_k, where $y_k = \frac{M}{m_k}$. They derived an ingenious general solution of this subsidiary problem, which was well–adapted to their abacus. We shall not go into the details here, but only remark that this type of problem is easily (if tediously) solved with a pocket calculator.

In Sun Tzu's problem, $M = 3 \times 5 \times 7 = 105$, $y_1 = 35$, $y_2 = 21$ and $y_3 = 15$. The three subsidiary problems are then

$$\begin{aligned} \text{to find } a_1, &\quad \text{such that } 35a_1 - 1 \text{ is divisible by } 3, \\ \text{and } a_2, &\quad \text{such that } 21a_2 - 1 \text{ is divisible by } 5, \\ \text{and } a_3, &\quad \text{such that } 15a_3 - 1 \text{ is divisible by } 7. \end{aligned}$$

The answers are easy to find: $a_1 = 2$, $a_2 = 1$ and $a_3 = 1$. An answer to Sun Tzu's problem is then given by

$$x = a_1 y_1 b_1 + a_2 y_2 b_2 + \cdots + a_n y_n b_n.$$

To see that x is indeed as required, we must check that for each k from 1 to n, x leaves a remainder of b_k when divided by m_k. Since each term other than the kth is divisible by m_k, and so contributes nothing to the remainder, we concentrate on the kth term: $a_k y_k b_k$. Since $a_k y_k$ leaves a remainder of 1 when divided by m_k, we deduce that $a_k y_k b_k$ leaves a remainder of b_k when divided by m_k, and hence x leaves a remainder of b_k when divided by m_k.

Now this x may not be the smallest positive solution, which can however be found by simply subtracting a suitable multiple of M.

The solution of Sun Tzu's problem can now be set out as follows:

k	1	2	3	
m_k	3	5	7	$M = 3 \times 5 \times 7 = 105$
b_k	2	3	2	
y_k	35	21	15	
a_k	2	1	1	
$a_k y_k b_k$	140	63	30	$x = 233$
				Subtract $2 \times 105 = \underline{210}$
				Solution $\underline{23}$

Examples of the Chinese Remainder Theorem occur frequently in ancient Chinese mathematical writing. In a treatise written in 1247 the following problem is posed:

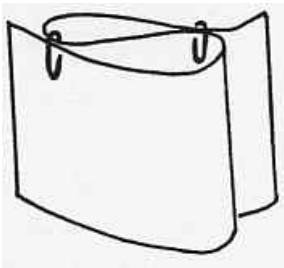
There are a certain number of bags of rice in a warehouse.
If they are shared by 19 restaurants, 9 bags will be left over.
If they are shared by 17 restaurants, 8 bags will be left over.
If they are shared by 12 restaurants, it will be just right.
How many bags of rice?

The first correct answer will win a BACKPACK (a set of all available numbers of *Mathematical Digest*)

. Apart from its interest as a problem in number theory the Chinese Remainder Theorem had a practical use for ancient Chinese astronomers, who needed to predict the simultaneous occurrence of several periodic astronomical events. Today the Chinese Theorem is used by computer programmers to store and manipulate long numbers. □

THE MAGIC PAPER CLIPS

Here's a neat party trick to impress a friend. It requires two paper clips and a R100 note, so make sure you have the paper clips handy and the persuade your friend to supply the R100 note. Fold the note and clip it as shown.



Hold the pinned note up for all to see, then pull the ends hard. The note will snap straight and the clips will shoot into the air. When the clips are found, they will be linked together as if by magic.

At this stage you should discourse learnedly on how the curvature in the Riemannian manifold was topologically transferred by a diffeomorphism to the paper clips. Your friend will be so stunned by the trick and impressed by your knowledge that he may not notice that you've pocketed his Riemannian manifold. □

NESTED SQUARE ROOTS

Problem: evaluate

$$\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\cdots}}}}.$$

This is an example of a "nested square root".

To find the value of the expression, set $X = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\cdots}}}}$.

Square both sides: $X^2 = 1+\sqrt{1+\sqrt{1+\sqrt{1+\cdots}}} = 1+X$.

This gives the quadratic equation $X^2 - X - 1 = 0$, from which it follows (using the quadratic formula) that $X = \frac{1}{2}(1 \pm \sqrt{5})$.

The negative value of X given by the quadratic formula may be rejected: square roots are not negative, so the original expression is clearly positive.

So $\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\cdots}}}} = X = \frac{1}{2}(1+\sqrt{5})$, which is the famous Golden Ratio.

The same idea can be used to evaluate the nested square root

$$\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\cdots}}}}.$$

Setting this expression equal to X and squaring leads to the quadratic equation $X^2 = 2 + X$, from which it follows that $X = 2$ (rejecting the negative solution as before).

In general, the value of the nested square root

$$\sqrt{n+\sqrt{n+\sqrt{n+\sqrt{n+\cdots}}}}$$

is $\frac{1}{2}(1+\sqrt{4n+1})$, which is an integer if (and only if) n is a product of two consecutive integers.

In 1911 the Indian mathematical genius Srinivasa Ramanujan proposed a much harder problem: the evaluation of the nested square root

$$\sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{1+\cdots}}}}$$

The key idea here is to note that the nested square root involves a function $f(n)$, where

$$f(n) = n\sqrt{1 + f(n+1)}$$

and the problem is to evaluate $f(1)$.

Squaring the equation we have $(f(n))^2 = n^2(1 + f(n+1))$, which suggests that f could be a quadratic function $an^2 + bn + c$.

That would mean that
$(an^2 + bn + c)^2 = n^2(1 + a(n+1)^2 + b(n+1) + c)$.

Multiplying out gives

$an^4 + 2abn^3 + (2ac + b^2)n^2 + 2bcn + c^2 = n^2(an^2 + (2a + b)n + 1 + a + b + c)$.

Since these two polynomials are equal for all values of n, their coefficients must be equal. That gives the equations

$2ab = 2a + b$ (equating the coefficients of n^3)

$2ac + b^2 = 1 + a + b + c$ (equating the coefficients of n^2)

$2bc = 0$ (equating the coefficients of n)

$c^2 = 0$ (equating the constant coefficients)

Solving these equations is quick and easy: $a = 1$, $b = 2$ and $c = 0$.

So $f(n) = n(n+2)$.

We now recall that $f(1) = \sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + \cdots}}}$ is what we need. This gives Ramanujan's conclusion: $\sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + \cdots}}} = 3$. $\square$

UNFASHIONABLE FACTORS

Can you factorize $x^3+y^3+z^3-3xyz$? The chances are you can't, for unless you've seen the answer you're unlikely to be able to work out the factors. You may not believe the expression factorizes, but it does:

$$x^3+y^3+z^3-3xyz=(x+y+z)(x^2+y^2+z^2-xy-yz-zx).$$

This factorization is no longer fashionable, but it was standard fare in Victorian times. You can find it, for example, in that classic of mathematics textbooks, Hall and Knight's *Higher Algebra*.

Today it's not part of the school curriculum, for it's a rather special form with not nearly as many applications as, say, $x^2-y^2=(x+y)(x-y)$. However, it's not an entirely useless snippet of knowledge, and it could prove useful in an Olympiad examination.

Here's a problem which comes out easily when you know the above factorization.

Prove that $x^3+y^3+z^3 \geqslant 3xyz \iff x+y+z \geqslant 0$.

In view of the factorization, the problem reduces to showing that the second factor is non-negative. This is easily accomplished by grouping and completing squares, for

$$x^2+y^2+z^2-xy-yz-zx=\frac{1}{2}(x-y)^2+\frac{1}{2}(y-z)^2+\frac{1}{2}(z-x)^2.$$

Now try the same problem from first principles. Remember, it's got two parts — and be careful with the inequalities!

Our second problem was set in the Chinese Mathematical Olympiad in 1978.

Solve (in integers):

$$\begin{aligned} x+y+z &= 0 \\ x^3+y^3+z^3 &= -18 \end{aligned}$$

The second equation is the same as

$$x^3+y^3+z^3-3xyz=-18-3xyz$$

and in view of our factorization and the first equation, that reduces to $xyz=-6$. Since x, y and z are integers, they can have the values ±1, ±2, ±3 or ±6. It's now simply a matter of trying the possibilities out, and we find six solutions for (x,y,z), namely $(1,2,-3)$ and its permutations.

Our unfashionable factorization also leads in a natural way to a formula for the solution of a cubic equation

$$x^3 + px + q = 0.$$

From our factorization it is clear that the only real solution of

$$x^3 + a^3 + b^3 - 3abx = 0$$

is $x - a - b$. So our aim is to put

$$x^3 + px + q = 0$$

into the form

$$x^3 - 3abx + a^3 + b^3 = 0$$

by finding a and b such that

$$\begin{aligned} -3ab &= p \\ \text{and } a^3 + b^3 &= q. \end{aligned}$$

These two equations are readily solved, and we have our solution:

$$x = -a - b = \left\{ -\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}} \right\}^{\frac{1}{3}} + \left\{ -\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}} \right\}^{\frac{1}{3}} \qquad \square$$

TONS PER KILOMETRE

> "When measured in tons per kilometre, about 94 percent of global trade is diesel-powered."

This snippet was picked up by reader David Allison in *Business Report*, 28 March 2011. The original source is a book review of Vaclav Smil's *Prime Movers of Globalization*.

The measure "tons per kilometre" (dividing tons by kilometres) makes no sense. What was probably meant was "kilometre-tons", obtained by multiplying the load in tons by the distance it is carried in kilometres. That would be a reasonable measure of how much freight is carried.

When you give the matter only a little thought, it becomes very clear that, in terms of kilometre-tons, most of the world trade is carried by sea, on diesel-powered ships.

The strange "tons per kilometre" is an interesting example of getting units wrong, rather like the common journalistic error of giving energy consumption in "kilowatts per hour" rather than "kilowatt-hours". $\square$

VARSITY READINESS TEST

How ready are you for varsity maths? Try the problems below (no calculators!) and then check out the correct answers on page 31. Any mistake you make will flag a weakness in your mathematical background requiring fixing before signing up for first-year university mathematics.

1. A local landfill can hold a million cubic metres of rubbish. Each truck that goes to the landfill delivers five cubic metres of rubbish. Six trucks per day go to the landfill every day of the week except Sundays. The number of years the landfill will last is about

   (1) 1 (2) 3 (3) 11 (4) 25 (5) 100

2. Thirty-six $1 \times 1 \times 1$ cubes are used to make a rectangular prism. How many different rectangular prisms can be made, using all 36 cubes?

   (1) 5 (2) 6 (3) 7 (4) 8 (5) 9

3. Which one of the following numbers is divisible by 7?

   (1) 111 (2) 2 222 (3) 33 333 (4) 444 444 (5) 5 555 555

4. Which of the following fractions is the biggest?

   (1) $\frac{2}{3}$ (2) $\frac{3}{4}$ (3) $\frac{5}{7}$ (4) $\frac{7}{10}$ (5) $\frac{9}{13}$

5. If the number $10^{2011} - 2011$ is written as an integer, the sum of its digits is

   (1) 18095 (2) 18096 (3) 18097 (4) 18098 (5) 18099

6. A tap leaks at a rate of one drop per second. A drop contains one-fifth of a millilitre of water. What is the total loss (to the nearest litre) from the dripping tap in one week?

   (1) 100 (2) 106 (3) 115 (4) 121 (5) 137

7. If $a \star b = a + ab$ then the value of $3 \star (3 \star 2)$ is

   (1) 24 (2) 18 (3) 30 (4) 9 (5) 12

8. If $x = \sqrt{5} - 1$ then $x^2 - 1$ is equal to

   (1) $\sqrt{3}$ (2) $5 - 2\sqrt{5}$ (3) 5 (4) $2\sqrt{5} - 2$ (5) $25 - 2\sqrt{5}$

9. If x and y are positive real numbers, then $(x+y)^{-1}(x^{-1}+y^{-1})$ is equal to

(1) $x^{-1}y^{-1}$ (2) $x^{-2}+y^{-2}$ (3) x^2+y^2
(4) $(x^{-1}+y^{-1})^2$ (5) $(x+y)^{-2}$

10. In rectangle $ABCD$, $AB = 2$ and $AD = 4$. The area of the circle with center A that passes through C is

(1) 25π (2) 20π (3) 16π (4) 12π (5) 9π

11. The solution of the inequality $\dfrac{x-1}{2x-3} < 0$ is

(1) $1 < x < \dfrac{3}{2}$ (2) $x < 1$ or $x > \dfrac{3}{2}$ (3) $\dfrac{2}{3} < x < 1$
(4) $x < \dfrac{2}{3}$ or $x > 1$ (5) $x < 1$

12. $\dfrac{2-\sqrt{3}}{2\sqrt{3}-3}$ is equal to

(1) $\sqrt{3}$ (2) $2+\sqrt{3}$ (3) $3-\sqrt{3}$ (4) $\dfrac{1}{\sqrt{3}}$ (5) $\sqrt{3}-1$

13. $\log a^2 - \log b^2$ is equal to

(1) $\log(a^2-b^2)$ (2) $2\log(a-b)$ (3) $(\log a - \log b)(\log a + \log b)$
(4) $2\log(\dfrac{a}{b})$ (5) $2\log(a+b)$

14. If $f(x) = 1 - \dfrac{1}{x}$ and $g(x) = 1 - x$ then $f(g(x)) - g(f(x))$ is equal to

(1) $\dfrac{1-x-x^2}{x-x^2}$ (2) $\dfrac{2(x-1)^2}{x}$ (3) $\dfrac{-1+x-x^2}{x(1-x)}$
(4) $\dfrac{1-x-x^2}{1-x+x^2}$ (5) 0

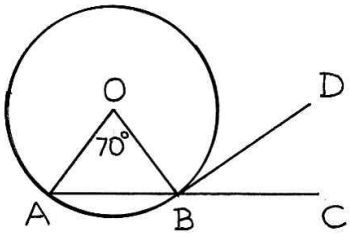
15. $\log_2 8 \times \log_8 64$ is equal to

(1) 2 (2) 4 (3) 5 (4) 6 (5) 8

16. A four-digit number N is a perfect square. If each of the digits is reduced by 1, the new number is also a perfect square with four digits. What is N?

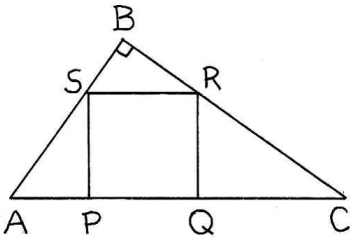
(1) 3136 (2) 3249 (3) 1111 (4) 2025 (5) 4247

17. In the figure, the circle with centre O passes through A and B, and BD is tangent to the circle. If $\angle AOB = 70°$, then $\angle DBC$ is equal to



(1) $30°$ (2) $35°$ (3) $40°$ (4) $45°$ (5) $50°$

18. In the figure $PQRS$ is a square inscribed in the right-angled triangle ABC. If $AP = 3$ and $QC = 12$, determine PQ.



(1) 5 (2) 6 (3) 7 (4) 8 (5) 9

19. For which positive value of k is the line $y = kx - 9$ tangent to the parabola $y = x^2$?

(1) 2 (2) 3 (3) 6 (4) 8 (5) 9

20. A solid gold sphere of radius 16 cm is melted down and the gold is used to make 8 solid gold spheres of the same size. What is the radius (in cm) of each of the small spheres?

(1) 8 (2) $4\sqrt{2}$ (3) 4 (4) $2\sqrt{2}$ (5) 2

21. If If $kx^2 - kx - 6$ is divisible by both $x + 1$ and $x + m$, find the value of $km^2 + km + 6$.

(1) 12 (2) 15 (3) 18 (4) 21 (5) 20

22. $\sqrt{5 + 2\sqrt{6}} - \sqrt{5 - 2\sqrt{6}}$ is equal to

(1) $2\sqrt{2}$ (2) $\sqrt{10}$ (3) $2\sqrt{2\sqrt{6}}$ (4) 3 (5) $2\sqrt{6}$

23. The smallest possible value of $x^2 + y^2 - 6y + 14$ is

(1) 1 (2) 2 (3) 3 (4) 4 (5) 5

24. If x is a real number, the maximum value of $\dfrac{x+1}{x^2+3}$ is equal to

(1) $\dfrac{1}{3}$ (2) $\dfrac{1}{6}$ (3) $-\dfrac{1}{6}$ (4) $\dfrac{1}{2}$ (5) $-\dfrac{1}{2}$

25. In $\triangle ABC$, $\angle C = 2\angle A$, $AC = 5$, $BC = 4$. Determine AB.

(1) $2\sqrt{10}$ (2) 6 (3) 7 (4) $2\sqrt{7}$ (5) $5\sqrt{2}$

IMO IN AMSTERDAM

The 52nd International Mathematical Olympiad (IMO) will take place in Amsterdam during July 2011. Teams from 102 countries will take part. A team consists of six high school students aged 19 or lower.

South Africa has taken part in the IMO every year since 1992.

The South African team is:

Ashraf Moolla (Rondebosch Boys High School)
Dalian Sunder (Star College)
Dylan Nelson (Benoni High School)
Kira Düsterwald Springfield Convent)
Robert Spencer (Westerford High School)
Sean Wentzel (Westerford High School).

The Team Leader is Professor Stephan Wagner (University of Stellenbosch) and the Deputy Leader is Dr Koos van Zyl (Rhodes University), both medallists at the IMO in their schooldays.

Professor John Webb (University of Cape Town) will attend the IMO in his capacity as Secretary of the IMO Advisory Board.

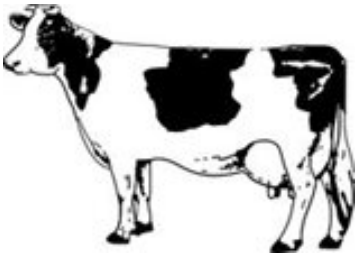
The selection process for the South African team was long and tough. A group of about 50 top contestants from the 2010 South African Mathematics Olympiad were invited to attend a training camp at the University of Stellenbosch in December last year, where they were given lectures on topics which lie outside the South African school curriculum, but comprise essential background knowledge for the IMO. The programme continued into 2011 with a series of monthly problem assignments. The next hurdle was a training camp at the University of the Free State in Bloemfontein in April 2011, with more lectures and a battery of selection tests. The team was chosen from this group.

The IMO consists of two papers, each containing just three problems to be completed in $4\frac{1}{2}$ hours. After the contest, the leaders and deputy leaders evaluate the solutions of their contestants and hold coordination sessions to ensure that the marking is correct and consistent. Meanwhile the teams enjoy a programme of activities and excursions. The IMO ends with an Awards Ceremony and Closing Banquet.

South African participation in the IMO is organized by the South African Mathematics Foundation and sponsored by Harmony Gold. □

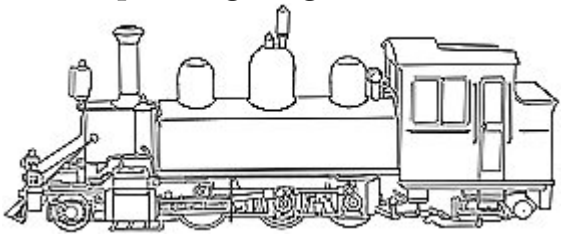
QUICKIES

1. There are 100 players in a knockout tennis tournament. How many matches must be played to find a winner?

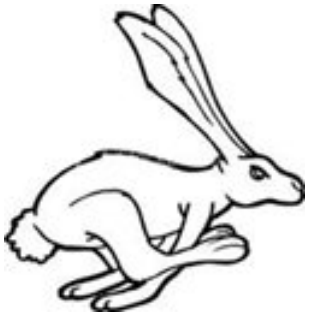


2. There are 100 cows in a field.
   All but 20 are taken off to be milked.
   How many cows are left in the field?

3. You are driving a train with eight coaches each carrying 100 passengers.
   At the first station half the passengers get off and 16 get on.
   At the second station half the passengers get off and 32 get on.
   At the third station half the passengers get off and 64 get on.
   At the fourth station half the passengers get off and 128 get on.
   At the fifth station all the passengers get off.



   What is the name of the engine-driver?

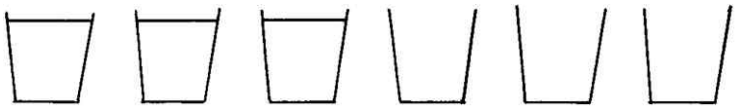


4. A rabbit is placed in a long box with a hole at each end, big enough for the rabbit to put its head through.

   The rabbit hops down the box at 1 km/h and puts its head through the hole at the end of the box. It then turns round and hops at 2 km/h to the other end, and puts its head through the hole there. Then it hops back, at 4 km/h, to the first hole, puts its head through, then hops back at 8 km/h to the other hole and puts its head through. It carries on in this way, hopping up to a hole, putting its head through, then hopping back at double its last speed and putting its head through the other hole.

   How fast must the rabbit hop to enable it to put its head through both holes at the same time?

5. Six glasses are placed in a row. The first three are full of water and the last three are empty. By moving just one glass, change the arrangement so that full and empty glasses alternate.



6. When the polynomial in x

$$(a-x)(b-x)(c-x)\cdots(z-x)$$

is multiplied out, what is the coefficient of x^{13}?

7. An explorer sets out from his base camp in a northerly direction. He continues in a straight line for 10 km then sets up his overnight camp.

   The next day he sets out from his overnight camp, again in a northerly direction. He continues in a straight line for 10 km and ends up at his base camp.

   How is that possible?

8. Mary is 24 years old. She is twice as old as Ann was when Mary was as old as Ann is now. How old is Ann?

9. What is significant about the arrangement 8 5 4 9 1 7 6 3 2 0?

10. There are eight notes in an octave. How many notes are there in two octaves?

Answers are on page 31. □

NEGATIVE MEMBERS

The *Business Report* in the *Cape Times* of Wednesday 20 April 2011 had a piece about the Bonitas Medical Aid on page 21. At the end of the first column the following statements are made:

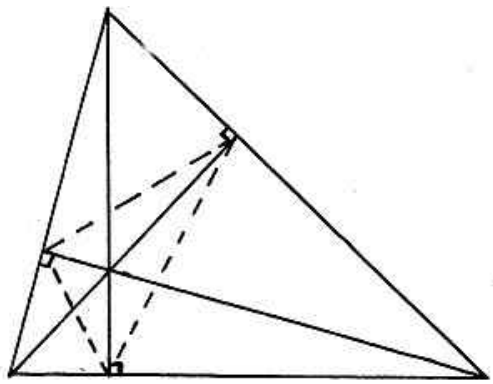
- Black ownership as a percentage of total ownership is 75.3 percent;
- Black female ownership as a percentage of total ownership is 76.5 percent.

The figures seem to imply that the number of black male members of the fund is negative. Has proof-reading died out?

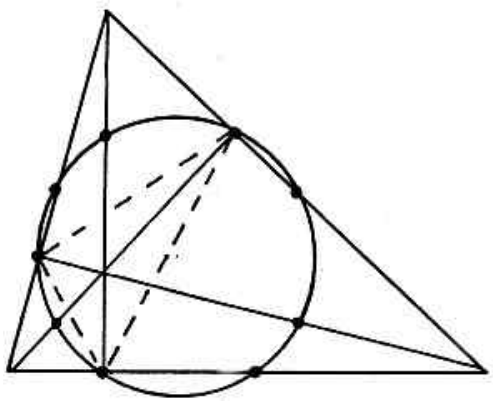
(Submitted by David Allison) □

THE NINE POINT CIRCLE

It is a standard result in geometry that the altitudes of a triangle are concurrent, i.e. they meet in a single point. That point is called the *orthocentre* of the triangle.

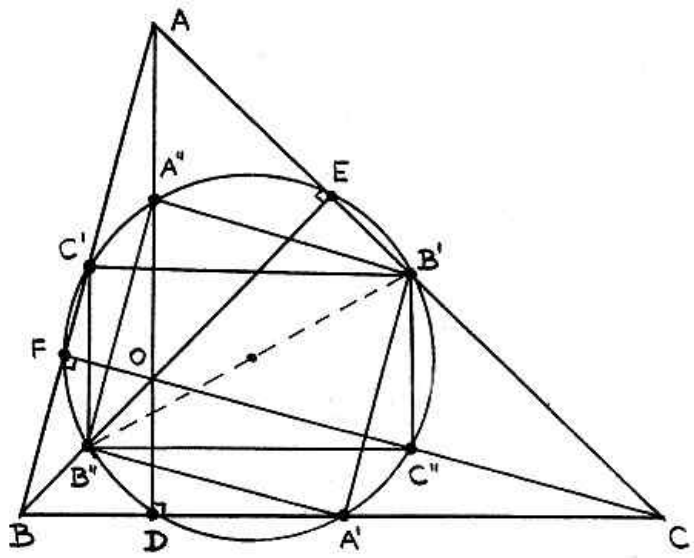


The feet of the altitudes are the vertices of a triangle called the *pedal triangle*. The circumcircle of the pedal triangle passes through six other significant points of the original triangle: the midpoints of the three sides, and the midpoints of the lines joining the orthocentre to the three vertices. This circle is called the Nine Point Circle of the triangle.



There are various proofs that those nine points do indeed lie on a circle. One of the simplest is due to the renowned geometer H S M Coxeter, and is given in his classic textbook *Introduction to Geometry*, published by John Wiley. It relies on the Midpoint Theorem, which says that a line joining the midpoints of two sides of a triangle is parallel to the third side.

To prove the Nine Point Circle theorem, we start with triangle ABC, in which the altitudes AD, BE and CF meet in the orthocentre O. The points A', B' and C' are the midpoints of BC, CA and AB respectively, and A'', B'' and C'' are midpoints of OA, OB and OC respectively.



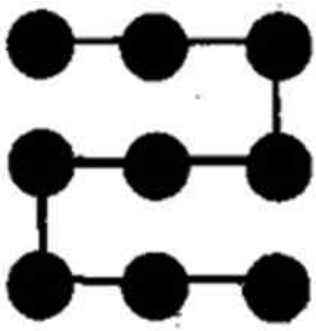
Consider quadrilateral $B'C'B''C''$. From the Midpoint Theorem applied to triangle ABC we deduce that $B'C' \| BC$, and from the same theorem applied to triangle OBC we deduce that $B'C' \| BC$. So $B'C' \| B''C''$. Similar reasoning applied to triangles ABO and ACO shows that $B''C' \| AO \| B'C''$. So $B'C'B''C''$ is a parallelogram. In fact, it is a rectangle, for $B'C'' \| AD$, $B'C' \| BC$ and $AD \perp BC$.

Now consider quadrilateral $A'B'A''B''$. By the same reasoning, it is a rectangle, and shares a diagonal ($B'B''$) with rectangle $B'C'B''C''$. The midpoint of this diagonal is the centre of a circle passing through the points A', C'', B', A'', C' and B''. That accounts for six of our nine points.

Since $A'A''$ is a diameter of this circle, and $\angle A'DA''$ is a right angle, D lies on the circle. Similarly, E and F lie on the circle, and the Nine Point Circle Theorem is proved. □

THE NINE DOT PROBLEM

In the diagram, nine dots have been joined by five straight lines, drawn without lifting the pencil from the paper.
Can you find a way of joining the dots with just four lines?



The answer is on page 21. □

SHARP CALCULATOR COMPETITION

The Sharp Calculator Competition is sponsored by Sharp Electronics (S.A.) (Pty) Limited, who have donated three calculators as prizes.

These prizes are offered for the three best solutions of the problems below. Solutions may be in English or Afrikaans. Partial solutions will be judged on their merits. The Editor reserves the right to make no award is no satisfactory entries are received, and his judgement will be final. The winner of an nth prize in a previous competition will only be eligible for an mth prize if $m < n$.

Only high school students may take part. Entrants must give their full name and home address, age, school and grade. Solutions must be certified by a mathematics teachers at the school as the entrant's own unaided work.

The closing date for receipt of entries is 1 September 2011, and the results will be announced in our October 2011 edition.

SHARP

1. Prove that for all positive integers $n > 1$
$$\frac{2^3-1}{2^3+1}\cdot\frac{3^3-1}{3^3+1}\cdot\frac{4^3-1}{4^3+1}\cdots\frac{n^3-1}{n^3+1} > \frac{2}{3}.$$
2. Prove that, if A, B and C are the angles of a triangle, then
$$\sin A + \sin B + \sin C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.$$
3. How many of the points of the Nine Point Circle (see page 14) of a triangle coincide when the triangle is (a) isosceles (b) right-angled (c) isosceles and right-angled (d) equilateral?

THE END

Steven Hawking can prove with a plot
Whether we will or will not
  Expand without limit
  Or end in an intimate
Space which is all in one spot.

□

HARVARD STUDENT'S LAMENT 1744

Now algebra, geometry,
Arithmetick, astronomy,
Opticks, chronology, and staticks,
All tiresome parts of mathematicks,
With twenty harder names than these
Disturb my brains, and break my peace.
All seeming inconsistencies
Are solv'd by A's, or solv'd by B's;
Our senses are depriv'd by prisms,
Our arguments by syllogisms.
If I should confidently write,
This ink is black, this paper is white,
They'd contradict it, and perplex one
With motion, light, and it's reflection,
And solve th' apparent falsehood by
The curious structure of the eye.

The *American Magazine and Historical Magzine*, February 1744.

□

LITERARY QUIZ

The word "lesson" came back to X as one he had heard before somewhere.
"There's a thing called Twy–stymes," he said.
"Y tried to teach it to me once, but it didn't."
"What didn't?" said Z.
"Didn't what?" said W.
X shook his head.
"I don't know," he said. "It just didn't. What are we talking about?"

Solve for X, Y, Z and W. The answer is on page 31. □

SHARP SOLUTIONS

1. *Triangle APB is isosceles with $\angle PAB = \angle PBA = 30°$. The point C lies between A and B. Two isosceles triangles AQC and CRB are constructed, with Q and R on the opposite side of AB from P, with $\angle CAQ = \angle ACQ = \angle RCB = \angle CBR = 30°$. Prove that triangle PQR is equilateral.*

   With the origin at A and ACB along the positive x-axis, let $AC = 2a$ and $CB = 2b$. Then the coordinates of P, Q and R are, respectively:

   $P(a+b, (a+b)/\sqrt{3}), \quad Q(a, -(a+b)/\sqrt{3}), \quad R(2a+b, -b\sqrt{3})$

   Then $PQ^2 = b^2 + ((2a+b)/\sqrt{3})^2 = \frac{4}{3}(a^2+b^2+4ab)$

   $PR^2 = a^2 + ((a+2b)/\sqrt{3})^2 = \frac{4}{3}(a^2+b^2+4ab)$

   $RQ^2 = (a+b)^2 + ((b-a)/\sqrt{3})^2 = \frac{4}{3}(a^2+b^2+4ab)$

   and hence triangle PQR is equilateral.

   Source: Indian Regional Mathematics Olympiad 2000, question 1.

2. *Solve the equation $m^3 - n^3 = 2m^2 - 3m + 1$, where m and n are integers.*

   From $m^3 - n^3 = 2m^2 - 3m + 1$ (1)

   we have

   $(m+1)^3 - n^3 = 3m^2 + 3m + 1 + 2m^2 - 3m + 1 = 5m^2 + 2 \geq 0$ (2)

   and

   $(m-1)^3 - n^3 = -3m^2 + 3m - 1 + 2m^2 - 3m + 1 = -m^2 \leq 0.$ (3)

   So $(m-1)^3 \leq n^3 < (m+1)^3$.

   Since m and n are integers, it follows that $n = m$ or $n = m - 1$.

   In the first case equation (1) reduces to $2m^2 - 3m + 1 = 0$, and hence $m = 1$ (the other root of the quadratic is not an integer).

   In the second case equation (2) gives $m = 0$.

   So there are just two solutions: $m = n = 1$ and $m = 0, n = -1$.

3. *For which value(s) of k is there at least one common tangent to the circle $x^2 + y^2 = 1$ and the parabola $y = x^2 + k$?*

   Let $(\cos\theta, \sin\theta)$ be a point on the circle. Then the tangent at this point to the circle has slope $-\cot\theta$ and equation $y - \sin\theta = -\cot\theta(x - \cos\theta)$, or $y = \sin\theta - \cot\theta(x - \cos\theta)$

   This tangent cuts the parabola $y = x^2 + k$ when

   $$x^2 + k = \sin\theta - \cot\theta(x - \cos\theta) \Leftrightarrow x^2 + x\cot\theta + k - \frac{1}{\sin\theta} = 0$$

$$\Leftrightarrow x^2 \sin\theta + x\cos\theta + (k\sin\theta - 1) = 0. \qquad (1)$$

For tangency to the parabola, the quadratic equation (1) must have exactly one real root, which means that its discriminant is equal to zero. So we have

$$\cos^2\theta - 4\sin\theta(k\sin\theta - 1) = 0 \Leftrightarrow 1 - \sin^2\theta - 4k\sin^2\theta + 4\sin\theta = 0$$

$$\Leftrightarrow (4k+1)\sin^2\theta - 4\sin\theta - 1 = 0 \qquad (2)$$

For equation (2) to have real solutions, its discriminant must be positive: $16 + 4(4k+1) \geq 0 \Leftrightarrow 4k + 1 \geq -4 \Leftrightarrow k \geq -\dfrac{5}{4}$.

There are now seven cases to consider.

When $k < -\frac{5}{4}$, the circle lies inside the parabola without intersecting it. There is no common tangent (figure 1).

When $k = -\frac{5}{4}$, the circle is tangent internally to the parabola and there are two common tangents (figure 2).

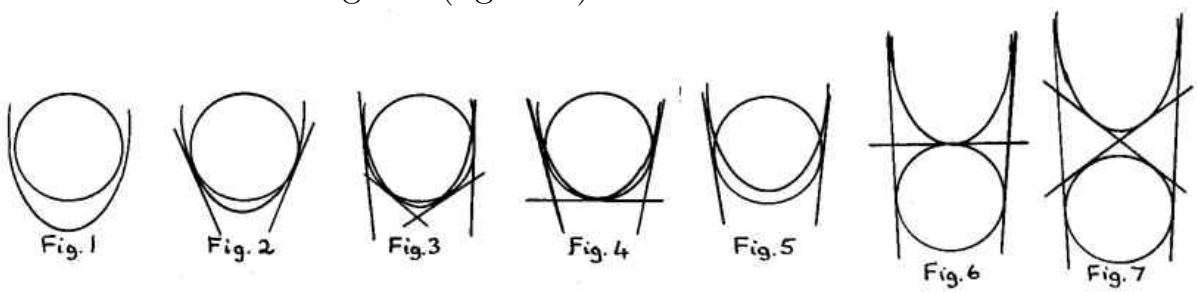
When $-\frac{5}{4} < k < -1$, the circle and the parabola intersect in four points and there are four common tangents (figure 3).

When $k = -1$ the circle and the parabola are tangent at $(0, -1)$. There is a common horizontal tangent at that point, and two further common tangents (figure 4).

When $-1 < k < 1$ the circle and the parabola intersect in two points, and there are two common tangents (figure 5).

When $k = 1$ the circle and the parabola are tangent at $(0, 1)$ and there is a common horizontal tangent at that point, and two further common tangents (figure 6).

When $k > 1$ the circle and the parabola do not intersect, and there are four common tangents (figure 7).



No solutions for all three problems were received, so no First Prize could be awarded.

Second Prize: Shaylan Lalloo (grade 10, Pearson High School)

Third Prize: Emma Belcher (grade 9, Springfield Convent). □

PRIZEWINNERS AND SOLUTIONS

Alphametric (April 2011, page 15):
The alphametric

$$\text{TWENTY} + \text{TWENTY} + \text{THIRTY} = \text{SEVENTY}$$

has two solutions:

$$546250 + 546250 + 593750 = 1686250$$

and

$$546750 + 546750 + 593250 = 1686750.$$

Partition Problem (April 2011, p. 26):
In how many ways can a positive integer n ($n \geq 3$) be split into the sum of three positive integers.

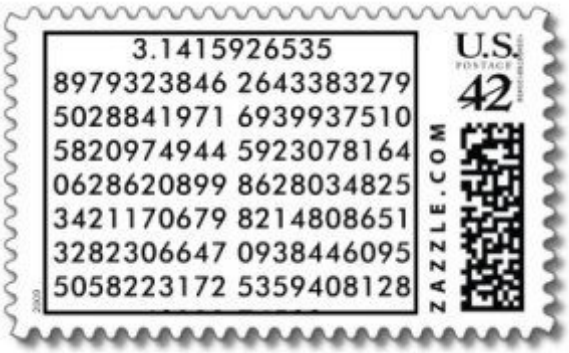
The answer to this rather difficult problem is the integer nearest to $n^2/12$. (See *A Course in Combinatorics*, by J.H. van Lint and R.M. Wilson, Exercise 15.1.)

Birthday Problem (April 2011, 26):
The days of the week on which the birthdays fall are: Ben – Friday, Eve – Saturday, Alf – Sunday, Dan – Monday, Guy – Tuesday, Eve – Wednesday, Fan – Thursday.

Storey Sums (April 2011, p. 30):
Dirk (age 6) lives on floor 1, Charlie (10) on 2, Ahmed (11) on 3 and Bongani (9) on 4.

Emma Belcher (grade 9, Springfield Convent) and Shaylan Lalloo (grade 10, Pearson High School) each win a BACKPACK (a set of all available back numbers of *Mathematical Digest*) for their solutions. □

PI ON A STAMP



THE CURSE OF 39

The number 39 has suddenly shot into prominence as a mark of shame in Afghanistan. In this deeply conservative society, 39 has become a symbol of immoral behaviour.

Car owners whose number plates contain 39 are mocked by children in the streets. If they try to sell the car, they have to drop the price substantially. The problem became especially pronounced in Kabul earlier this year when the government started a new series of licence plates beginning with 39. The head of Kabul's traffic department drew up a notice assuring drivers that 39 is not a cursed number. The notice lists the occurrences of 39 in the Koran, and gives a formula turning 39 into "Allah".

Despite the notice, Afghans are still queueing up to change their car numbers, only to encounter more bad luck: officials who demand a substantial bribe to give them a new 39-free number.

The furore over 39 is starting to spread from cars to other spheres of life. People are embarrassed if their apartment number is 39, or if their cellphone number includes 39. If you are 39 years old, you are advised to give your age as "under 40"

Nobody knows why 39 is regarded as immoral or unlucky in Afghanistan, or why the panic has spread so rapidly. □

BIBLICAL MATHS

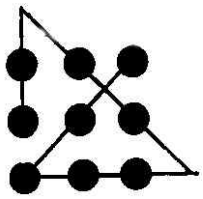
Which of the following words can be found in the Bible (the King James Version)?

BRACKETS COMPLEX CUBE GROUP EXPONENT LOGARITHM MATRIX QUATERNIONS REAL SOLUTION

The answer is on page 30 . □

NINE DOTS IN FOUR LINES

The nine dots can be joined with four connected lines as shown. Can you find a way of joining the dots with just three lines?



The answer is on page 25. □

A CIRCLE–SQUARING CARDINAL

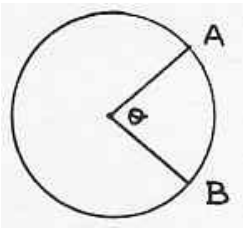
Nicholas Cusanus (1401–1464) was the son of a poor German fisherman who grew up to become a noted theologian and scholar. He was made a Cardinal in 1448.
Cusanus is known today as a "Reformer before the Reformation". Indeed, 100 years later Giordano Bruno was burnt at the stake for opinions which were derived from Cusanus' teachings.

Cusanus anticipated Copernicus (who was born nine years after Cusanus died) in maintaining that the earth was not fixed, but rotated. He is remembered also for two notable failures: he predicted that the world would end in 1734, and wrote a book which claimed to square the circle.

Though Cusanus was a dud prophet, some of his mathematics turned out to be of interest. Though he was mistaken in thinking he had squared the circle (it's as impossible to square the circle with ruler and compasses as it is to checkmate with only a knight in chess), Cusanus came up with a rather good approximate formula for finding the length of an arc of a circle.

Suppose that, in a circle of radius 1, an arc AB subtends an angle θ at the centre. The formula of Cusanus is

$$\text{arc} \quad AB = \frac{3\sin\theta}{2+\cos\theta} \ .$$



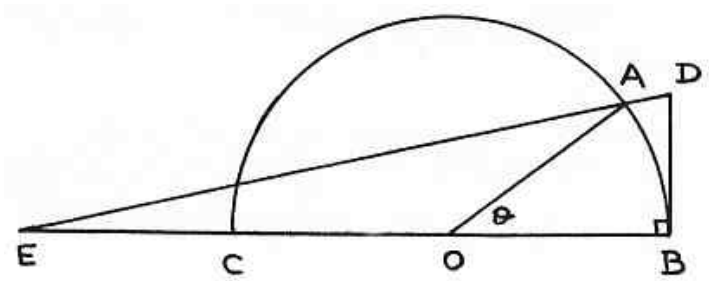
The formula is obviously not accurate, for with $\theta = 90°$ it gives the length of a quarter of the circumference as $\frac{3}{2}$. That is equivalent to using 3 as the value of π. However, the formula is quite accurate for small values of θ. If $\theta = 30°$, the formula give the arc length as

$$\frac{3\sin 30°}{2+\cos 30°} = \frac{3\cdot\frac{1}{2}}{2+\frac{\sqrt{3}}{2}} = \frac{3}{4+\sqrt{3}} = 0,5233\ldots$$

while the correct answer is $\pi/6 = 0.5235\ldots$.

How is the formula derived? Cusanus gave a complicated discussion, but 22 years later Christiaan Huygens and Willebrord Snell, two Dutch physicists

renowned for their work in optics, found a simpler approach.



In the figure, $EC = CO = OB = 1$ and $\angle AOB = \theta$. If θ is small, the length of arc AB is very close to BD, and it's a nice little exercise in trigonometry to show that $BD = \dfrac{3\sin\theta}{2+\cos\theta}$. □

SHARP COMMENT

Dr Geoff Smith (University of Bath, U.K.) has noted that there is a much shorter and neater solution to the second problem in the Sharp Competition (*Mathematical Digest* **162**, January 2011, page 18).

$$\textit{If } \frac{(a-b)(c-d)}{(b-c)(d-a)} = k, \textit{ determine the value of } \frac{(a-c)(b-d)}{(a-b)(c-d)}.$$

The key insight is to note that each of the two expressions is invariant under translation and scaling, i.e. they remain unchanged if we replace the variables a, b, c, d by $a+t, b+t, c+t, d+t$ or by ta, tb, tc, td, respectively.

So we may put $a = 0$ and $d = 1$.

The problem then reduces to proving that

$$\textit{If } \frac{(-b)(c-1)}{(b-c)} = k, \textit{ determine the value of } \frac{(-c)(b-1)}{(-b)(c-1)}.$$

If we denote the second term by x, it follows that

$$\frac{1}{k} + x = \frac{b-c}{b-bc} + \frac{c-bc}{b-bc} = 1,$$

and so $x = 1 - \dfrac{1}{k} = \dfrac{k}{k-1}$. □

TRIG PRODUCTS AND SUMS

The trig identities below convert products of trig functions to sums.

$$\sin A \sin B = \frac{1}{2}(\cos(A-B) - \cos(A+B))$$
$$\cos A \cos B = \frac{1}{2}(\cos(A-B) + \cos(A+B))$$
$$\sin A \cos B = \frac{1}{2}(\sin(A+B) + \sin(A-B))$$
$$\cos A \sin B = \frac{1}{2}(\sin(A+B) - \sin(A-B))$$

The first two formulas are easily established by adding and subtracting the formulas for the expansions of $\cos(A \pm B)$.

Doing the same for the expansions of $\sin(A \pm B)$ gives the other two formulas.

Now put $A + B = P$ and $A - B = Q$.

Then $A = \dfrac{P+Q}{2}$ and $B = \dfrac{P-Q}{2}$, and we have formulas for converting sums into products:

$$\cos Q - \cos P = 2 \sin \frac{P+Q}{2} \sin \frac{P-Q}{2}$$
$$\cos Q + \cos P = 2 \cos \frac{P+Q}{2} \cos \frac{P-Q}{2}$$
$$\sin P + \sin Q = 2 \sin \frac{P+Q}{2} \cos \frac{P-Q}{2}$$
$$\sin P - \sin Q = 2 \cos \frac{P+Q}{2} \sin \frac{P-Q}{2}$$

These formulas are sometimes useful. They are not worth memorizing, but if you need one, it can be quickly worked out. □

SPACE MATH

NASA astronomer and educator Sten Odenwald has created the website Space Math @ NASA (`http://spacemath.gsfc.nasa.gov/2`), an online educational program helping students understand connections with mathematics in their everyday lives. □

823 AGAIN

This year, July has 5 Fridays, 5 Saturdays and 5 Sundays. This happens once every 823 years. This is called money bags. So, forward this to your friends and money will arrive within 4 days. Based on Chinese Feng Shui the one who does not forward ..... will be without money. Not taking any chances!

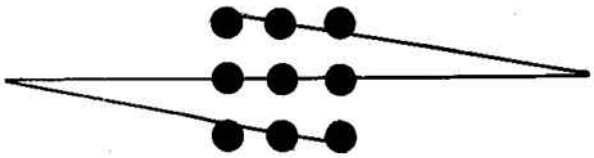
The curious 823 legend is being kept alive in the silly world of email chain letters, as the above extract shows. It is perfectly true that July 2011 has 5 Fridays, 5 Saturdays and 5 Sundays. However, such a combination occurs far more often than every 823 years. The last occurrence was in July 2005 while the next occurrence will take place in July 2016. The message is just a revamped version of very similar – and equally erroneous – messages about August and October 2010 (see *Mathematical Digest* **162**, January 2011).

And, in any case, there is nothing even remotely unusual about months that have such "interesting" combinations of days. In fact, any month that has 31 days will have three consecutive days that occur five times in the month. Since there are seven 31-day months in a year, such combinations are commonplace.

At the end of this year December will have five Thursdays, five Fridays and five Saturdays. And the next month, January 2012, will have five Sundays, five Mondays and five Tuesdays. And at the end of 2012 December will have have five Saturdays, five Sundays and five Mondays, followed by January 2013 with five Tuesdays, five Wednesdays and five Thursdays. □

NINE DOTS IN THREE LINES

The nine dots can be joined with three lines as shown below.



Can you think of a way of joining the nine dots using just one line?

The answer is on page 27 □

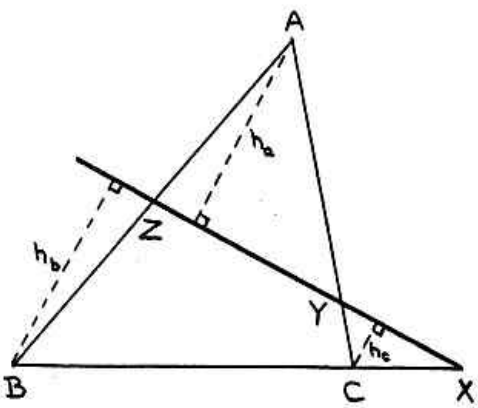
THE THEOREM OF MENELAUS

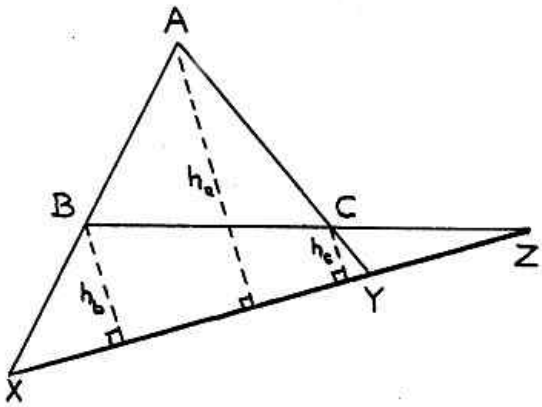
Menelaus of Alexandria (not to be confused with Homer's Menelaus, the husband of Helen of Troy) was a mathematician who wrote a treatise on spherical geometry in about 100 A.D. In it he quoted a theorem which now bears his name.

The Theorem of Menelaus concerns a triangle ABC with three points on its sides (produced if necessary): X on BC, Y on AC, Z on AB. The theorem says that if X, Y and Z are collinear (i.e. they lie in a straight line) then

$$\frac{BX}{CX} \cdot \frac{CY}{AY} \cdot \frac{AZ}{BZ} = 1.$$

For X, Y and Z to be collinear, either just one, or all three of the sides must be produced, giving rise to two possible situations.





Drop perpendiculars onto XYZ from A, B and C, of lengths h_a, h_b and h_c respectively.

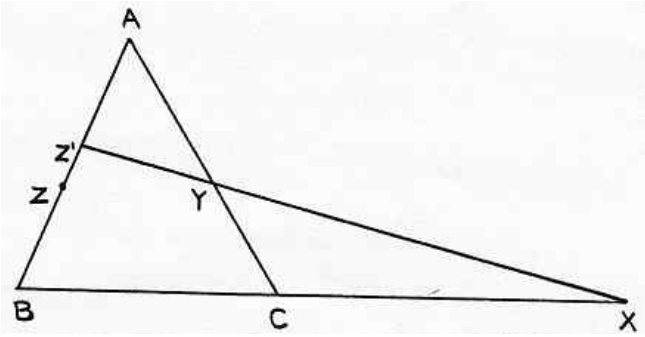
Then $\dfrac{B}{CX} = \dfrac{h_b}{h_c}$, $\dfrac{CY}{AY} = \dfrac{h_c}{h_a}$ and $\dfrac{AZ}{BZ} = \dfrac{h_a}{h_b}$,

and it follows that $\dfrac{BX}{CX} \cdot \dfrac{CY}{AY} \cdot \dfrac{AZ}{BZ} = \dfrac{h_b}{h_c} \cdot \dfrac{h_c}{h_a} \cdot \dfrac{h_a}{h_b} = 1.$

The converse of the theorem of Menelaus is also true.

Suppose that X, Y and Z lie on the sides (perhaps produced) BC, CA and AB respectively of a triangle ABC, such that $\frac{BX}{CX} \cdot \frac{CY}{AY} \cdot \frac{AZ}{BZ} = 1$. Then X, Y and Z are collinear.

We begin the proof by producing XY to cut AB in Z'.



Then $\dfrac{BX}{CX} \cdot \dfrac{CY}{AY} \cdot \dfrac{AZ'}{BZ'} = 1$ by Menelaus' Theorem.

But $\dfrac{BX}{CX} \cdot \dfrac{CY}{AY} \cdot \dfrac{AZ}{BZ} = 1$ by assumption.

Hence $\dfrac{AZ'}{BZ'} = \dfrac{AZ}{BZ}$.

The final step is the deduction that $Z = Z'$, which shows that X, Y and Z are collinear. □

MALEMA'S MATHS

Julius Malema is concerned about the pace of land reform, as the following quotations show.

"At the rate we are going now, we will only be able to expropriate 5 percent of land every 20 years, meaning it will take us 100 years to expropriate 20 percent of the land."

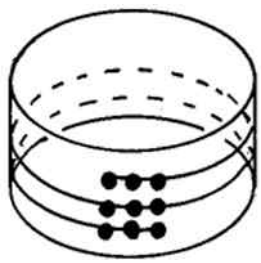
(*Cape Times*, 6 May 2011)

"Indications are very clear that by 2014 we will only have transferred 5%. So what this means is that every 20 years we transfer only 5%. It means in 100 years we would have transferred less than 20%."

(*Sunday Times*, 12 June 2011) □

NINE DOTS IN ONE LINE

If the nine dots are drawn on a strip of paper which is folded and joined together, they can be joined by just one line.
Can you think of a situation of nine dots joined, with zero lines?



The answer is on page 29. □

HOW TO LOSE AT ROULETTE

Now that casinos are becoming increasingly popular with South African holidaymakers, winning systems for roulette devised by self-described "mathematical geniuses" are being advertised at exorbitant prices. Have they any value?

The answer is a categorical and unequivocal NO. Roulette is easily analyzed using only elementary probability theory, and there is simply no winning system. Of course, it is quite possible for you to play the game for a few spins of the wheel and come out on top, regardless of what system you use. A winning system, however, must guarantee that you win in the long run.

Roulette is, in principle, no different from flipping a coin with a friend and betting on the result. Suppose you agree that, as your friend flips the coin, you call Heads (or Tails). If your call is correct, your friend pays you 10c. If not, you pay your friend 10c.

On each flip of the coin, you have a probability of $\frac{1}{2}$ of winning 10c, and a probability of $\frac{1}{2}$ of losing 10c. The total expectation is $\frac{1}{2}\times 10+\frac{1}{2}\times(-10) = 0$. It's a perfectly fair game. In the long run you neither win nor lose.

Now let's change the rules of this game a little. Each time you call correctly, you win 8c. Each time you are wrong, you pay 10c. Your expectation is now negative:

$$\tfrac{1}{2} \times 8 + \tfrac{1}{2} \times (-10) = -1.$$

On average you lose one cent per flip. Of course, you can easily have a winning streak, but in the long run you will end up the loser. What's more, there is no possible "winning system" of calling Heads or Tails which can turn your negative expectation into a positive one.

We now move from this simple and transparent example of flipping a coin, to roulette. The coin — a randomizing device with two equally likely outcomes — is replaced by a spinning roulette wheel and a little ivory ball which bounces around inside it. When the wheel comes to rest, the ball settles into one of the 37 compartments, numbered from 0 to 36. Like the coin, the wheel is a randomizing device, with 37 equally likely outcomes

instead of two. The gambler bets on these outcomes, or combinations of them.

Let us examine what happens when you place a bet on your favourite number. If it comes up, the bank pays you odds of 35 to 1, i.e. you get your stake back plus 35 times your stake. If your number doesn't come up, you lose your stake. On average, you lose 36 times and win only once in 37 spins of the wheel. Your expectation for each stake of one unit is

$$\tfrac{1}{37} \times 35 + \tfrac{36}{37} \times (-1) = -\tfrac{1}{37}$$

i.e. on average you lose about 2.7% of your stake on every spin of the wheel.

Though there is a confusingly large number of ways of betting at roulette, each way can be analyzed as above, and the odds offered by the bank ensure that every stake has a negative expectation. Any combination of stakes has a negative expectation, for the simple reason that when you add negative numbers you get a negative answer. There is just no way you can play roulette, and win. □

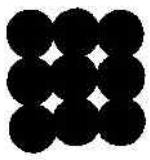
DUSNER NUMBERS

The Indonesian language *Dusner* has an unusual number system: it is in base five. The number word for six (*rindi yoser*) is a combination of the words for five (*rindi*) and one (*yoser*).

The word for ten is *sampur* and twenty is *snontujoser*. Thirty is *snontujoser e sampur* (twenty and ten), forty is *snontunoru* (two times twenty) and so on up to one hundred, *utin*, which also means "a very large number". That is as far as Dusner counting goes.

Dusner is a dying language. There are only three native speakers left. □

NINE DOTS JOINED WITH NO LINES



A BRIEF HISTORY OF GRAVITY

Bruce Elliot

It filled Galileo with mirth
To watch his two rocks fall to Earth.
  He gladly proclaimed,
  "Their rates are the same,
And quite independent of girth!"

Then Newton announced in due course
His own law of gravity's force:
  "It goes, I declare,
  As the inverted square
Of the distance from object to source."

But remarkably, Einstein's equation
Succeeds to describe gravitation
  As spacetime that's curved,
  And it's this that will serve
As the planets' unique motivation.

Yet the end of the story's not written;
By a new way of thinking we're smitten.
  We twist and we turn,
  Attempting to learn
The Superstring Theory of Witten! □

BIBLICAL MATHS ANSWERS

Only two of the ten mathematical words can be found in the King James Bible.

MATRIX can be found in four places in the Old Testament: Exodus 13:12 and 34:19, and Numbers 3:12 and 18:15. In each case the word means *womb*.

QUATERNIONS can be found in the New Testament (Acts 12.4). A *quaternion* was a squad of four Roman soldiers. When the Apostle Peter was thrown into prison, King Herod assigned "four quaternions of soldiers to keep him". □

ANSWERS

Quickies (page 12)

1. To find one winner, 99 must be knocked out. So 99 matches must be played.
2. 20
3. Don't know? Have you forgotten your name?
4. The problem can only be solved by splitting hares.
5. Pick up glass 2, empty it into glass 5 and replace it in its original position.
6. Since one of the factors of the polynomial is $(x - x)$, all the coefficients are zero.
7. His base camp is situated near the North Pole, less than 10 km from it. So starting off in a northerly direction and walking in a straight line would take him to the North Pole and beyond. The next day he starts in a northerly direction from his overnight camp and retraces his steps past the North Pole, arriving at his base camp after 10 km.
8. 18
9. When written out in words, the numbers are in alphabetical order.
10. 15. □

Literary Quiz (page 17)
X = Pooh
Y = Christopher Robin
Z = Rabbit
W = Piglet

The extract comes from *The House at Pooh Corner*, by A.A. Milne. □

Varsity Readiness Test (page 8)

54422 43212 14434 12231 11542

□

mathematical digest

No. 165 October 2011

International Mathematical Olympiad Amsterdam 2011

mathematical digest

Number 165 **October 2011**

A high school student doing Euclidean geometry gets no credit for pulling out a ruler and measuring the triangle, even though that guarantees a correct answer. The point of the lesson is to inculcate a method that later can be used to calculate the unmeasurable, such as the distance to the moon.

(Steven Pinker, in *How the mind works*) □

Department of Mathematics and Applied Mathematics
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Cover: For news of the IMO see page 17.

IMO 2014 IN SOUTH AFRICA

The International Mathematical Olympiad began in 1959 in Romania, and for a number of years was dominated by Eastern Bloc countries. For the obvious political reasons, South Africa was not able to take part.

The political situation changed quickly in the early 1990s, and in 1991 South Africa had an Observer at the IMO in Sweden. Since 1992 South Africa has been a regular participant in the IMO.

The IMO takes place in a different country every year, but has never been held on the African continent. At every IMO (even in Sweden in 1991) South Africans have been asked when we will host the event. The cautious stock response has always been that the country was not yet ready.

The IMO is a big event. It costs a lot of money. The tradition is that the host country pays all the expenses (except the international travel) of participating teams. In the Netherlands this year the price tag was not far short of two million Euros. A country also needs to have good infrastructure (airline connections, hotels). In these respects South Africa, having hosted the FIFA World Cup in 2010, is clearly ready.

But the most important requirement is academic expertise: a body of mathematicians with substantial IMO experience is needed. A small problems committee of top experts is needed to draw up a bank of suitable problems, all of which must be original. A much larger group is needed to mark the papers, in consultation with the Team Leaders. Over the years, the number of South African IMO medallists has been growing, and they are now taking a significant role in training the country's teams. With a solid body of expertise being built up, in early 2011 it was decided to make a formal bid to host the IMO in South Africa in 2014.

The bid was presented at the IMO in Amsterdam in July 2011. It was endorsed by the IMO Advisory Board and approved by the full Jury of 101 countries at its final meeting in Amsterdam. It was clear that countries are looking forward with enthusiasm to taking part in the first IMO in Africa.

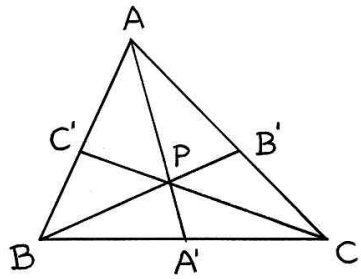
An African IMO will require substantial African representation. In the recent IMO in Amsterdam only five of the 54 African countries (South Africa, Nigeria, Tunisia, Morocco, and Ivory Coast) took part. One of the special targets of IMO2014 will be to increase African participation. □

CEVA'S THEOREM

Giovanni Ceva (1647-1734) was an Italian mathematician whose name is today attached to a theorem in elementary geometry.

If P is a point inside triangle ABC, and the lines AP, BP and CP are produced to meet sides BC, CA and AB in A', B' and C', respectively, then

$$\frac{AC'}{C'B}.\frac{BA'}{A'C}.\frac{CB'}{B'A} = 1.$$



He published the result in his book *De lineis rectis* (On Straight Lines). Ceva's Theorem is closely connected with Menelaus' Theorem (see *Mathematical Digest* **164** (July 2011, pp. 26–27)), which Ceva rediscovered and published.

The proof of Ceva's Theorem relies on a simple fact about areas of triangles: if two triangles have the same altitude, then the ratio of their areas is equal to the ratio of their bases.

We shall use the notation $|XYZ|$ to denote the area of triangle XYZ.

Applying the area observation to the triangles $BA'A$ and $A'CA$, and then to triangles $BA'P$ and $A'CP$ we have

$$\frac{BA'}{A'C} = \frac{|BA'A|}{|A'CA|} = \frac{|BA'P|}{|A'CP|}\,. \qquad (1)$$

We now use a simple result in algebra:

$$\text{If } \frac{a}{b} = \frac{c}{d}, \text{ then both are equal to } \frac{a-c}{b-d}.$$

(Why? Put $\frac{a}{b} = \frac{c}{d} = k$. Then $a = bk$ and $c = dk$, and subtracting these two equations gives the result.)

Applying the result to equation (1), we have

$$\frac{BA'}{A'C} = \frac{|BA'A| - |BA'P|}{|A'CA| - |A'CP|} = \frac{|BPA|}{|CPA|}. \qquad (2)$$

Similarly, using triangles $CB'B$ and $B'AB$ we get

$$\frac{CB'}{B'C} = \frac{|CPB|}{|APB|} \qquad (3)$$

and

$$\frac{AC'}{C'A} = \frac{|APC|}{|BPC|} \qquad (4)$$

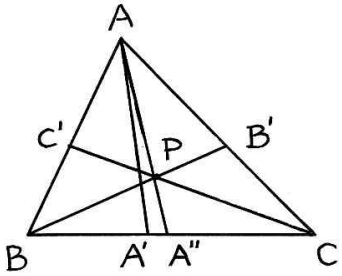
Multiplying equations (2), (3) and (4) together gives

$$\frac{BA'}{A'C}.\frac{CB'}{B'C}.\frac{AC'}{C'A} = \frac{|BPA|}{|CPA|}.\frac{|CPB|}{|APB|}.\frac{|APC|}{|BPC|} = 1,$$

which proves Ceva's Theorem.

The converse of Ceva's Theorem is also true:

> If A', B' and C' are points on the sides of BC, CA and AB, respectively, such that
> $$\frac{AC'}{C'B}.\frac{BA'}{A'C}.\frac{CB'}{B'A} = 1$$
> then the lines AA', BB' and CC' are concurrent (i.e. they meet in the same point).



Suppose that the lines BB' and CC' meet in the point P. Join AP and produce it to meet BC in A''. The proof now amounts to showing that A' and A'' coincide.

We are given that

$$\frac{AC'}{C'B}.\frac{BA'}{A'C}.\frac{CB'}{B'A} = 1. \qquad (5)$$

We also know, from Ceva's Theorem, that

$$\frac{AC'}{C'B}.\frac{BA''}{A''C}.\frac{CB'}{B'A} = 1. \qquad (6)$$

Comparing equations (5) and (6), gives

$$\frac{BA'}{A'C} = \frac{BA''}{A''C}$$

from which it follows that $A' = A''$. □

NUMBER QUIZ

What is the significance of the number 9 192 631 770?

The answer can be found on page 30. □

THE COLLATZ CONJECTURE

Pick any positive integer.
If it is even, divide it by 2.
If it is odd, multiply it by 3 and add 1.
Repeat the process.

For example, with starting point 7 you get the sequence

$$7, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1$$

after which the last three numbers repeat.

Try another number, say 23:

$$23, 70, 35, 106, 53, 160, 80, 40, 20, 10, 5, 16, 8, 4, 2, 1.$$

Once again, the process ends on the 4, 2, 1 cycle.

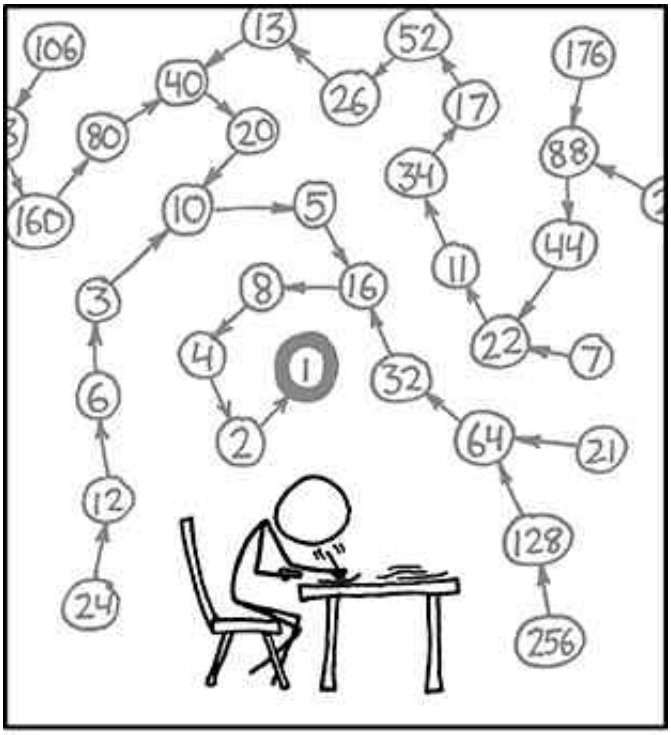
If you try a few more numbers, the same happens.

Back in 1937 a young German mathematician, Lothar Collatz, noticed this phenomenon, and conjectured that, whatever starting number you choose, the sequence always settles down to 4, 2, 1. He was, however, unable to prove that his conjecture was true. Nor could he find a number where the sequence carried on without descending to earth.

A simple-minded argument in terms of probability suggests that the conjecture is plausible. When the sequence hits an odd number, it rises by a factor of 3, and adding 1 makes the next number even. So at the next step the sequence drops by a factor of 2, and half the time it will drop by a factor of four, or a quarter of the time by a factor of 8. As soon as the sequence hits a power of 2, it plummets to 1.

The way the sequence rises and falls has given it the popular name of "Hailstone numbers", by analogy with the way hailstones rise and fall in clouds, accumulating layers of ice before finally descending to earth.

For the last 70 years the Collatz Conjecture has been an unsolved problem. It was already 34 years old when it was featured in the very first edition of *Mathematical Digest* back in 1971. The Hungarian mathematician Paul Erdös recognized the problem's difficulty, and offered a prize of $500 for a solution. Saying "Mathematics is not yet ready for such problems" he obviously felt his money was safe.



THE COLLATZ CONJECTURE STATES THAT IF YOU PICK A NUMBER, AND IF IT'S EVEN DIVIDE IT BY TWO AND IF IT'S ODD MULTIPLY IT BY THREE AND ADD ONE, AND YOU REPEAT THIS PROCEDURE LONG ENOUGH, EVENTUALLY YOUR FRIENDS WILL STOP CALLING TO SEE IF YOU WANT TO HANG OUT.

The Collatz Conjecture has a reputation for being addictive. It made the rounds of American universities during the Second World War and got the reputation of being a German conspiracy to divert American mathematicians from the war effort. When it continued to distract mathematicians in the 1960s, the blame was put on the Soviet Union for promoting the problem and sabotaging Western science.

The problem is very easy to set up on a computer:

```
procedure collatz(x)
if x=1 then halt
elseif even(x) then collatz(x/2)
elseif odd(x) then collatz(3*x+1)
```

Run the program with an input of 27. It goes on for quite a long way, but eventually the numbers descend to 4, 2, 1. An input of $x = 319\ 804\ 831$ gives an even longer run.

The Collatz Conjecture has been checked for numbers up to $20 \times 2^{58} = 5\ 764\ 607\ 523\ 034\ 234\ 880$. But without a proper mathematical proof there

is always the possibility that a larger number could break the rule.

Collatz never returned to his "$3x + 1$ problem" after the 1930s; his later work was in numerical analysis. He died in 1990.

Earlier this year a former student of Collatz, Gerhard Opfer (University of Hamburg), announced that he had submitted a 32-page proof of the Collatz Conjecture to the journal *Mathematics of Computation.* Instead of waiting for the journal's referees and editors to check his argument, he put the proof online.

It immediately hit the headlines, and then hit a barrage of objections. Flaws in the argument were pointed out, and Opfer withdrew the paper.

As one mathematician wryly commented "Such is mathematics in the Internet age - no longer are proofs brought down from the mountain top in their perfection but they are thrown to the crowd to survive being torn apart." □

COIN CHOICE CONUNDRUM

Fifty coins, of various denominations, are lined up on a table. Jake and Julie take turns in selecting a coin from either end of the line, with Jake making the first choice.

Prove that Jake can play in a way that guarantees that he gets at least half the money.

The answer is on page 30. □

HISTORY QUIZ

Which famous scientist was born on Christmas Day?
The answer is on page 30. □

EXPRESSO MATHS

Top winners in the 2011 UCT Mathematics Competition were interviewed live on the early-morning SABC3 television show *Expresso* on 6 June.

View the clip at `http://www/youtube.com/watch?v=VOFrb2_PfEQ`. □

GOLDBACH VARIATIONS

The Goldbach Conjecture is a famous unsolved problem of mathematics. It was first proposed by a German mathematician Christian Goldbach, in a letter to the famous Swiss mathematician Leonhard Euler. Noting the following sums,

$$4 = 2 + 2,\ 6 = 3 + 3,\ 8 = 3 + 5,\ 10 = 3 + 7,\ 12 = 5 + 7,\ 14 = 3 + 11$$

$$16 = 3 + 13,\ 18 = 5 + 13,\ 20 = 3 + 17,\ 22 = 11 + 11,\ 24 = 7 + 17$$

Goldbach wondered whether *every* even number could be expressed as the sum of two prime numbers.

In his reply, Euler wrote "I regard this as a completely certain theorem, although I cannot prove it."

And to this day, nobody has been able to prove the truth of Goldbach's Conjecture.

There are a number of variations of Goldbach's conjecture, which are all easy enough for our readers prove:

1. Every natural number greater than 11 is the sum of two composite numbers
2. Every natural number greater than 5 is the sum of a composite number and a prime number.
3. Every even natural number greater than 8 is the sum of two composite numbers.
4. Every even number greater than 38 can be expressed as the sum of two odd composite numbers.

Hints can be found on page 30. □

VOTE QUOTE

It does not matter how they vote. It matters how we count.

(Joseph Stalin) □

QUICK SQUARE ROOTS

Suppose we want to calculate $\sqrt{47}$, and no calculator is handy. Here's a way of doing it your head.

We start by noting that $6^2 = 36 < 47 < 49 = 7^2$, from which it follows that $6 < \sqrt{47} < 7$. That shows that the whole number part of $\sqrt{47}$ is 6.

The fractional part is then calculated as

$$\frac{47-36}{6+7} = \frac{11}{13}.$$

So the square root of 47 is $6\frac{11}{13}$. This is $6.846\ldots$, which is not a bad approximation to the correct value $6.855\ldots$.

Another example: to calculate $\sqrt{71}$, we find that the integer part is 8 (since $8^2 = 64 < 71 < 81 = 9^2$), and the fractional part is

$$\frac{71-64}{8+9} = \frac{7}{17}.$$

So the square root of 71 is $8\frac{7}{17} = 8.411\ldots$, which is again a good approximation to the correct value $8.426\ldots$.

In general, suppose you want to find $\sqrt{A}$.

The first step is to set A between two consecutive squares n^2 and $(n+1)^2$, which is the same as writing A in the form $n^2 + d$, where

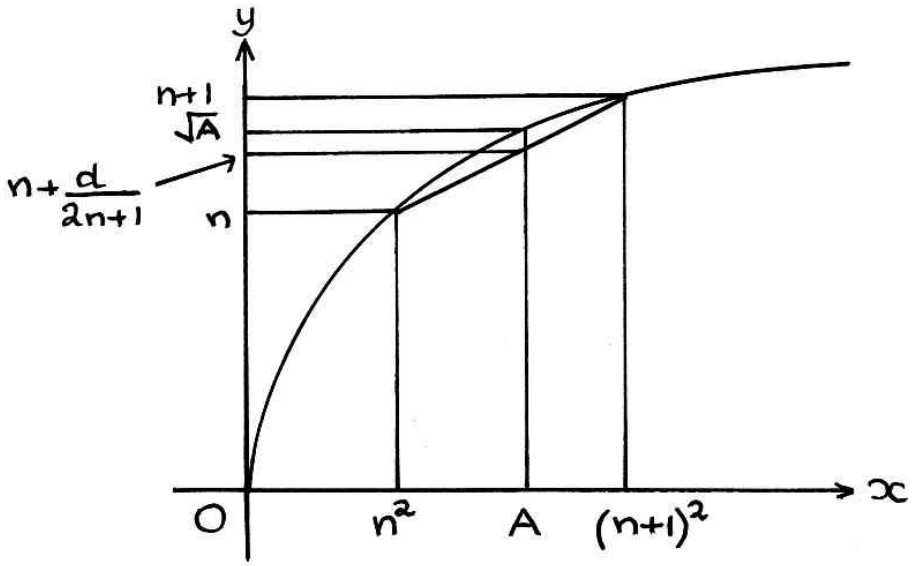
$$n^2 < n^2 + d < (n+1)^2.$$

The algorithm gives

$$n + \frac{d}{2n+1}$$

as the approximate square root.

The graph below of the parabola $y = \sqrt{x}$ explains why the algorithm works.



Consider the straight line joining the point (n, n^2) to $((n+1), (n+1)^2)$. It has slope $1/(2n+1)$ and equation

$$y = n + \frac{x - n^2}{2n+1}.$$

Putting $x = A = n^2 + d$ gives $y = n + \dfrac{d}{2n+1}$.

Since the line lies below the parabola, the algorithm gives an approximate square root which is always a little too small.

In geometrical terms, the algorithm replaces the arc of the parabola with a straight line segment. The true square root is given by the y-coordinate on the parabola, while the approximate square root is obtained from the straight line.

Because a line segment is involved, this method is known as *linear interpolation*.

Could this method be modified to find approximate cube roots? □

TV QUIZ

> "The best number is 73. Why? 73 is the 21st prime number. Its mirror (37) is the 12th and its mirror (21) is the product of multiplying, hang on to your hats, 7 and 3. In binary, 73 is a palindrome, 1001001, which backwards is 1001001."

Who said this, in which popular TV series? The answer is on page 31. □

SHAKESPEARE'S PSALM

The 400th anniversary of the publication of the King James translation of the Bible has been celebrated during 2011.

Because of the beauty of its prose and poetry, the King James Bible has had a profound impact on the English language.

The translation was the work of eminent English clerics and theologians, and relied extensively on previous translations of the Old and New Testaments into English.

There is widely-spread story that William Shakespeare was one of the scholars involved in the translation. To be sure, Shakespeare was the greatest playwright and poet of his time, so it is feasible that he could have been asked to take part in the massive work. But a little thought shows that this was unlikely. Shakespeare was not nobly born and not university-educated. Moreover, acting and the theatre were not regarded as respectable professions in those days.

Nevertheless, there is a piece of numerical evidence that Shakespeare was one of the translators, and took trouble to leave his signature in the translation in a very clever way.

Go to Psalm 46, and count 46 words from the beginning. You end on the word *shake*. Now count 46 words from the end, and you find *spear*. Put them together and you have the playwright's name.

Why 46? The translation was completed in 1610, the year in which Shakespeare turned 46.

There is an extra twist to the tale. Some serious doubts have been raised that Shakespeare really wrote the works attributed to him. The true author, so the theory goes, was Francis Bacon, the pioneer scientist, scholar and man of letters. Bacon moved in all the top circles of the English establishment, and is far more likely to have been consulted by the Bible translators than the low-class uneducated actor from Stratford. So Bacon wrote the plays and sonnets, but published them under the pseudonym Shakespeare.

While working on the translation, Bacon then had the idea of hiding his name somewhere in the King James Bible, but could find no suitable place. After all "Bacon" is just not kosher, so it would be difficult to find a place for it in the Old Testament. So he decided to hide his playwright-pseudonym instead in Psalm 46. □

NUMBER PATTERN

$$\begin{aligned} 47 + 2 &= 49 \\ 47 \times 2 &= 94 \\ 497 + 2 &= 499 \\ 497 \times 2 &= 994 \\ 4997 + 2 &= 4999 \\ 4997 \times 2 &= 9994 \\ 49997 + 2 &= 49999 \\ 49997 \times 2 &= 99994 \end{aligned}$$

WEST CUMBRIAN COUNTING

Joseph rhymed off the count from one to twenty, in his own, the West Cumbrian dialect, singing it almost, as the words demanded:

> 'Yan, tyan, tethers, methera, pimp, sethera, lethera, hovera, dovera, dick. Yan-a-dick, tyan-a-dick, tethera-dick, methera-dick, bumfit.'
>
> 'Bumfit!' Mr Lenty interrupted ecstatically. 'Oh, thou Bumfit! My Bumfit! Now why can't we *still* say Bumfit. Fifteen doesn't hold a candle to it. Bumfit! Oh – go *on*, Joseph.'
>
> 'Yan-a-bumfit, tyan-a-bumfit, tethera-bumfit, methera-bumfit, giggot.'
>
> 'Giggot!' said Mr Lenty. 'Twenty. And-the-days-of-thy-years-are-tethera-giggots-and-dick. Now isn't that better than three score and ten? Tethera giggots and dick. It *sounds* like a lifetime, doesn't it? I could hear you repeat that all evening.'

(From "*A Place in England*" by Melvyn Bragg.) □

GRAEME WEST

Graeme West, who played a major role in setting up the programme to train students for participation in the International Mathematical Olympiad, died on Saturday 10 September after complications from surgery.

Graeme attended St Stithian's College and entered the University of Cape Town as an actuarial student in 1985. He moved over to mathematics and graduated B.Sc (1987), B.Sc. (Hons.) (1988), M.Sc. (1990) and Ph.D. 1993.

In 1991, while working on his Ph.D (under the supervision of Dr Jurie Conradie) he discovered the world of the International Mathematical Olympiad (IMO). Emerging from decades of academic boycotts and isolation, South Africa was invited that year to send an Observer to the IMO in Sweden, with a view to full participation the following year. To tackle the daunting task of participating in what is arguably the most challenging intellectual activity for high schools, a nationwide Talent Search was set up at the University of Cape Town to select and prepare teams for the IMO.

Graeme, a consummate problem-solver, took an immediate interest in the Talent Search and was soon playing a major role. He drew up an extensive data bank of challenging problems, which he used in a programme of monthly assignments for aspirant IMO team members. He played an important part in running vacation training camps at Stellenbosch, Rhodes and other universities for the best young mathematicians in the country.

Graeme's rapport with young mathematicians made him an obvious choice for Deputy Team Leader of South Africa's first IMO teams, a role he filled with distinction from 1992 to 1997.

The first two years of South Africa's participation in the IMO were undistinguished. But in 1994 at the IMO in Hong Kong the Talent Search and training camps paid off. South Africa won three Bronze Medals and an Honourable Mention, and was ranked 27th out of 69 countries.

David Hatton, who was in the IMO teams of 1995 and 1996, wrote:

> "It was Graeme's passion and confidence that inspired me to work at getting into the national Mathematics Olympiad team, and

> thereafter his example that motivated me to pursue Olympiad training myself. Without Graeme my life would have been very different and I'm eternally grateful. I'm very sad to hear about his passing."

From 1994 to 1999 Graeme edited the annual reports of the IMO programme, invaluable sources of IMO training material that are still available.

Graeme wrote three training manuals for the Olympiad enthusiast (on Graph Theory, Inequalities and Functional Equations) that are still in use today, both in South Africa and overseas.

József Pelikán, the Hugnarian IMO Team Leader, said

> "Graeme's booklet 'Functional Equations for the Olympiad Enthusiast' is a masterpiece of writing on mathematical problem solving. The chapter on 'Pointers to solving functional equations' and the subsequent examples illustrating the pointers could serve as the standard model for future similar notes on solving mathematical problems. For sure, I myself benefitted from them a lot."

Graeme was a gifted coach, and the successes that South Africa achieved in the IMO in those early years are a tribute to his work. □

UK PRIZE PROBLEMS

The U.K. magazine *Mathematical Gazette* has a regular "Student Problems" section, with prizes for the best entries.

Three Cape Town students featured in a recent edition: Robert Spencer and Sean Wentzel (both of Westerford High School) were highly commended for their entries, and Kira Düsterwald (Springfield Convent Senior School) was awarded a £20 prize.

The problems are below.

1. Find the greatest common divisor of all numbers of the form $p^4 - 1$, for p a prime greater than 5.

2. Let the triangle ABC be inscribed in a circle and let the tangents at A,, B and C intersect the lines BC, CA and AB at P, Q and R, respectively. Prove that P, Q and R are collinear. □

CHATTING OUR WAY TO BETTER MATHEMATICS

Laurie Butgereit
CSIR

MXit is fast becoming a household name to the youth of South Africa. This mobile instant messaging platform provides instant messaging and social networking between users at a minimal cost. The popularity of MXit amongst teenagers led researchers at the CSIR Meraka Institute to develop a mobile tutoring service, Dr Math, which allows users to receive maths tutoring by accessing tutors on their cellphones via the MXit network. Dr Math provides real-time support and assistance with mathematics homework and revision. The service, likened to a text-based call centre, was initiated in 2007 and has had over 25 000 registered users to date, mainly through word-of-mouth advertising.

Tutors are people trained to act as Dr Math. They have a strong mathematical background, are required to understand how to use the medium and conform to a strict code of conduct. Dr Math tutors are managed in a similar way to call centre agents and are available Sundays to Thursdays between 14:00 and 20:00. There are currently 110 volunteers from around the country—most are engineering students from the University of Pretoria.

Dr Math was initially developed as a stand-alone service. The CSIR has, however, over the past six months developed a generic set of tools that enables anyone to set up a similar service to Dr Math. The toolset is known as 'C3TO' (Chat Call Centre Tutor Online), which currently uses the MXit platform as a channel, but can utilise other platforms that support Google Talk. This means that a service like Dr Math can be made available via most mobile instant messaging platforms (e.g. MTN's NokNok, Vodacom's The Grid, Mig33, Yahoo Messenger, MSN Instant Messenger) and also via PCs running a Google Talk-compliant instant messaging client.

There are additional features available via Dr Math, which can be used by learners when the tutors are not online. This information includes definitions of mathematical terms (such as asymptote and parabola) and the definitions of formulae such as trigonometric identities, geometric calculations of area and perimeters, and logarithmic identities. Games and competitions, which hone mathematics skills, have been implemented and

have proved to be very successful and popular with learners. A scientific calculator has also been made available.

The CSIR strongly supports the notion that the Dr Math service is an investment into the growth of mathematics skills of learners in South Africa. The CSIR is currently working on licensing Dr Math to an independent organisation that will be able to offer the service nationally and grow the initiative to achieve its maximum impact. The intention is that learners should be able to use the service free of charge or at minimal cost.

Potential tutors can register online at

`http://c3to.meraka.csir.co.za/c3to`.

To ensure ethics clearance, tutors will be emailed a code of conduct which they must sign and return along with a copy of their ID or passport before they will be authorised to chat with pupils. The safety and security of the minor pupils are most important.

Learners can access Dr Math by registering on MXit and adding Dr Math to their contacts and navigating to Tradepost/MXit/Cares/Education/Dr Math.

Enquiries: Laurie Butgereit `lbutgereit@csir.co.za`

UCT MATHS COMPETITION BOOK

The 2011 UCT Mathematics Competition took place in April, with 6636 participants from 134 high schools in the Western Cape.

The annual report has just been published. The 96-page book comprises all the questions papers, detailed solutions, a full item analysis of all the problems, graphs of the performance in each grade and the names of the winners.

aurecon

With the support of Aurecon, a free copy of the book is sent to the 200+ teachers who took part in the organization of the event. Further copies may be ordered — see the back of the magazine for details. □

SHARP CALCULATOR COMPETITION

The Sharp Calculator Competition is sponsored by Sharp Electronics (S.A.) (Pty) Limited, who have donated three calculators as prizes.

These prizes are offered for the three best solutions of the problems below. Solutions may be in English or Afrikaans. Partial solutions will be judged on their merits. The Editor reserves the right to make no award if no satisfactory entries are received, and his judgement will be final. The winner of an nth prize in a previous competition will only be eligible for an mth prize if $m < n$.

Only high school students in grade 11 or lower may take part. Entrants must give their full name and home address, age, school and grade. Solutions must be certified by a mathematics teacher at the school as the entrant's own unaided work.

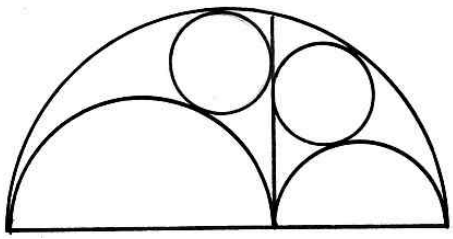
The closing date for receipt of entries is 1 November 2011, and the results will be announced in our January 2012 edition.

SHARP

1. A wine dealer has three casks of capacities 7, litres, 9 litres and 20 litres. The first two are empty, and the third is full of wine. He wants to measure out exactly one litre of wine by pouring wine from one cask to the other using just the three measures, and without wasting any wine. How should he do this?

2. Prove that $n^4 - 14n^3 + 35n^2 + 50n$ is divisible by 24 for every positive integer n.

3. Three semicircles are tangent to each other as shown, and the straight line is perpendicular to their diameters. Prove that the two small circles, which are tangent to the semicircles and to the straight line, are equal in radius.



□

IMO IN AMSTERDAM

The 52nd International Mathematical Olympiad took place in Amsterdam, the Netherlands, in July this year.

South Africa has taken part in the IMO every year since 1992, so this was the country's 20th participation.

The 2011 team was

Ashraf Moolla (Rondebosch Boys' High School)
Dalian Sunder (Star College)
Dylan Nelson (Benoni High School)
Kira Düsterwald Springfield Convent)
Robert Spencer (Westerford High School)
Sean Wentzel (Westerford High School).
Team Leader: Dr Stephan Wagner (University of Stellenbosch)
Deputy Leader: Dr Koos van Zyl (Rhodes University)

Professor John Webb (UCT) attended in his capacity of Secretary of the IMO Advisory Board.

Sean won a Silver Medal, and Kira and Ashraf won Bronze Medals. Robert and Dylan won Honourable Mentions for solving at least one problem perfectly. These successes took South Africa's medal tally up to a total of 42 medals (one Gold, 9 Silver, 32 Bronze), with 37 Honourable Mentions.

The top ten countries were China (1), USA (2), Singapore (3), Russia (4), Thailand (5), Turkey (6), North Korea (7), Romania (8), Taiwan (8) and Iran (10).With 101 competing countries, South Africa achieved a creditable 41st position.

The question papers and full results can be found on the IMO website `www.imo-official.org`.

In 2012 the IMO will take place in Mar del Plata, Argentina, 4–16 July.

Participation in the IMO is a project of the South African Mathematics Foundation. For details, go to `www.samf.ac.za`. □

SHARP SOLUTIONS

1. *Prove that for all positive integers $n > 1$*

$$\frac{2^3-1}{2^3+1}\cdot\frac{3^3-1}{3^3+1}\cdot\frac{4^3-1}{4^3+1}\cdots\frac{n^3-1}{n^3+1} > \frac{2}{3}.$$

Noting that $$\frac{n^3-1}{n^3+1} = \frac{n-1}{n+1}\cdot\frac{n^2+n+1}{n^2-n+1}$$

we can write the product out in the form

$$\left(\frac{1}{3}\cdot\frac{7}{3}\right)\left(\frac{2}{4}\cdot\frac{13}{7}\right)\left(\frac{3}{5}\cdot\frac{21}{13}\right)\left(\frac{4}{6}\cdot\frac{31}{21}\right)\left(\frac{5}{7}\cdot\frac{43}{31}\right)\left(\frac{6}{8}\cdot\frac{57}{43}\right)\left(\frac{7}{9}\cdot\frac{73}{57}\right)\times\cdots$$
$$\cdots\times\left(\frac{N-2}{N}\right)\left(\frac{(N-1)^2+(N-1)+1}{(N-1)^2-(N-1)+1}\right)\cdot\left(\frac{N-1}{N+1}\right)\left(\frac{N^2+N+1}{N^2-N+1}\right).$$

Noting that $(N-1)^2+(N-1)+1 = N^2-N+1$, we see that we may cancel terms, so that the product collapses to

$$\frac{2}{3}\cdot\frac{N^2+N+1}{N(N+1)}$$

which is clearly greater than $\frac{2}{3}$.

This problem appeared in the German magazine $\sqrt[Die]{Wurzel}$, March/April 2011, p.77.

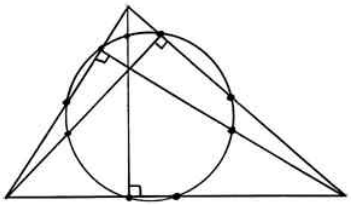
2. *Prove that, if A, B and C are the angles of a triangle, then*

$$\sin A + \sin B + \sin C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.$$

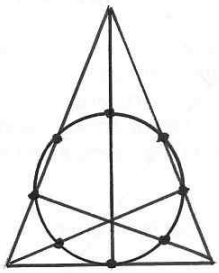
Using the products and sums formulas on page 24 of *Mathematical Digest* (July 2011), we have:

$$\begin{aligned}
&\sin A + \sin B + \sin C\\
&= \sin A + \sin B + \sin(A+B) \qquad (\text{since } A+B+C = 180^\circ)\\
&= 2\sin\frac{A+B}{2}\cos\frac{A-B}{2} + 2\sin\frac{A+B}{2}\cos\frac{A+B}{2}\\
&= 2\sin\frac{A+B}{2}\left(\cos\frac{A-B}{2} + \cos\frac{A+B}{2}\right)\\
&= 2\cos\frac{C}{2}\left(2\cos\frac{A}{2}\cos\frac{B}{2}\right)\\
&= 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.
\end{aligned}$$

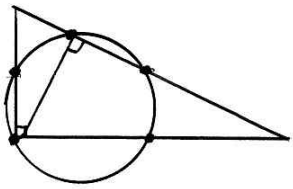
3. *How many of the points of the Nine Point Circle of a triangle coincide when the triangle is (a) isosceles (b) right-angled (c) isosceles and right-angled (d) equilateral?*



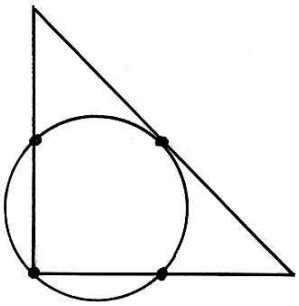
(a) In an isosceles triangle (which is not equilateral or right-angled), the nine points are reduced to eight.



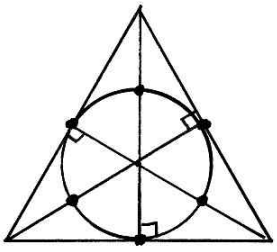
(b) In a right-angled triangle (which is not isosceles) the nine points are reduced to five.



(c) In an isosceles right-angled triangle, the nine points are reduced to four.



(d) In an equilateral triangle the nine points are reduced to six.



SHARP

Sharp calculators are awarded to:

First Prize: Nicholas Cheng (Fairmont High School)
Second Prize: Robert Spencer (Westerford High School)
Third Prize: Ntamo Abongile (Sijongephambili High School)
Good entries were received from James Nevin (Bergvliet High School) and previous prizewinners Shaylan Lalloo (Pearson High School) and Emma Belcher (Springfield Convent). □

THE 2011 INTERPROVINCIAL MATHEMATICS OLYMPIAD

Western Province teams dominated the 2011 Interprovincial Mathematics Olympiad, winning both senior and junior divisions.

The first Interprovincial Mathematics Olympiad (IPMO) took place in 1990, with just two teams (Western Province and Natal) taking part. It grew rapidly and today covers the whole country.

The Olympiad is in two divisions: Junior (grades 8 and 9) and Senior (grades 10, 11 and 12). Teams of 10 meet in different venues all round the country on a Saturday afternoon in September. Participating provinces may enter as many teams as they like.

The first paper is an individual multiple-choice paper, consisting of 15 questions to be tackled in one hour and worth 100 marks. During a break for refreshments, the teams get together to discuss strategy for the second round, a paper consisting of ten quite challenging questions requiring just an answer, worth 100 marks (with no part marks). A team works together and submits just one set of answers.

The papers are quickly marked and the results are sent to the central organizer, who puts them together and sends the final scores back to the waiting teams.

The 2011 IPMO took place on 10 September, with 55 teams (23 junior and 22 senior). Most provinces entered more than one team in each division. For the first time, teams from Swaziland took part.

In the Junior section the Western Province A and B teams took the top two places, with 1388 and 1256 points, respectively, followed by Free State A on 1087 and KwaZulu-Natal Midlands very close behind on 1077. In the Senior division the Western Province A team's score of 1503 points narrowly beat the University of KwaZulu-Natal A team 's score of 1454, with Boland A third on 1303 and KwaZulu-Natal Midlands A close behind on 1275.

The winning teams:

Western Province Juniors:
Jonathan Alp (grade 9, Parklands College)
Emma Belcher (grade 9, Springfield Convent High School) (Captain)
Warren Black (grade 9, Bishops)

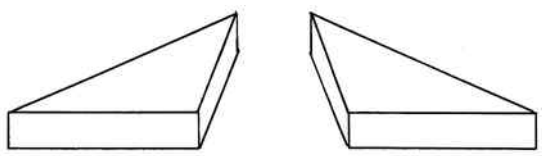
Nicolene Greeff (grade 9, Herschel Girls' School)
Chris Kim (grade 9, Reddam House Constantia)
Soo-Min Lee (grade 8, Bishops)
Murray McKechnie (grade 9, Bishops)
Daniel Mesham (grade 9, Bishops)
Michelle van der Merwe (grade 9, Herschel Girls' School)
Robin Visser (grade 9, St George's Grammar School)

Western Province Seniors:
Khadija Brey (grade 11, Wynberg Girls' High School)
Nicholas Cheng (grade 11, Fairmont High School)
Mickey Chew (grade 11, Elkanah House)
Lauren Denny (grade 10, Rustenburg Girls' High School)
Kira Düsterwald (grade 12, Springfield Convent High School)
Melissa Greeff (grade 12, Herschel Girls' School)
Ashraf Moolla (grade 12, Rondebosch Boys' High School)
James Nevin (grade 10, Bergvliet High School)
Robert Spencer (grade 11, Westerford High School)
Sean Wentzel (grade 12, Westerford High School) (Captain)

The 2011 Interprovincial Mathematics Olympiad papers were set by a team in the Department of Mathematics of the University of KwaZulu Natal, under the direction of Dr Sudan Hansraj. It is a project of the South African Mathematics Foundation, and is sponsored by the Actuarial Society of South Africa. □

CUTTING A CAKE

You have a triangular cake and a box in the shape of its mirror image. Since the cake has icing on the top you can't put it in the box upside down.



Can you cut the cake and rearrange the pieces to fit neatly into the box without turning them over? What is the smallest number of pieces into which you need to cut the cake?
The answer is on page 31. □

COMPUTER OLYMPIAD WINNERS

In the hotly contested Final Round of the Standard Bank Computer Programming Olympiad all the medals and prize money went to boys who are also stars in other Olympiads.

More than 31 000 participants from 459 schools took part in the first round, and 3000 won through to the second round.

The top 17 students were invited to the final round, which consisted of two five-hour papers written at the University of Cape Town over the weekend 3–4 September. The Awards dinner took place the next day.

Vaughan Newton (Bishops, grade 12) was this year's Gold Medallist. This was his third Computer Programming Olympiad Final: he won Silver in 2010 and was a runner-up in 2009. Vaughan was a member of the South African team that took part in the International Olympiad in Informatics in Thailand this year. Competing against participants from 80 countries, he won a Bronze Medal there.

There were two Computer Olympiad Silver Medallists: Stephen Barnes (grade 11, Pearson High School) and Sean Wentzel (grade 12, Westerford High School).

Both of them are high achievers in other Olympiads. Stephen won the Science Olympiad in 2009 while Sean not only won the Computer Programming Olympiad in 2010, but also achieved bronze at the International Olympiad in Informatics that year and silver at the International Mathematical Olympiad in Amsterdam this year.

Computer Olympiad Bronze Medals were won by Yaseen Hamdulay (grade 12, Wynberg Boys' High School), Ashraf Moolla (grade 12, Rondebosch Boys' High School) and Robert Spencer (grade 11, Westerford High School). Both Ashraf and Robert took part in the International Mathematical Olympiad in Amsterdam this year, winning a Bronze Medal and an Honourable Mention, respectively.

This was Ashraf's second bronze at the Computer Olympiad, following in the footsteps of his brothers Saadiq (silver 2007, bronze 2006) and Haroon (bronze 2008, silver 2007).

The runners-up were: Christiaan Kruger (grade 12, Duineveld High School, Upington), Mose Kwon (grade 12, Pretoria Boys High School), Bertus Malan (grade 12, Pretoria Boys High School), Christiaan Müller (grade

12, Paul Roos Gymnasium), Dylan Nelson (grade 11, Benoni High School), Guy Paterson-Jones (grade 10, Bishops), Dineo Sathekge (grade 12, CBC Mt Edmund), Ann Wait (grade 11 Nico Malan High School) and Grant Zietsman (grade 11, Pretoria Boys High School).

Computer Olympiad

South African Computer Olympiad: a project of the Computer Society of South Africa, Sponsored by Standard Bank

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LEAGUE TABLE

When football leagues were first started, a team was awarded two points for winning a game, and one point for a draw. Because a draw is a rather boring result, it was decided to encourage teams to play for a win rather than the safety of a draw by increasing the points for a win to three.

The change in awarding league points had a surprising result. It turns out that it is possible for a team to come bottom of the league in the old system, and top in the new system.

Suppose there are 13 teams in the league, each playing every other team just once. One of the teams, Team A, wins five of its matches and loses the other 7. All other matches in the league are drawn.

With 2 points for a win, Team A scores 10 points. The teams that beat A score 13 points and the teams that lose to A score 11 points. On the old league table, Team A is bottom.

What happens under the second system? With 3 points for a win, Team A scores 15 points. The teams that beat A score 14 points and the teams that lose to A score 11 points. On the new league table, A is top. □

SAMO THIRD ROUND PROBLEMS

The third and final round of the South African Mathematics Olympiad was written on Wednesday 7 September. The problems of the senior paper (for grades 10, 11 and 12) are below.

1. Consider the sequence $2, 3, 5, 6, 7, 8, 10, \ldots$ of all positive integers that are not perfect squares. Determine the 2011th term of this sequence.

2. Suppose that x and y are real numbers that satisfy the system of equations
$$2^x - 2^y = 1,$$
$$4^x - 4^y = \frac{5}{3}.$$
Determine $x - y$.

3. We call a sequence of m consecutive integers a *friendly* sequence if its first term is divisible by 1, its second by 2, ..., the $(m-1)^{\text{th}}$ by $m-1$, and in addition, the last term is divisible by m^2. Does a friendly sequence exist for (a) $m = 20$ (b) $m = 11$?

4. An airline company is planning to introduce a network of connections between the ten different airports of Sawubonia. The airports are ranked by priority from first to last (with no ties). We call such a network *feasible* if it satisfies the following conditions:

   - All connections operate in both directions.
   - If there is a direct connection between two airports A and B, and C has a higher priority than B, then there must also be a direct connections between A and C.

   Some of the airports may not be served, and even the empty network (no connections at all) is allowed. How may feasible networks are there?

5. Let $\mathbb{N}_0$ denote the set of all nonnegative integers. Determine all functions $f : \mathbb{N}_0 \to \mathbb{N}_0$ with the following two properties:

   (a) $0 \leq f(x) \leq x^2$ for all $x \in \mathbb{N}_0$;

   (b) $x - y$ divides $f(x) - f(y)$ for all $x, y \in \mathbb{N}_0$ with $x > y$.

6. In triangle ABC, the incircle touches BC in D, CA in E and AB in F. The bisector of $\angle BAC$ intersects BC in G. The lines BE and CF intersect in J. The line through J perpendicular to EF intersects BC in K. Prove that
$$\frac{GK}{DK} = \frac{AE}{CE} + \frac{AF}{BF}.$$

The results will be announced at a gala Awards Dinner in Cape Town on 15 October.

Dates for 2012:

Closing date for entries: 3 February
First round: 19 March
Second round: 23 May
Third round: 6 September

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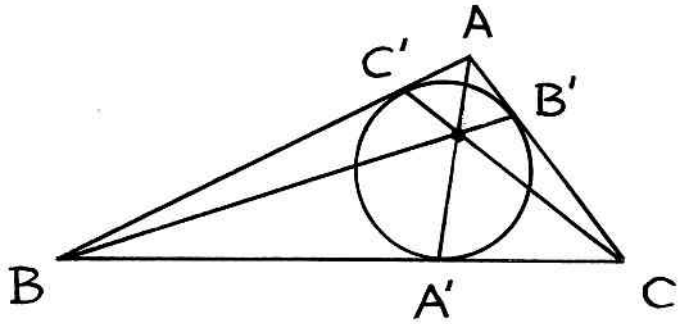
THE GERGONNE POINT

In triangle ABC shown, the inscribed circle touches the three sides at A', B' and C'. If the lines AA', BB' and CC' are drawn, they meet in a point, called the *Gergonne point* of the triangle, after the French mathematician Joseph Gergonne (1771–1859) who discovered this result.



Can you prove Gergonne's Theorem? A simple argument can be found on page 31. □

UNATTAINABLE SCORES

In a certain game, scores of either a points or b points are made, where a and b are positive integers. Which total scores are attainable?

For example, suppose that $a = 3$ and $b = 5$. Then scores of 1, 2, 4 and 7 are unattainable, but all scores of 8 or more are attainable, since

$$8 = 3 + 5, \quad 9 = 3 + 3 + 3, \quad 10 = 5 + 5$$

are attainable, and any higher score can be attained by adding a suitable multiple of 3 onto one these scores.

On the other hand, if $a = 2$ and $b = 6$, no odd scores are attainable.

More generally, if a and b have a common factor $k > 1$, then no scores that are not multiples of k are attainable. So from now on we shall assume that a and b are relatively prime (i.e. they have no common factor greater than 1.)

From the Euclidean Algorithm for finding the highest common factor of two numbers, it follows that there are integers p and q such that that $ap + bq = 1$. We will need this result to prove the theorem below.

Theorem

If a and b are two relatively prime positive integers, then

1. $ab - a - b$ is unattainable.
2. If $N > ab - a - b$, then N is attainable.
3. If $ab - a - b = M + N$, and M is unattainable, then N is attainable.

Proof

1. Suppose $ab - a - b$ is attainable. Then we can find non-negative integers s and y such that $ab - a - b = ax + by$. Then

$$a(x + 1) = b(a - y - 1). \tag{1}$$

Since a and b have no common factor, it follows that

$$x + 1 = mb \quad \text{and} \quad y + 1 = na \tag{2}$$

where m and n are positive integers.

Substituting the equations in (2) into equation (1) gives $amb = b(a - na)$. Since a and b are positive, it follows that $m = 1 - n$, which is not possible since m and n are positive integers.

2. The argument here is by "backwards induction".

We first note that there are arbitrarily large scores that are attainable (any multiples of a and b, for example). We shall show that if a number $N \geq ab - a - b + 2$ is attainable, then so is $N - 1$. That will prove that all scores greater than $ab - a - b$ are attainable.

So suppose that N is attainable, where $N \geq ab - a - b + 2$.
Then $N = ax + by$, where x and y are non-negative integers.
So $ax + by - 1 + a + b \geq ab + 1$
Since a and b are relatively prime, there are integers p and q such that $ap + qb = 1$. It follows that

$$ax + by - pa - qb + a + b \geq ab + 1$$

and

$$a(x - p + 1) - b(q - y - 1) \geq ab + 1.$$

Dividing through by ab (which we know is positive) gives

$$\frac{x - p + 1}{b} - \frac{q - y - 1}{a} \geq 1 + \frac{1}{ab}.$$

Assuming that p is negative and q is positive, we deduce that there is an integer t such that

$$\frac{q - y - 1}{a} < t < \frac{x - p + 1}{b}.$$

So $q - y - 1 < at$ and $x - p + 1 > bt$, from which it follows that

$$at - q + y \geq 0 \quad \text{and} \quad x - bt - p \geq 0.$$

Since

$$(x - bt - p)a + (at - q + y)b = (ax + by) - (ap + bq) = N - 1,$$

we have shown that $N - 1$ is attainable.

3. Suppose that $M + N = ab - a - b$ and M is not attainable.
Since a and b are relatively prime, it follows that $M = ax + by$, for some z, y. Since the possible values of x occur at intervals of b, we may assume that $0 \leq x \leq b - 1$. Since we have assumed M is not attainable, $y < 0$.

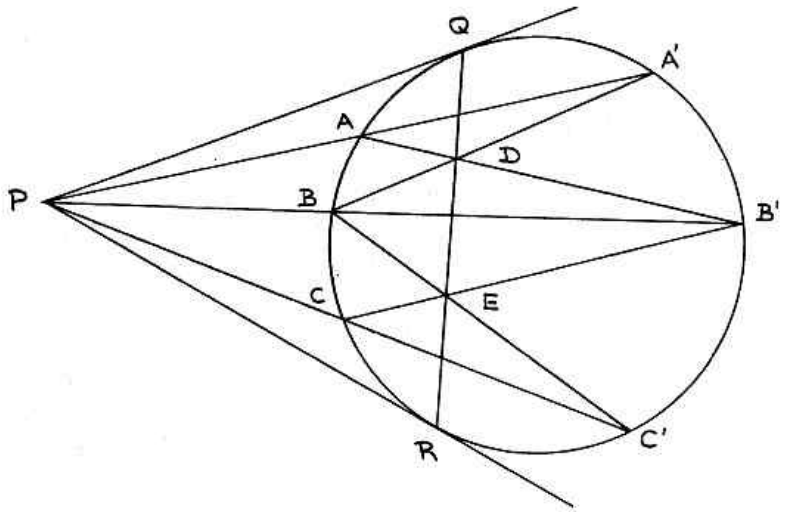
Now $N = ab - a - b - M = ab - a - b - ax - by = a(b - 1 - x) + b(-y - 1)$, where $b - 1 - x \geq 0$ and $-y - 1 \geq 0$. Thus N is attainable.

Exercise (IMO 1983, problem 3): Let a, b and c be positive integers, no two of which have a common divisor greater than 1.

Show that $2abc - ab - bc - ca$ is the largest integer that cannot be expressed in the form $xbc + yca + zbc$, where x, y and z are positive integers. $\square$

TANGENTS WITH RULER ONLY

The construction of the two tangents to a circle from a point outside the circle is a standard piece of ruler-and-compasses school geometry. However, it is an interesting fact that the construction can be carried out with only a ruler. This is pointed out by C Premkumar Yesudian in an article in the *Mathematical Gazette* of October 1982.



The method is to draw three secants from P, cutting the circle in A and A', B and B' and C and C' as shown. Next draw chords AB' and $A'B$, intersecting in D, and chords BC' and $B'C$, intersecting in E. Produce DE in both directions to cut the circle in Q and R. Then PQ and PR are the required tangents.

Exactly the same construction will work for drawing tangents to an ellipse, parabola or hyperbola. The proof that the construction works lies in the field of projective geometry. □

SHARING SANDWICHES

Alf and Ben are on a hike, and stop to eat their sandwiches. Alf has 9 sandwiches and Ben has 15.

Their friend Charlie arrives, and has nothing to eat. However, he has R24. He offers to give all his money to Alf and Ben in exchange for sharing their sandwiches. They agree, and share the 24 sandwiches equally.

How should Alf and Ben share the money?

The solution is on page 31. □

GILBREATH'S CONJECTURE

Write down the list of prime numbers:

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, \ldots$$

Now write down the differences between successive terms:

$$1, 2, 2, 4, 2, 4, 2, 4, 6, 2 \ldots$$

Repeat the process, subtracting successive terms (larger minus smaller).

$$1, 0, 2, 2, 2, 2, 2, 2, 4 \ldots$$

and again:

$$1, 2, 0, 0, 0, 0, 0, 2, \ldots$$

Carry on writing down the sequences of differences. After a while you will notice that every new sequence starts with 1. Does that always happen?

Norman Gilbreath noticed this property in 1958, but was unable to prove that all the sequences start with 1. He was not the first to make this observation: Francois Proth had noticed the pattern in 1878 and thought he had proved it, but there was an error in his argument.

Gilbreath's conjecture has been checked for several billion sequences of differences, but nobody has been able to prove that every sequence starts with 1. □

SIMPLIFY WITH SYMBOLS

"When a twelfth-century youth fell in love he did not take three paces backwards, gaze into her eyes, and tell her she was too beautiful to live. He said he would step outside and see about it. And if, when he got out, he met a man and broke his head—the other man's head, I mean—then that proved that his—the first fellow's—girl was a pretty girl. But if the other fellow broke *his* head—not his own, you know, but the other fellow's—then the other fellow to the second fellow, that is, because of course the other fellow would only be the other fellow to him, the first fellow who—well, if he broke his head, then *his* girl—not the other fellow's, but the fellow who *was* the ... Look here, if A broke B's head then A's girl was pretty girl; but if B broke A's head, then A's girl wasn't a pretty girl, but B's girl was."

(Jerome K. Jerome) □

ANSWERS

The Rice Problem (July 2011, page 3):
Emma Belcher (grade 9, Springfield Convent) and Shaylin Lalloo (grade 10, Pearson High School) used the Chinese Remainder Theorem to find the smallest number of bags of rice: 1776. In place of the offered set of back numbers of *Mathematical Digest*, which they both already have, they have been awarded 2012 subscriptions to the magazine.

Number Quiz (page 3):
The *second* is defined as 9 192 631 770 vibrations of the microwave radiation emitted by caesium-133 atoms during a specified atomic rearrangement.

Coin Choice Conundrum (page 6):
Jake can win by numbering the coins from left to right. He then adds up the value of the coins in the even-numbered places, and the value of the coins in the odd-numbered places. Suppose the even-numbered coins have the greater sum. Jake then selects coin number 50, forcing Julie to choose an odd-numbered coin (either 1 or 49). Jake continues to choose even-numbered coins, forcing Julie always to take an odd-numbered coin. So Jake wins.
However, it is a far more difficult problem to devise a strategy for Jake that guarantees he wins the largest amount possible.

History Quiz (page 6):
Isaac Newton was born on Christmas Day, 1642.

Goldbach Hints (page 7):

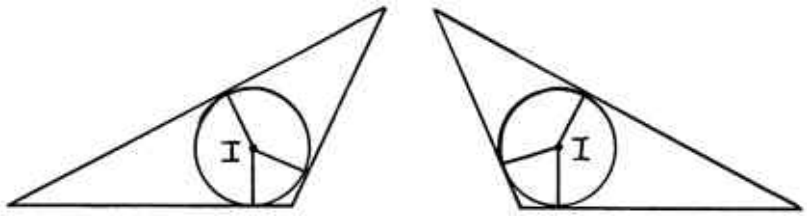
1. $2n - 1 = 2(n - 4) + 9$; $2n = 2(n - 4) + 8$
2. $2n - 2 = 2(n - 1) + 2$; $2n + 1 = 2(n - 1) + 3$
3. $2n = 2(n - 2) + 4$; $2n + 1 = 2(n - 4) + 9$
4. It is not difficult to check that 38 cannot be expressed as the sum of two composite odd numbers. The next five even numbers can, however, be so expressed:

   $$40 = 15 + 25, 42 = 15 + 27, 44 = 35 + 9, 46 = 25 + 21, 48 = 15 + 33$$

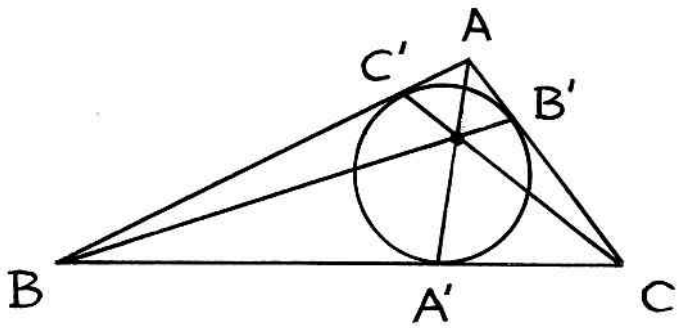
   For numbers above 50, add a multiple of 5 to one of the sums above.

TV Quiz (page 9):
Sheldon Cooper, in *Big Bang Theory*.

Cutting a Cake (page 21):
Let I be the incentre of the triangle (the centre of the inscribed circle, which is the point where the internal bisectors of the angles of the triangle meet). Cut the cake along the perpendiculars from I to the three sides. The pieces can then be rotated and fitted into the box as shown.



The Gergonne Point (page 25):
The key observation is that the two tangents to a circle from a point outside the circle are equal in length.



So $AC' = AB'$, $BA' = BC'$ and $CB' = CA'$. It follows that

$$\frac{AC'}{C'B} \cdot \frac{BA'}{A'C} \cdot \frac{CB'}{B'A} = 1$$

and so, by Ceva's Theorem (page 2), AA', BB' and CC' are concurrent.

Sharing Sandwiches (page 28):
Since Alf initially has 9 sandwiches and Ben has 15 it looks as though the money should be split in the ratio $9 : 15$, with Alf taking R9 and Ben R15. That, however, would not be a fair division. Of the total of 24 sandwiches, each of the three gets 8. So Alf must give one of his 9 sandwiches to Charlie, and Ben must hand over 7. So the money should be shared in the ratio $1 : 7$, with Alf getting R3 and Ben getting R21. $\square$

mathematical digest

No. 166 January 2012

Pythagoras in Babylon

mathematical digest

Number 166 **January 2012**

I was just going to say, when I was interrupted, that one of the many ways of classifying minds is under the heads of arithmetical and algebraical intellects. All economical and practical wisdom is an extension or variation of the following arithmetical formula: $2 + 2 = 4$. Every philosophical proposition has the more general character of the expression, $a + b = c$. We are mere operatives, empirics, and egotists, until we learn to think in letters instead of figures.

(Oliver Wendell Holmes, in *The Autocrat of the Breakfast Table*) □

Department of Mathematics and Applied Mathematics
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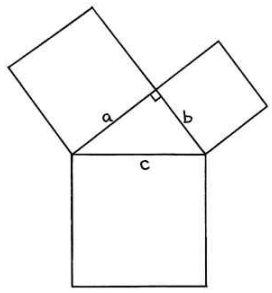
Cover: The clay tablet shown was found in Iraq and is between 3 500 and 4 000 years old. It gives, in Babylonian cuneiform writing, a list of Pythagorean triples. (See p.1).

PYTHAGOREAN TRIPLES



In the 6th century B.C. the legendary Greek mathematician Pythagoras is said to have discovered the most famous theorem of all mathematics: in a right-angled triangle with sides a and b and hypotenuse c,

$$a^2 + b^2 = c^2.$$

The converse is also true: If a, b and c are positive real numbers such that $a^2 + b^2 = c^2$, then they are the sides of a right-angled triangle.

If moreover a, b and c are positive *integers* such that $a^2 + b^2 = c^2$, they are said to form a *Pythagorean triple*.

The standard examples of Pythagorean triples met at school are $(3, 4, 5)$ and $(5, 12, 13)$. Multiples of these examples such as $(9, 12, 15)$ and $(10, 24, 26)$ are also Pythagorean triples. Less familiar Pythagorean triples are

$$(7, 24, 25), (9, 40, 41), (11, 60, 61) \ldots$$

from which one can detect a pattern and generate more examples

$$(13, 84, 85), (15, 112, 113), (17, 144, 145) \ldots.$$

(Hint for finding the pattern: square and halve the first term.)

So there are infinitely many Pythagorean triples. However, those we have listed so far exclude $(8, 15, 17)$, $(20, 21, 29)$ and many others. Is it possible to draw up a complete list of Pythagorean triples?

More than a thousand years before Pythagoras, mathematicians of ancient Babylon had puzzled over this problem, and had recorded a list of triples on a clay tablet. How they found these numbers is not known.

A useful way of considering the problem is to divide the equation

$$a^2 + b^2 = c^2$$

by c^2, to get

$$\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1$$

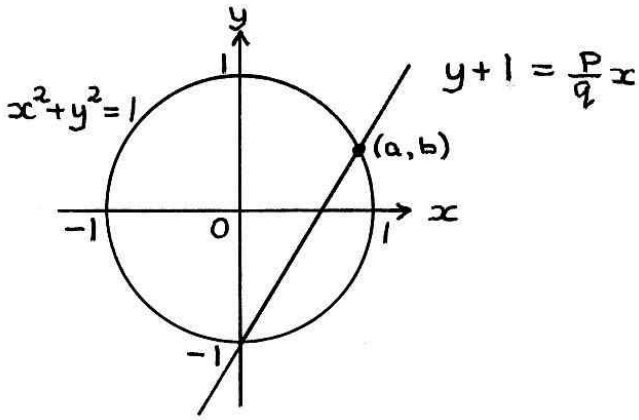
and to interpret the problem geometrically:

Find all points on the circle $x^2 + y^2 = 1$ with *rational* coordinates.

It is in fact sufficient to find all such points in the first quadrant of the circle.

Suppose that (a, b) is such a point. Then the line through $(0, -1)$ and (a, b) has slope $\dfrac{b+1}{a}$, which is a rational number greater than 1.

Conversely, suppose we have a line through $(0, -1)$ with rational slope p/q (where p and q are integers with no common factor and $p > q > 0$). It will cut the circle again in the first quadrant.



The equation of the line is $y = \left(\dfrac{p}{q}\right)x - 1.$ (1)

This line cuts the circle when $x^2 + \left(\dfrac{px}{q} - 1\right)^2 = 1$

which simplifies to $x\left(\left(\dfrac{p^2+q^2}{q^2}\right)x - \dfrac{2p}{q}\right) = 0$. So $x = 0$ or $\dfrac{2pq}{p^2+q^2}$.

Substituting these values into the equation of the line gives the points $(0, -1)$ and $\left(\dfrac{2pq}{p^2+q^2}, \dfrac{p^2-q^2}{p^2+q^2}\right)$.

We knew about the first point. It's the second point that is important: it satisfies the equation

$$\left(\frac{2pq}{p^2+q^2}\right)^2 + \left(\frac{p^2-q^2}{p^2+q^2}\right)^2 = 1$$

which is equivalent to

$$(2pq)^2 + (p^2 - q^2)^2 = (p^2 + q^2)^2.$$

Since p and q were assumed to be integers, we see that the numbers

$$2pq, \quad p^2 - q^2, \quad p^2 + q^2$$

form a Pythagorean triple. All you have to do is plug in different values of p and q, with $p > q$ and p and q having no common factor, and you will get all Pythagorean triples (a, b, c) with a, b and c having no common factor. These are called *primitive* Pythagorean triples. Multiplying through by different integers gives all possible Pythagorean triples. □

PI RIDDLES

- What do you get if your divide the circumference of a Halloween lantern by its diameter?
- What do you get if you divide the circumference of an igloo by its diameter?
- What do you get if you divide the circumference of the sun by its diameter?
- What are you doing when thinking about pi?
- How would you describe someone who dislikes jokes about pi?

Answers are on page 28. □

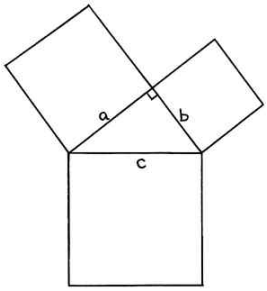
QUICK QUIZ

1. Which mathematician invented the cat flap?
2. Who signed the patent for the Toblerone chocolate design?
3. Which Russian maths teacher won a Nobel Prize for literature?
4. In which novel by Charles Dickens does the ghost of Pythagoras appear?
5. In Mathematics, what is the equivalent of a Nobel Prize?

Answers are on page 28. □

THE 3-4-5 TRIANGLE

The Theorem of Pythagoras states that the area of the square on the hypotenuse of a right-angled triangle is equal to the sum of the areas of the squares on the other two sides.
There are many ways of proving the theorem. A number have appeared in this magazine over the years, but there are over a hundred more.



An important application of Pythagoras' Theorem is mentioned by the Roman writer Marcus Vitruvius Pollio, better known as Vitruvius, in his ten-volume treatise *De architectura*. He wrote

> Pythagoras demonstrated the method of forming a right triangle without the aid of the instruments of artificers: and that which they scarcely, even with great trouble, exactly attain, nay be performed by his rules with great facility.
>
> Let three rods be procured, one three feet, one four feet, and the other five feet long; and let them be so joined as to touch each other at their extremities; they will then form a triangle, one of whose angles will be a right angle. For, if on the the lengths of each of the rods, squares be described, that whose length is three feet will have an area of nine feet; that of four, of sixteen feet; and that of five, of twenty-five feet: so that the number of feet obtained in the two areas of the square of three and four feet, added together, are equal to those contained in the square, whose side is five feet.

Constructing a right angle in this way is often referred to as an application of the Theorem of Pythagoras. That is not correct: it is an application of the *converse* of the theorem, which states:

> If the area of the square on one side of a triangle is equal to the sum of the areas of the squares on the other two sides, then the angle opposite the first side is a right angle.

The proof of the converse of Pythagoras is often skipped at school, even when the proof of the theorem itself is included.

So here's a challenge: How would you prove, without citing the theorem of Pythagoras or its converse, that in the specific case of a 3-4-5 triangle the largest angle is a right angle?

The proof is not obvious. Think about it before turning to page 31. □

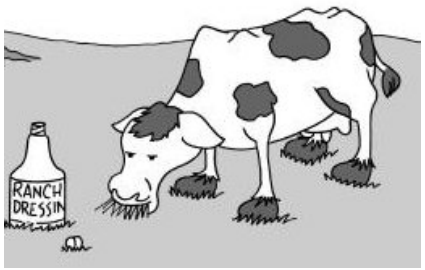
THE GRAZING OXEN PROBLEM

A standard ratio and proportion problem is the story about grazing oxen.

> If the grass in a field will feed ten oxen for a week, how long would five oxen take to eat the grass?

The answer is, obviously, two weeks.

However, that answer does not allow for the fact that in real life the grass is growing while it is being eaten, so will last longer. Indeed, if the field is big enough and the grass grows fast enough, then the oxen could graze on it forever.



Isaac Newton recognized that taking the growth of the grass into account makes the problem rather harder. He posed the following problem in his book *Arithmetica Universalis*, published in 1706.

> If the number of oxen a eat up the meadow b in the time c, and the number of oxen d eat up as good a piece of pasture e in the time f, and the grass grows uniformly; to find how many oxen will eat up the like pasture g in the time h.

The problem used to be standard in Victorian school textbooks of arithmetic, with specific numbers replacing the symbols. Since this was regarded as "arithmetic", algebraic methods were not allowed, and everything had to be written in words.

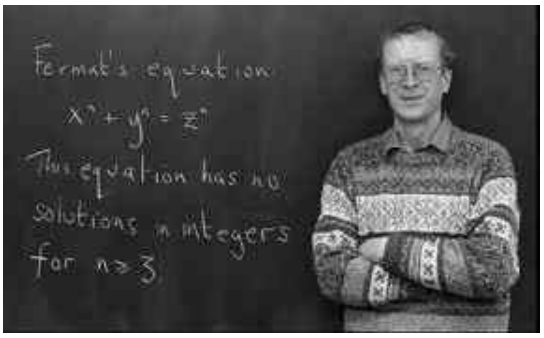
The answer (in algebraic terms) can be found on page 28. □

FERMAT'S LAST THEOREM

There are no positive integers x, y, z and n, with $n > 2$ such that

$$x^n + y^n = z^n.$$

This is the famous *Last Theorem* of the 17th Century French mathematician Pierre de Fermat, which remained unproved for 350 years until finally cracked by Andrew Wiles. The full proof is a very long story, and involves some extremely sophisticated mathematics.



It is interesting that a quite simple argument can crack the special case of $n = 4$. In fact, we shall prove a slightly stronger result:

Theorem: There are no positive integers x, y and z such

$$x^4 + y^4 = z^2. \tag{1}$$

Proof: If there are solutions, there must be one with z as small as possible. Starting with such a "minimal" solution, we will then produce another solution with a smaller value for z, and that will be a contradiction.

So suppose that $x^4 + y^4 = z^2$, where z is minimal. Then the three numbers x, y, z have no common divisor, for if $p > 1$ was such a divisor, then $(x/p, y/p, z/p^2)$ would be a solution of equation (1) with a smaller z-value.

Note that x and y cannot both be odd (since the square of an odd number is of the form $4A + 1$, and the sum of two odd squares therefore leaves a remainder of 2 on division by 4, and therefore cannot be a square). So let us suppose that x is odd and y is even (from which it follows that z is odd).

Now we note that equation (1) says that (x^2, y^2, z) is a primitive Pythagorean triple (see: *Pythagorean Triples*, pp 1–3).

We can therefore write

$$x^2 = p^2 - q^2, \quad y^2 = 2pq, \quad z = p^2 + q^2$$

where p and q are positive integers with no common factor. Since $2pq$ is a square, either p or q is even: we may suppose that q is even.

We now have

$$p\left(\frac{q}{2}\right) = \frac{2pq}{4} = \frac{y^2}{4} = \left(\frac{y}{2}\right)^2.$$

Since p and $q/2$ are integers with no common factor, and their product is a square, both p and $q/2$ are squares.

Now (x, q, p) is another Pythagorean triple, and may be written

$$x = r^2 - s^2, \qquad q = 2rs, \qquad p = r^2 + s^2$$

where r and s are positive integers with no common factor. Since $rs = q/2$ is a square, both r and s are squares. So we may put

$$r = u^2, \qquad s = v^2, \qquad p = w^2.$$

Substituting into $p = r^2 + s^2$ gives

$$w^2 = u^4 + v^4,$$

another solution of equation (1).

Now we note that $w \leq w^4 < w^4 + q^2 = p^2 + q^2 = z$, which is a contradiction, since we began under the assumption that z was minimal.

The contradiction forces us to the conclusion that there are no solutions, in positive integers, of $x^4 + y^4 = z^2$. □

READER'S PROBLEM

Submitted by Prashant Bagga, S.D.Jain Modern School, Surat

There are three farmers viz. A, B, and C. A sold a quarter of his land to B, then B sold one third of his land to C. After that, A, B and C had the same amount of land.

In what proportion was the ownership of the land at the beginning?

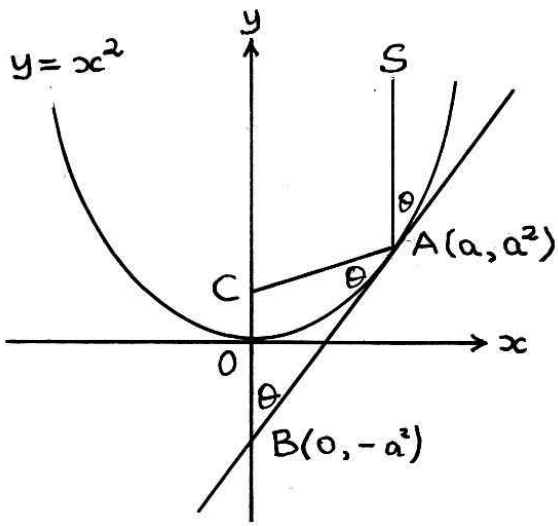
See page 28 for the answer. □

THE BURNING MIRROR

About 2200 years ago a Greek mathematician by the name of Diocles wrote a treatise entitled *On burning mirrors*. In it he showed that a mirror in the shape of a paraboloid (a surface obtained by rotating a parabola about its axis of symmetry) will focus the sun's rays to one point, creating enough intense heat, said Diocles, to cremate bodies in temples.

Whether mirrors were ever used by Greek priests for that purpose is uncertain. Another use for burning mirrors was attributed to Archimedes (a contemporary of Diocles), who was said to have used burning mirrors to set fire to Roman ships besieging his home city of Syracuse. That story dates from several centuries later, and is highly unlikely to be true.

Why does a paraboloidal mirror focus the the sun's rays to a point? The answer turns out to be a neat little piece of mathematical reasoning, combining one of the easiest bits of calculus (the derivative of x^2) with one of the easiest bits of geometry (the isosceles triangle).



Suppose that a cross-section of the paraboloidal mirror is given by the equation $y = x^2$, and that the sun is shining directly down the axis of symmetry of the parabola. Consider a ray of light from S (the sun) that strikes the parabola at the point $A(a, a^2)$ and is reflected to cut the y-axis at $C(0, c)$. We shall show that the value of c is independent of a. In other words, all the reflections of the suns rays will pass through the same point on the y-axis.

Since the equation of the parabola is $y = x^2$, the slope of the tangent at A is $2a$. The tangent at A therefore has equation

$$y - a^2 = 2a(x - a)$$

Putting $x = 0$ shows that the tangent cuts the y-axis at the point $B(0, -a^2)$.

By the basic law of reflection, the ray of light from the sun and its reflection make the same angle θ at A. Since AS is parallel to the y-axis, $\angle ABC = \theta$. So triangle ABC is isosceles.

From the equation $AC = CB$ we have

$$\sqrt{a^2 + (c - a^2)^2} = c + a^2$$

which becomes (squaring)

$$a^2 + c^2 - 2a^2c + a^4 = c^2 + 2a^2c + a^4$$

from which it follows easily that $c = \frac{1}{4}$, which is independent of a.

So all rays from the sun are reflected to pass through the point $(0, \frac{1}{4})$. $\square$

Reference

Mark B. Villarino, Burning mirrors on opening day, *Mathematical Gazette*, **95**(November 2011), pp. 512–514.

ANOTHER SHARP COMMENT

Sharp Problem 2 (*Mathematical Digest*, October 2010) was

$$\textit{If } \frac{(a-b)(c-d)}{(b-c)(d-a)} = k, \textit{ determine the value of } \frac{(a-c)(b-d)}{(a-b)(c-d)}.$$

A dreadfully complicated solution was given in the January 2011 issue of the magazine, and a much more elegant solution was provided by Dr Geoff Smith in the July 2011 edition.

Professor Johan Meyer (University of the Free State) has sent in another very simple solution.

Since $(a-c)(b-d) = (a-b)(c-d) - (b-c)(d-a)$,

$$\frac{(a-c)(b-d)}{(a-b)(c-d)} = \frac{(a-b)(c-d)}{(a-b)(c-d)} - \frac{(b-c)(d-a)}{(a-b)(c-d)} = 1 - \frac{1}{k}. \qquad \square$$

THE WOODEN SPOON

The term "wooden spoon" is commonly used for a sporting team that has finished last. What is not commonly known is that the phrase has a mathematical origin.

The University of Cambridge has a reputation for mathematical excellence, and for centuries success in the Cambridge Mathematical Tripos (as the examination is known) has been a benchmark of mathematical prowess. It was, and still is, an extremely tough examination. The successful students are ranked in three classes: First (called *Wranglers*), Second (*Senior Optimes*) and Third (*Junior Optimes*).

The Mathematical Tripos results are traditionally announced in a public ceremony in the University Senate House. Until 1909 they were read out in ranked order, and much excitement surrounded the announcement of the name of the Senior Wrangler, a term first used in 1791 to describe the top student. From 1793 the bottom Junior Optime was given the title of "the Wooden Spoon". As a prank, students would lower a huge wooden spoon on ropes from the balcony of the Senate House to the embarrassed bottom Junior Optime of that year, amid great hilarity among the students and disapproving sniffs from the dons.

The Wooden Spooner was actually not the bottom student of all. He was the last of those qualifying for Mathematics Honours. Below him were the "Gulph": those who had passed, but were deemed unworthy of Honours. Below them came the failures, and below them those that did not even attempt the exam.

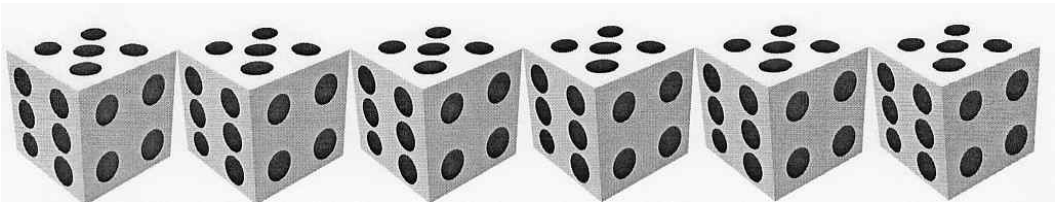
In the 18th Century the Cambridge colleges brewed their own beer, and the original Wooden Spoon was a large malting shovel. As years went by, the Wooden Spoon became a highly decorated trophy, and proudly waved by its recipient. In 1909 the practice of ranking the Tripos students was terminated, and there have been no Wooden Spoons since then.

The winner of the last Wooden Spoon in 1909 was Cuthbert Lempriere Holthouse of St John's, a keen oarsman who had rowed in the college First Boat for three years. He is shown here with his spoon, the handle of which is a rowing oar. Cuthbert is clearly proud of the fact that he spent more time on the river than working on maths.

In 1824 the Classics Tripos was instituted at Cambridge, and there was a debate on the name of the award to be given to the bottom student. It so happened that the bottom student that year was named Wedgewood, and the award was duly named the *Wooden Wedge.* Mr Wedgewood was, however, no slouch: he also took mathematics, sat the Mathematical Tripos, and was ranked Eighth Wrangler. □

SIX DICE PROBLEMS

1. If you throw six dice, what is the probability that at least one will be a six?
2. If you throw six dice, what is the probability that they will show six different numbers?
3. If six dice are thrown, what is the probability that at least two will show the same number?



4. What is the chance of throwing six dice, and getting at least two sixes?
5. If you throw six dice, what will be the average total obtained?
6. After throwing six dice 666 times, you have not seen a single six on any of them. What is the chance that you will get a six on at least one of the dice if you throw them once more?

Answers are on page 28. □

MORE ABOUT CEVA

Ceva's Theorem featured in *Mathematical Digest* **165**, October 2011. Here are some further ideas on this useful result, and its close relation Menelaus' Theorem (*Mathematical Digest* **164**, July 2011).

- Ceva was Italian, and the correct pronunciation of his name is not *seever*, but *chay-vah*.

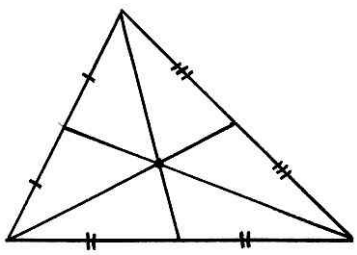
- There is a trig version of Ceva's Theorem, which is easily established by means of the sine rule:

  If P is a point inside triangle ABC, and the lines AP, BP and CP are produced to meet sides BC, CA and AB in A', B' and C', respectively, then
  $$\frac{\sin\angle ABB'}{\sin\angle CBB'}\cdot\frac{\sin\angle BCC'}{\sin\angle ACC'}\cdot\frac{\sin\angle CAA'}{\sin\angle BAA'}=1$$
  and the converse also holds.

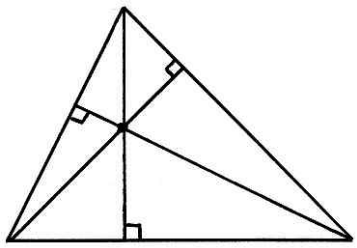
- Ceva's Theorem can also be used to prove Menelaus' Theorem. Or, if you like, Menelaus' Theorem can be proved from Ceva's Theorem.

- Ceva's Theorem can be used to provide quick proofs of a number of *concurrency theorems*:

  The medians of a triangle are concurrent. The point of concurrency is called the *centroid* of the triangle.

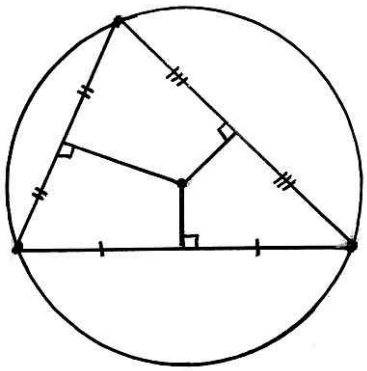


  The altitudes of a triangle are concurrent. The point of concurrency is called the *orthocentre* of the triangle.

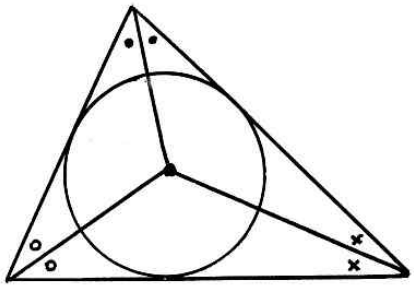


  The perpendicular bisectors of the sides of a triangle are concurrent.

The point of concurrency is the centre of the circumscribed circle of the triangle and is called the *circumcentre* of the triangle.



The internal bisectors of the angles of a triangle are concurrent. The point of concurrency is the centre of the inscribed circle of the triangle and is called the *incentre* of the triangle.



- Recent research has shown that Giovanni Ceva was not the first to discover the theorem named after him. It had been found 600 years earlier by Al-Mutaman ibn Hud, who was not only a brilliant mathematician but also the King of the Moorish city of Saragossa from 1082 to 1085. □

LIGHT BANTER AT CERN

The barman says "We don't allow faster than light neutrinos in here."
A neutrino walks into a bar.

There have been better jokes, but not many that managed to combine, in two lines, quantum mechanics, special relativity and time travel.

(*The Times*, 5 October 2011) □

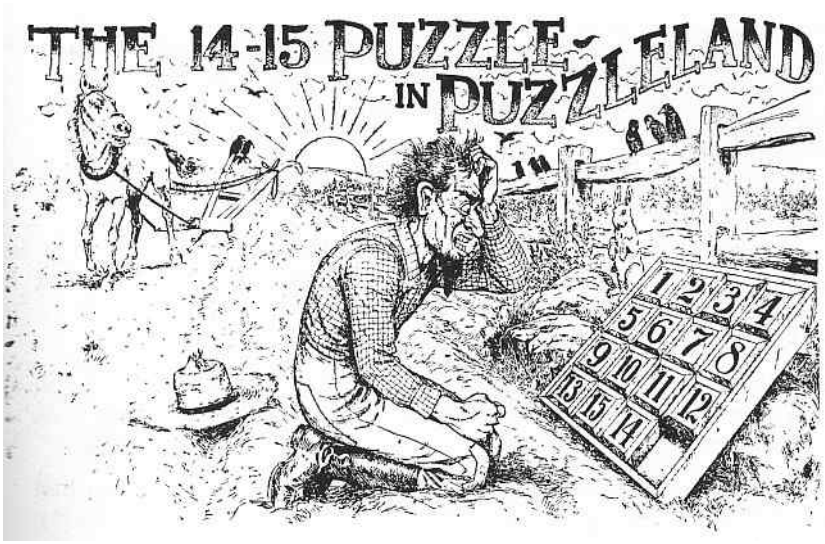
THE 14-15 PUZZLE

The 14-15 Puzzle has been around for over a hundred years, and is still marketed today, in many forms. Of all mathematical puzzles, only Rubik's Cube has surpassed it in popularity.

The puzzle consists of a 4×4 frame containing 15 numbered blocks, and one vacant space into which a neighbouring block can be slid. The blocks are numbered as shown, with 14 and 15 switched, and the task is to slide the blocks around until they are in proper numerical order.

1	2	3	4
5	6	7	8
9	10	11	12
13	15	14	

The puzzle was devised by Noyes Chapman, a New York postmaster, in 1880 and became wildly popular in the U.S.A. and Europe for a few months. Some 16 years later the puzzle was revived by the American puzzlist Sam Loyd, who falsely claimed to be its inventor. Martin Gardner published a collection of Loyd's puzzles, in which the illustration below appears.



Loyd offered a \$1000 prize for a solution. This was a clever marketing ploy, since when the puzzle first appeared several mathematicians had pointed out that it cannot be solved.

Any rearrangement, or permutation, of the blocks has to be done by a series of transpositions (a transposition is a simple switch of two blocks). Though a permutation can be performed by many different sequences of

transpositions, any two such sequences must have the same parity: if one requires an even number of transpositions, so does the other, and similarly for odd numbers of transpositions. This observation, which requires a proof, means that permutations can be classified as either even or odd, depending on the parity of the number of transpositions needed to execute it.

The purpose of the puzzle is to switch the blocks marked 14 and 15, and a single transposition is, perforce, odd. However, any manipulation of the sliding blocks which returns the vacant space to its starting position will take an even number of moves, since any move up must be followed by a move down at some later stage, and similarly any move to the left or right must be followed later by a move to the right or left, respectively. So only even permutations can return the vacant square to its starting position, and a single transposition is odd. Since a rearrangement cannot be both even and odd, it follows that the puzzle cannot be solved.

You don't need to buy this puzzle, because it is the same as a Patience card game. Take the court cards (Ace, King, Queen and Jack) of each suit, and remove the Jack of clubs. Shuffle the remaining 15 cards and deal them out in a 4×4 square, with the bottom right hand corner empty. Now try to slide the cards around until they are in suit order from the top down: A-K-Q-J of spades, A-K-Q-J of hearts, A-K-Q-J of diamonds and A-K-Q of clubs.

It turns out that half the time the puzzle can be solved: when the cards laid down represent an even permutation. Odd permutations cannot be sorted out.

Reference

Martin Gardner, *The mathematical puzzles of Sam Loyd*, Dover 1959. □

POETRY QUIZ

For he, by geometric scale
Could take the size of pots of ale
And wisely tell the time o' th' day
The clock doth strike, by algebra.

In which poem does this verse appear, and who is the poet?
The answer is on page 29. □

OLYMPIAD MEDALLISTS

The results of the 2011 South African Mathematics Olympiad were announced at a gala function in Cape Town on 15 September 2011.

Junior Winners

1. Chris Kim (grade 9, Reddam House College)
2. Robin Visser (grade 9, St George's Grammar School)
3. Nashlen Govindasamy (grade 9, Star College)
4. Roger Song (grade 8, Jeppe Boys' High School)
5. Deon du Plessis (grade 9, Hoërskool Diamantveld)
6. Soo-Min Lee (grade 8, Bishops)
7. Josiah Meyer (grade 9, Hoërskool Jim Fouché)
7. Ryan Broodryk (grade 9, Westerford High School)
9. Warren Black (grade 9, Bishops)
10. Adam Herman (grade 9, Rondebosch Boys' High School)
10. Michael Thomson (grade 8, Bishops)

Senior Winners

1. Dylan Nelson (grade 11, Benoni High School)
2. Sean Wentzel (grade 12, Westerford High School)
3. Kira Düsterwald (grade 12, Springfield Convent)
4. Ashraf Moolla (grade 12, Rondebosch Boys' High School)
5. Dhaneshwar Dalian Sunder (grade 11, Star College)
6. Robert Spencer (grade 11, Westerford High School)
6. Stephen Barnes (grade 11, Pearson High School)
8. Shaylan Lalloo (grade 10, Pearson High School)
9. Lauren Denny (grade 10, Rustenburg Girls' High School)
10. Lloyd Mahadeo (grade 10, Star College)

Important dates for 2012

Closing date for entries: 3 February
First round: 19 March
Second round: 23 May
Third round: 6 September

Contact details

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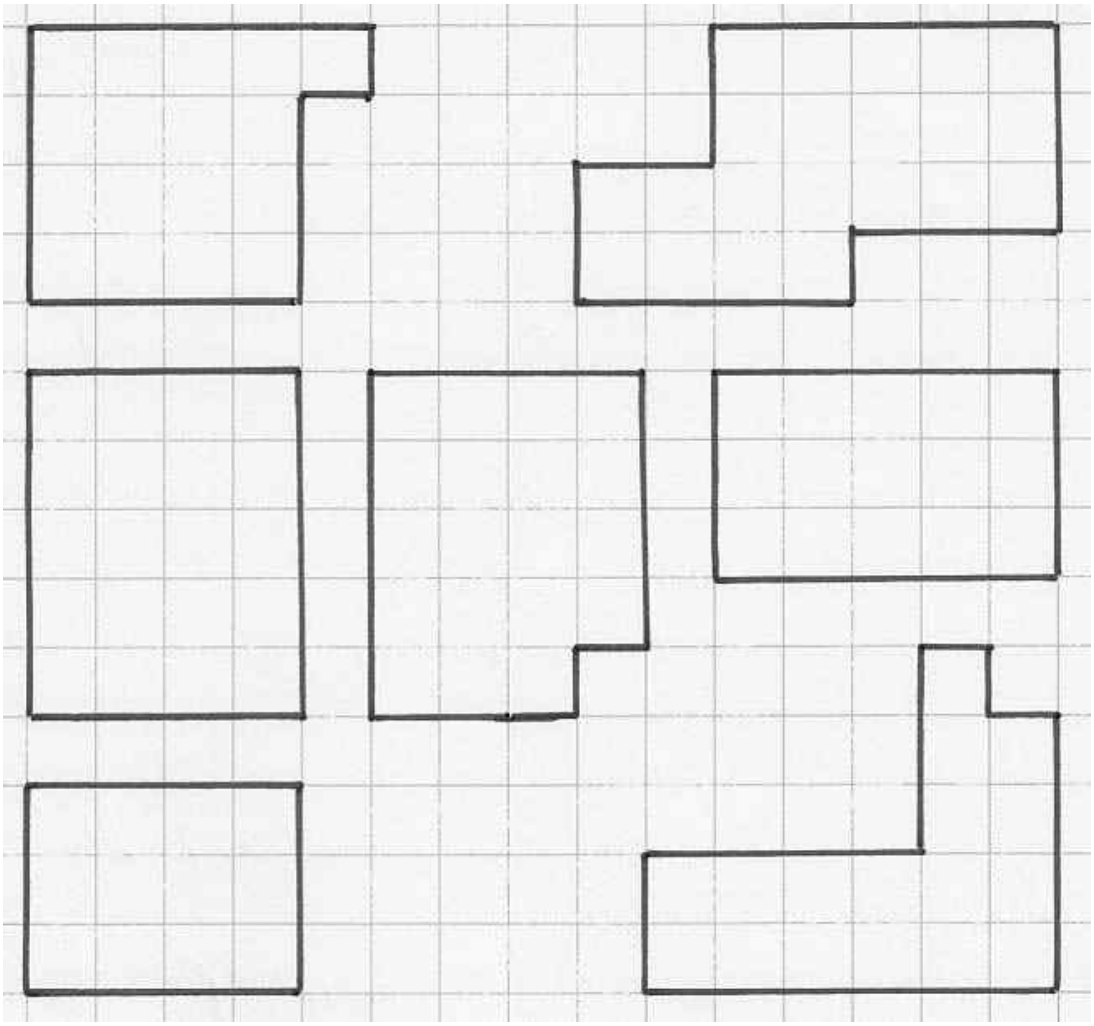
TURKISH SQUARES PUZZLE

Seven puzzle pieces are shown below. Copy the pieces onto stiff card and cut them out.



- The first problem is to put them together to make an 11×11 square.
- Next, with six of the seven pieces you have to make a 10×10 square.
- Follow that up by choosing five of the pieces to make a 9×9 square.
- Finally, with four pieces you must make an 8×8 square.

The pieces may be rotated, but not turned over.

The puzzle was designed by the Turkish puzzlist Emrehan Halici. He is a software entrepreneur, a member of MENSA, a pioneer in computer-based music technology, a former President of the Turkish Chess Federation and a member of the Turkish National Assembly.

Solutions are on page 29. □

SHARP SOLUTIONS

1. *A wine dealer has three casks of capacities 7 litres, 9 litres and 20 litres. The first two are empty, and the third is full of wine. He wants to measure out exactly one litre of wine by pouring wine from one cask to the other using just the three measures, and without wasting any wine. How should he do this?*

   With the contents of the three casks given by a triplet of numbers, the sequence of 11 pourings below will measure out exactly one litre:

   $(0,0,20) \to (7,0,13) \to (7,7,6) \to (5,9,6) \to (5,0,15) \to (0,5,15)$
   $\to (7,5,8) \to (3,9,8) \to (3,0,17) \to (0,3,17) \to (7,3,10) \to (1,9,10)$.

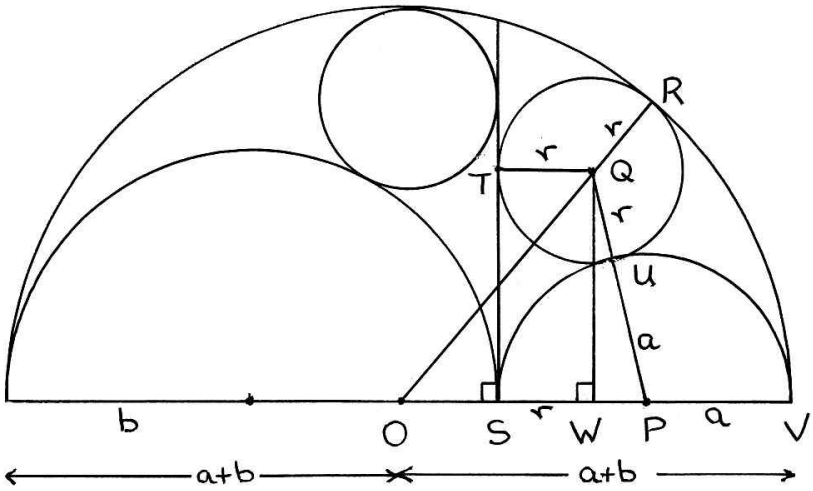
2. *Prove that $n^4 - 14n^3 + 35n^2 + 50n$ is divisible by 24 for every positive integer n.*

   By the Factor Theorem we have

   $$n^4 - 14n^3 + 35n^2 + 50n = n(n+1)(n-5)(n-10).$$

   We start by noting that either n or $n+1$ is even.
   If n is even, so is $n-10$, and since these two factors differ by 10, one of them is a multiple of 4. So their product is a multiple of 8.
   If $n+1$ is even, so is $n-5$, and since these factors differ by 6, one of them is a multiple of 4. So their product is a multiple of 8.
   If either n or $n+1$ is a multiple of 3, we are done.
   Suppose neither is. Then $n+2$ is a multiple of 3, and so $n-10 = n+2-12$ is also a multiple of 3. So at least one of the factors is a multiple of 3.
   So the product of the four factors is a multiple of 3 and a multiple of 8, and hence is a multiple of 24.

3. *Three semicircles are tangent to each other as shown, and the straight line is perpendicular to their diameters. Prove that the two small circles, which are tangent to the semicircles and to the straight line, are equal in radius.*



In the diagram, the large semicircle has centre O, and the vertical line is ST. We shall concentrate on the small circle on the right. The vertical line ST is tangent to this circle at T. The centre of the small semicircle is P and the centre of the small circle on the right is Q. The perpendicular from Q meets the diameter SV at W.

Suppose that the radii of the two small semicircles are a and b as shown. Then the radius of the large semicircle is $a+b$. We shall calculate the radius r of the circle on the right.

We have the following results:

$SW = QT = QU = QR = r$
$PQ = QU + UP = r + a$
$OS = OV - SV = (a+b) - 2a = b - a$
$OW = OS + SW = b - a + r$
$OQ = OR - QR = a + b - r$
$PW = PS - SW = a - r$

From the right-angled triangle QWP we have:

$WQ^2 = PQ^2 - PW^2 = (r+a)^2 - (a-r)^2 = 4ar.$

From the right-angled triangle OWQ we have $OQ^2 = OW^2 + WQ^2$, giving $(a+b-r)^2 = (b-a+r)^2 + 4ar$
$\Leftrightarrow a^2 + b^2 + r^2 + 2ab - 2ar - 2br = b^2 + a^2 + r^2 - 2ab - 2ar + 2br + 4ar.$
This simplifies to $4ab = 4br + 4ar$, from which it follows that

$$r = \frac{ab}{a+b}.$$

The crucial observation now is that this expression is symmetric in a and b, so when the radius of the small circle on the left is calculated it will be the same.

The first two problems appeared in the Dutch magazine *Pythagoras* (January 2011, page 32). The third problem is ancient: it was first posed and solved by Archimedes, who gave it the name *Arbelos*, the Greek word for a semicircular shoemaker's knife.

SHARP

Excellent entries, with solutions of all three problems, were received from Robert Spencer (grade 11, Westerford High School) and Shaylan Lalloo (grade 10, Pearson High School). Both are awarded First Prizes: top level Sharp calculators. □

PARTY PIECES

Five girls (Ann, Bea, Con, Dot and Eve) sang several songs at a party. Ann sang the fewest (five) and Eve sang the most (eight). Each song was sung by four girls. How many different songs were sung at the party?

The answer is on page 30. □

GAME PROBLEM

A lion can eat a springbok in 4 hours. A hyena takes 6 hours and a jackal takes 10 hours. How long would it take for a lion, hyena and jackal to eat a springbok if they ate together?

The answer is on page 30. □

LIMERICK PROBLEM

The calculation below can be written as a limerick. How?

$$(12 + 144 + 20 + 3\sqrt{4}) \div 7 + (5 \times 11) = 9^2 + 0$$

The answer is on page 30. □

CALLING ALL CODEBREAKERS

In 1941, during the darkest days of the Second World War, the British Government was desperately searching for recruits for the top-secret codebreaking unit at Bletchley Park. Trawling through the mathematics departments of Oxford and Cambridge universities was one obvious solution. Another was more original.

Cryptic crosswords have always required a devious but ultimately logical mind. In 1941 they were very popular, and there were races to see who could crack a crossword in the shortest time. So when a donor came forward offering to donate an amount of £100 to the Minesweeper's Fund if anybody could crack the tough *Daily Telegraph* cryptic crossword in under 12 minutes, there was a good response.

About thirty enthusiastic "cruciverbalists" turned up at the newspaper's offices in Fleet Street, and were presented with a crossword to solve. The 12 minute limit was spectacularly broken by four contestants, with the winner taking a mere 7 minutes 57.5 seconds.

A few days later the contestants were rather surprised to receive a letter stamped **CONFIDENTIAL**, inviting them to an interview. The letter came from the Head of MI8, the military intelligence section concerned with the top secret codebreaking centre at Bletchley Park. They were told that their sort of twisted brains that could crack crossword puzzles in record time could be put to better use. They were put through a basic course in codebreaking, and in due course arrived in Bletchley Park.

With the crossword puzzlists, an assortment of mathematicians, chess champions and linguists joined the project. The recruitment programme was stunningly successful. It is now generally agreed that the work of the Bletchley Park codebreakers, led by the mathematical genius Alan Turing, shortened the Second World War by at least two years.

Today, 70 years on, the British government is again looking for codebreakers to turn them into internet spies. This time the search is online, not secret. According to a report in the *Business Times* (2 December 2011), a public invitation has been issued to expert codebreakers to take part in a contest to unscramble codes on the website `www.canyoucrackit.co.uk`. Competitors are faced with a grid of numbers and letters along with a countdown clock. This time the limit is not 12 minutes. They are given seven hours to unscramble the code. □

BOOKSHELF

Five books are on the shelf: a focused study at an elementary level of an intriguing branch of topology, a delightful ramble though the wonderland of mathematics, a scholarly yet eminently readable cultural history, a fascinating biographical novel of two genius-outcasts and a book on Problem Solving.

Knots Unravelled, by Meike Akveld and Andrew Jobbings. Arbelos 2011.

Knots are part of life and culture: parcels, shoelaces, carpets, neckties, knitting patterns, hair braids, koeksisters. Sailors, mountain climbers and lacemakers are all skilled in tying complicated knots. Knots, mere playthings in primary school, can be found in the frontiers of current mathematical research, such as polymer chemistry, quantum gravity and DNA analysis.

In mathematics, the study of knots is a branch of topology. How can you tell whether two knots are the same? Is there a sensible way of classifying knots? In *Knots unravelled*, the authors lead the reader through the basic theory of knots, with interludes of excursions into culture and the visual arts. There are activities and investigations, with hints and solutions at the back.

While *Knots unravelled* needs to be read from the beginning, carefully, with pencil and paper, and doing the exercises, the next book is different.

Alex's Adventures in Numberland, by Alex Bellos Bloomsbury 2010.

You can open this book it anywhere and just start reading. For example, let's just take the first 20 digits of pi, split them into page numbers fairly arbitrarily and see what turns up.

p. 314: Craps (that's the dice game, if you were wondering)
p. 159: The transcendence of π
p. 265: Perfect and amicable numbers
p. 358: Curious ways of representing numbers as spatial forms
p. 97: Colour-coding Euclid
p. 93: The Platonic solids
p. 23: A chimpanzee that can do arithmetic
p. 84: Pythagoras' Theorem

That leads seamlessly to a very different book.

Pythagoras, by Kitty Ferguson. Faber & Faber 2011.

Pythagoras is the theme of Kitty Ferguson?s book of that title, and the subtitle tells you what it is all about: *His lives and the legacy of a rational universe*. The plural signals that this is not a biography, but a comprehensive examination spread over 2600 years of the influence of the legendary Greek mathematician, separating fact from fiction and providing a grand tour of the history of mathematics from the remarkable work of the Greeks, through the medieval world into the Renaissance and the modern world of Albert Einstein and Stephen Hawking.

A Madman Dreams of Turing Machines, by Janna Levin. Weidenfeld & Nicholson 2008

Kurt Gödel shocked both mathematicians and philosophers by proving in 1931 that there are statements in mathematics that are undecidable. Alan Turing, a brilliant mathematician and pioneer of computing, shortened WWII by two years by cracking the German Enigma cipher. Janna Levin's story brilliantly weaves the lives of these two troubled geniuses into the fabric of the first half of the 20th century. The book reads like a novel, and like a novel has no index. But the narrative sticks closely to the truth and, like a real biography, is backed up by several pages of references. You won't get a good explanation of Gödel's famous Incompleteness Theorem, or an understanding of how a Turing Machine works, but you will get a a deeper understanding of the sad lives of two geniuses.

Problem Solving, by Derek Holton. The Mathematical Association 2011.

The core of this book is a collection of problems, some familiar, which the author has used over the years in enrichment classes in mathematics. The purpose is to show the reader how to tackle problems, and to show teachers how to teach the art of problem solving. The problems are not the mechanical exercises of routine school maths, but may take an hour or more to crack. The author emphasises that Mathematics is not a matter of memorising tedious sets of unconnected rules and formulas, but a creative activity. Each problem is carefully investigated and explored, giving the reader insights into the processes used by research mathematicians to create new mathematics. After the creative phase, the need for a convincing proof is stressed. Although the target reader is the maths teacher, the book can be read and appreciated by bright high school pupils. □

VARSITY ALGEBRA TEST

How ready are you to handle varsity maths? One of the most important skills required in university mathematics is fluency in algebra. Weakness in algebra will affect your ability to handle several other subjects at university level, notably physics, engineering and actuarial science.

The test below consists of questions to test your algebraic manipulation skills. If a question seems long and complicated, look at it again carefully, as a little insight or lateral thinking may show you a quick solution.

There is no recommended time limit for the test. However, you are recommended to work through the whole test in one sitting, without looking up the answers. There is no pass mark. Any problem that you can't do, or get wrong, is an indication of a weakness in your algebra that needs fixing.

No calculators allowed!

1. If $a + b = 7$ and $ab = 4$, what is the value of $a^2 + b^2$?

   (1) 28 (2) 40 (3) 41 (4) 49 (5) 55

2. A car travels x km on a litre of petrol and petrol costs y rands per litre. The cost of the petrol, in rands, for a journey of z km, is

   (1) $\dfrac{yz}{x}$ (2) $\dfrac{xz}{y}$ (3) $\dfrac{xy}{z}$ (4) xyz (5) $\dfrac{xyz}{x+y+z}$

3. If $2a - 3b = 4$ and $7a + 2b = 9$, then $a + b$ is equal to

   (1) $\frac{3}{7}$ (2) 1 (3) $-\frac{5}{2}$ (4) $\frac{7}{4}$ (5) -3

4. If $a^2 = a + 3$, then a^5 is equal to

   (1) $7a + 33$ (2) $6a + 31$ (3) $11a + 14$ (4) $19a + 21$ (5) $40a + 57$

5. Which of the following is not a factor of $6x^4 - 23x^3 + 10x^2 + 29x - 10$?

   (1) $3x - 1$ (2) $x - 2$ (3) $x + 1$ (4) $2x - 3$ (5) $2x - 5$

6. $\dfrac{3}{x-2} - \dfrac{2}{x+1}$ is equal to

   (1) $\dfrac{x+1}{x^2-x-2}$ (2) $\dfrac{x+7}{x^2-x-2}$ (3) $\dfrac{5x-1}{x^2+x-2}$

   (4) $\dfrac{x-2}{x^2-x-2}$ (5) $\dfrac{5x-1}{x^2+x+2}$

7. $\dfrac{5\sqrt{3}-3\sqrt{5}}{\sqrt{5}-\sqrt{3}}$ is equal to

(1) 0 (2) -1 (3) $\sqrt{15}$ (4) $\sqrt{5}+\sqrt{3}$ (5) $\sqrt{5}-\sqrt{3}$

8. $\sqrt{a^{144n^6}}$ is equal to

(1) a^{12n^3} (2) a^{12n^4} (3) a^{72n^4} (4) a^{72n^6} (5) a^{72n^3}

9. $\sqrt{64a^{16}b^{12}+100a^8b^{16}}$ is equal to

(1) $8a^8b^6+10a^4b^{16}$ (2) $2a^4b^6\sqrt{16a^8+25b^4}$ (3) $2a^8b^{12}\sqrt{16a^8+25b^4}$

(4) $8a^4b^3+10a^4b^8$ (5) $16a^8b^{12}\sqrt{4a^8+5b^4}$

10. $(a+b)^{-1}(a^{-1}+b^{-1})^{-1}$ is equal to

(1) $(a+b)^2$ (2) $(a+b)^{-1}$ (3) $(ab)^{-1}$ (4) a^2+b^2 (5) ab

11. Jacob ran for 10 km at 6 km/h and then ran back along the same route at 10 km/h. What was his average speed, in km/h, for the whole route?

(1) 7 (2) 7.5 (3) 7.75 (4) 8 (5) 8.25

12. If $\dfrac{1}{x^2+7x+5}=2$, then the value of $\dfrac{1}{x^2+7x+7}$ is

(1) $\frac{1}{2}$ (2) $\frac{2}{3}$ (3) $\frac{2}{5}$ (4) $\frac{1}{4}$ (5) 4

13. If $3 \le m \le 10$ and $12 \le n \le 21$, then the difference between the largest and smallest possible values of $\dfrac{m}{n}$ is

(1) $\dfrac{29}{42}$ (2) $\dfrac{29}{5}$ (3) $\dfrac{19}{70}$ (4) $\dfrac{19}{12}$ (5) $\dfrac{19}{84}$

14. If p, q and r are positive integers such that $p+\dfrac{1}{q+\dfrac{1}{r}}=\dfrac{10}{7}$, then r is equal to

(1) 2 (2) 3 (3) 4 (4) 5 (5) 6

15. $\dfrac{(x^2-1)(x^3-1)}{(x-1)^2}$ is equal to

(1) x^3+x^2+x+1 (2) x^3+2x^2+2x+1 (3) x^3-x^2+x-1

(4) x^3-2x^2-2x+1 (5) x^3-2x^2+2x-1

16. If $x+y=12$ and $x^3+y^3=1188$, the value of xy is

(1) 11 (2) 15 (3) 18 (4) 20 (5) 27

17. If $xy=7$ and $x+y=3$, what is the value of $(x+1)(y+1)$?

(1) 10 (2) 11 (3) 12 (4) 13 (5) 14

18. If $f(x)$ is a quadratic function of x such that $f(-2)=-6$, $f(0)=-4$ and $f(1)=0$, what is the value of $f(-1)$?

(1) -5 (2) -4 (3) -6 (4) -3 (5) -2

19. The coefficient of x^2 in the expansion of $(2x-1)(3x+5)(7x-3)$ is

(1) 56 (2) 47 (3) 27 (4) 61 (5) 31

20. When $42x^3+31x^2-56x+15$ is divided by $(2x-1)(5+3x)$ the quotient is

(1) $2x-3$ (2) $3x-7$ (3) $6x+5$ (4) $14x+1$ (5) $7x-3$

21. If $2^a=5$ and $5^b=10$, then ab is equal to

(1) $\log_2 10$ (2) 5 (3) 10 (4) $\log_5 20$ (5) 25

22. $\log_5 500$ is equal to

(1) $3+2\log_5 2$ (2) $5+\log_5 10$ (3) $10+\log_5 50$

(4) $\log_5 200+\log_5 300$ (5) $2\log_5 250$

23. The roots of the quadratic equation $3x^2-5x-11=0$ are a and b. Which of the quadratic equations below has roots $2a$ and $2b$?

(1) $5x^2-7x-13=0$ (2) $6x^2-10x-11=0$

(3) $3x^2-10x-44=0$ (4) $12x^2-10x-11=0$

(5) $x^2-3x-9=0$

24. $\dfrac{x+1}{x^2+3x+2} + \dfrac{x+2}{x^2+5x+6}$ is equal to

(1) $\dfrac{2x+3}{2x^2+8x+8}$ (2) $\dfrac{2x+3}{x^4+15x^2+12}$

(3) $\dfrac{2x+5}{(x+2)(x+3)}$ (4) $\dfrac{3x+2}{(x+1)(x+2)(x+3)}$

(5) $\dfrac{2x+3}{x^3+5x^2+9x+6}$

25. When the expression $(a+2b+3c+4d)(a-2b+3c-4d)$ is multiplied out and simplified, how many terms does the final expression have?

(1) 16 (2) 12 (3) 10 (4) 8 (5) 6

Answers are on page 30. □

ALL NUMBERS ARE EQUAL

We are going to prove the surprising result that all numbers are equal. You don't believe it?

Watch very carefully!

Consider the consecutive numbers x and $x+1$.

Then $x^2 - x(2x+1) = (x+1)^2 - (x+1)(2x+1)$.
(Both sides are equal to $-x^2 - x$.)

Complete the square on both sides by adding $\left(\frac{1}{2}(2x+1)^2\right)$:

$x^2 - x(2x+1) + \left(\frac{1}{2}(2x+1)^2\right) = (x+1)^2 - (x+1)(2x+1) + \left(\frac{1}{2}(2x+1)^2\right)$

Factorize:

$\left(x - \frac{1}{2}(2x+1)\right)^2 = \left(x+1 - \frac{1}{2}(2x+1)\right)^2$

Take square roots of both sides:

$x - \frac{1}{2}(2x+1) = x+1 - \frac{1}{2}(2x+1)$

and it clearly follows that $x = x+1$.

So any two consecutive numbers are equal. It follows that all numbers are equal. What went wrong? □

SOLUTIONS

Pi Riddles (page 3):
1. Pumpkin pi 2. Eskimo pi 3. Pi in the sky
4. Transcendental meditation 5. Irrational

Quick Quiz (page 3):
1. Isaac Newton 2. Albert Einstein 3. Alexander Solzhenitsyn
4. The Haunted House 5. A Fields Medal

The Grazing Oxen Problem (page 5):
The number of oxen is $\dfrac{gbdfh - ecagh - bdcgf + ecfga}{befh - bceh}$.

Reader's Problem (page 7):
The ownership of the land was originally in the proportion $8 : 7 : 3$.

Six Dice Problems (page 11):
1. The answer lies in changing the question: What is the probability that none of the six dice will be a six?

Since the chance of throwing a six with a single die is $\frac{1}{6}$, the chance of throwing any number other than six is $\frac{5}{6}$. So the chance of *not* throwing a six with six dice is

$$\left(\frac{5}{6}\right)^6 = \frac{15625}{46656} = 0.3349.$$

The chance of throwing at least one six is therefore $1 - 0.3349 = 0.6651$, or about two-thirds.

2. There are $6\times5\times4\times3\times2\times1 = 6! = 720$ different ways of the dice showing six different numbers. On the other hand, there are $6 \times 6 \times 6 \times 6 \times 6 \times 6 = 6^6 = 46656$ different outcomes when six dice are thrown. So the probability of getting six different numbers when throwing six dice is

$$\frac{720}{46656} = 0.015$$

– less than 2%.

3. The chance that all dice are different is 0.015 (shown above). So the chance of getting at least two the same is $1 - 0.015 = 0.985$ – more than 98%.

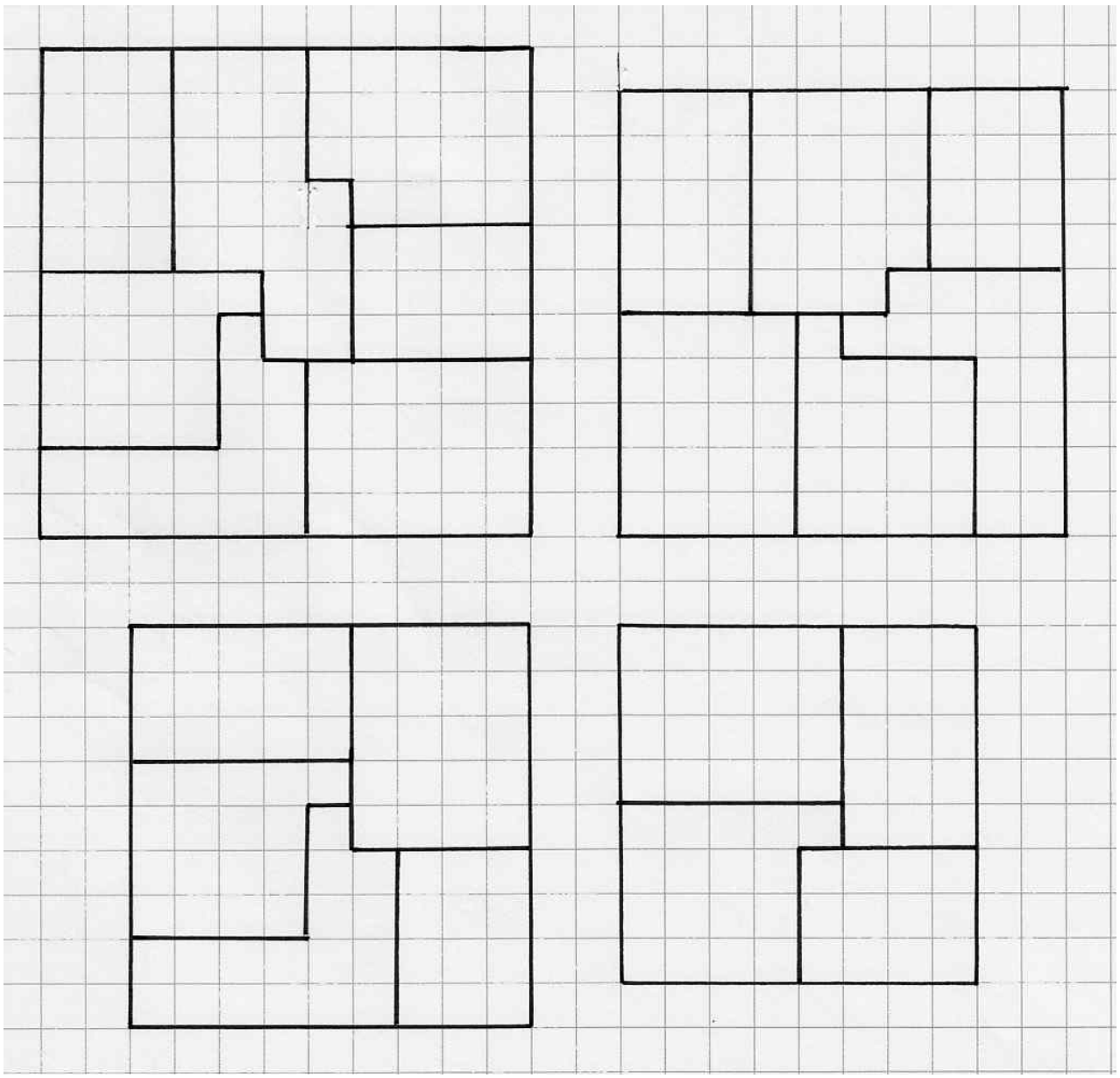
4. The chance of getting two dice the same is 0.985. Since the chance of getting two sixes is the same as the chance of getting two ones, two twos, two threes, two fours or two fives are all the same as the chance of getting two sixes, the chance of getting two sixes is $\frac{1}{6} \times 0.985 = 0.164$.

5. The average score when throwing one die is $\frac{1}{6}(1+2+3+4+5+6) = 3.5$. Since the six dice are independent, the average total on throwing them will be $6 \times 3.5 = 21$.

6. In such an unlikely scenario, there is a good chance that the dice are loaded so that a six never turns up. Or perhaps none of the dice has a six on it. The answer is very probably zero.

Poetry Quiz (page 15):
Hudibras, by Samuel Butler.

Turkish Squares Puzzle (page 17):



Party Pieces (page 20):
Since Ann sang 5 songs and Eve sang 8, Bea, Con and Dot sang either 6 or 7 songs. The total number of songs sung is thus one of the following:

$$5+6+6+6+8=31$$

$$5+6+6+7+8=32$$

$$5+6+7+7+8=33$$

$$5+7+7+7+8=34.$$

Since each song was sung by four girls, the total number of performances was a multiple of 4, so the only possible answer is 32.

Game Problem (page 20):
Lion, hyena and jackal never eat together.

Limerick Problem (page 20):

A dozen, a gross and a score
Plus three times the square root of four
    Divided by seven
    Plus five times eleven
Equals nine squared and not a bit more.

Varsity Algebra Test (page 22):

31244 23423 23122 22355 11335

MUSIC, MATHS AND DARWINISM

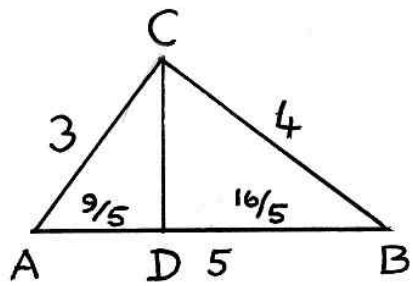
Till the fifteenth century little progress seems to have been made in the science and practice of music; but since that era it has advanced with marvellous rapidity, its progress being curiously parallel with that of mathematics, inasmuch as great musical geniuses appeared suddenly among different nations, equal in their possession of that special faculty to any that have since arisen. As with the mathematical so with the musical faculty—it is impossible to trace any connection between its possession and survival in the struggle for existence.

(Alfred Russell Wallace, in *Darwinism*) □

THE 3-4-5 SOLUTION

In triangle ABC, $AB = 5$, $BC = 4$ and $CA = 3$. Without using the Theorem of Pythagoras, or its converse, we shall prove that the triangle is right-angled (at C).

The proof begins with a construction. Let D be the point on AB such that $AD = \frac{9}{5}$ and $DB = \frac{16}{5}$. Join DC.



Now consider triangles ADC and ACB.

We have

$$\angle DAC = \angle CAB$$

and

$$\frac{AD}{AC} = \frac{\frac{9}{5}}{3} = \frac{3}{5} = \frac{AC}{AB}.$$

It follows that triangles ADC and ACB are similar, and therefore equiangular. Hence $\angle ADC = \angle ACB$. (1)

We now repeat the argument for triangles BDC and BCA.

We have

$$\angle DBC = \angle CBA$$

and

$$\frac{BD}{BC} = \frac{\frac{16}{5}}{4} = \frac{4}{5} = \frac{BC}{BA}.$$

It follows that triangles BDC and BCA are similar, and therefore equiangular. Hence $\angle BDC = \angle BCA$. (2)

From (1) and (2) it follows that $\angle ADC = \angle ACB$. Since the angles are supplementary, they must equal $90°$, and hence $\angle ACB = 90°$.

This argument is in fact a specific case of the proof of the converse of Pythagoras' Theorem. For the general case, start with triangle ABC, such that $a^2+b^2 = c^2$, and construct D on AC such that $AD = \dfrac{b^2}{c}$ and $DB = \dfrac{a^2}{c}$. Follow the same similarity arguments to deduce that $\angle ACB = 90°$. $\square$

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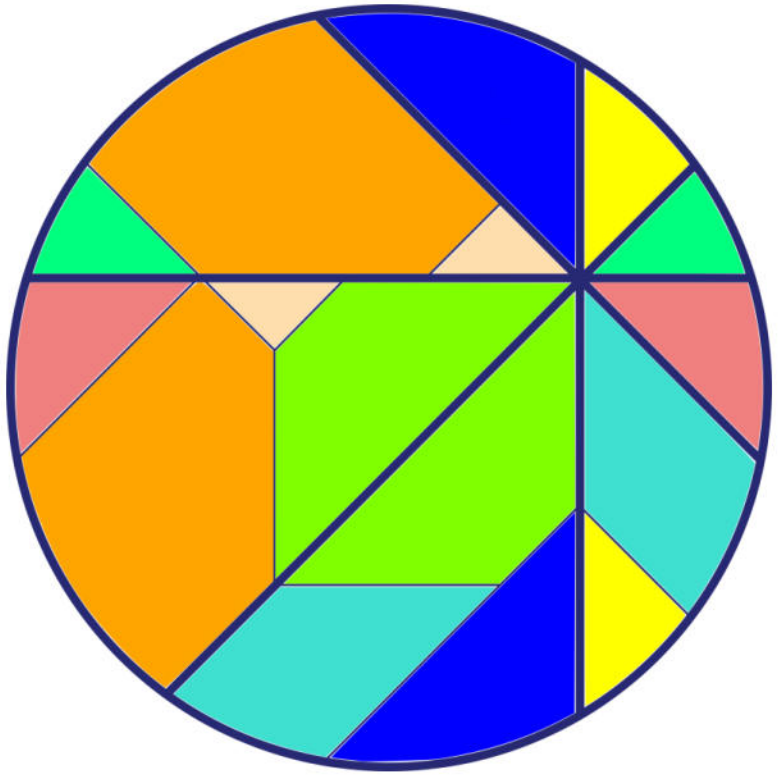

The Pizza Dissection

mathematical digest

Number 167 **April 2012**

Mathematical concepts come from snapping together old concepts in a useful new arrangement. But those old concepts are assemblies of still older concepts. Each subassembly hangs together by the mental rivets called chunking and automaticity: with copious practice, concepts adhere into larger concepts, and sequences of steps are compiled into a single step. Just as bicycles are assembled out of frames and wheels, not tubes and spokes, and recipes say how to make sauces, not how to grasp spoons and open jars, mathematics is learned by fitting together overlearned routines. Calculus teachers lament that students find the subject difficult not because derivatives and integrals are abstruse concepts—they're just rate and accumulation—but because you can't do calculus unless algebraic operations are second nature, and most students enter the course without having learned the algebra properly and need to concentrate every drop of mental energy on that. Mathematics is ruthlessly cumulative, all the way back to counting to ten.

(Steven Pinker, in *"How the mind works"*) □

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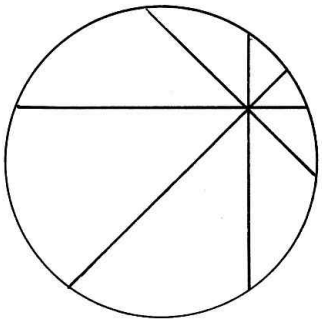
Cover: How can a badly-sliced pizza be cut up and shared fairly? See: *the Pizza Theorem* (page 1).

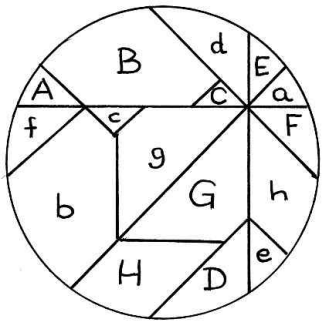
THE PIZZA THEOREM

John and Jane order a pizza to share. When they open the box they see that the pizza has been sliced badly: although the four cuts are at equal angles of 45° to each other, and through the same point, that point is clearly not the centre of the pizza, and the eight pieces are all different shapes. How should they share the slices, so that they get equal shares?



The simple (and surprising) answer is that they should number the pieces from 1 to 8 round the point of intersection of the slices as shown. If John takes the even-numbered pieces and Jane takes the odd-numbered pieces, they will get exactly half the pizza each. Moreover, they will each get the same amount of crusty edge.

Why does this work? The problem, posed by L.J. Upton in 1968, can be answered in different ways, some involving rather technical algebra and calculus. The dissection (below, and also on our front cover) is quite the most ingenious solution. It is reasonably clear how the dissection is made, but why it works needs more careful examination.



What happens when three or more people want to share a pizza? With n people, slice the pizza by $2n$ cuts through any common point P with straight cuts at equal angles, thus dividing the pizza into $4n$ pieces. Number the pieces consecutively from 1 to $4n$. If each person then takes every nth piece, they all get equal shares. $\square$

THE RAPUNZEL NUMBER

The story of Rapunzel, the princess with magically long hair, can be found in Grimm's Fairy Tales, and has more recently been retold in the Disney movie *Tangled*. It has recently got tangled with some significant developments in mathematical physics.

It all started when Unilever, the multinational corporation whose products include shampoos, invited Professor Raymond Goldstein, of the Department of Physics of Cambridge University, to collaborate in research on the basic properties of hair, including problems of tangling.

Goldstein saw that the research would involve some interesting physics, and put together a team he described as "a bunch of balding, middle-aged men" to tackle the problem of producing gorgeous hair for girls. Every girl would like to know what happens if she lets her hair grow, and pulls it back into a ponytail. Will it fan out and swing sexily from side to side, or just hang down boringly straight?

Predicting how some 100 000 hairs on the average girl's head will behave as they grow longer is not easy. It has intrigued scientists for over 500 years, and has connections with fluid dynamics. It was Leonardo da Vinci who noted that the waviness of long hair closely resembles the patterns of eddies in a flowing stream.

The Cambridge physicists have now come up with a formula which predicts the shape of a ponytail. The researchers had to take into account the elasticity, density and curliness of an individual's hair and used mathematical techniques from the field of statistical physics to calculate the effect of gravity on tight bundles of long hair. Raymond Goldstein could not resist naming a key element of their formula "the Rapunzel Number".

A short ponytail of springy hair with a low Rapunzel number fans outward. A long ponytail with a high Rapunzel number hangs down, as the pull of gravity overwhelms the springiness.

The research is not as frivolous as it might appear. The mathematical techniques the Cambridge team has developed will help to understand the structure of materials made up of bundles of fibres, such as wool and cotton.

The Rapunzel Number will have significant application in the world of computer graphics and movie animation, where realistic depiction of flowing hair has been a challenging problem.

So even if Unilever is unable to use the Cambridge research to develop shampoos that make long hair even more swinging and sexy, the graphics artists who design their television commercials will use the Rapunzel Number to give just the effect they want their customers to believe. And they may well claim that "99% of Cambridge physicists recommend" their new product.

What would Leonardo da Vinci have done if he had known about the Rapunzel number? Would he have given Mona Lisa a ponytail? □

THINK OF TWO NUMBERS

"Think of a number" tricks usually rely on some easy algebra. Here is a version where you start with two numbers.

Find a friend who can be relied on to do simple arithmetic without a calculator. Then say:

> "Think of two numbers, the first a two-digit number and the second a single-digit number. But don't tell me what they are.
> Multiply the first number by 5.
> Add 8.
> Multiply your answer by 2.
> Add the second number.
> Subtract 5.
> What is your answer?"

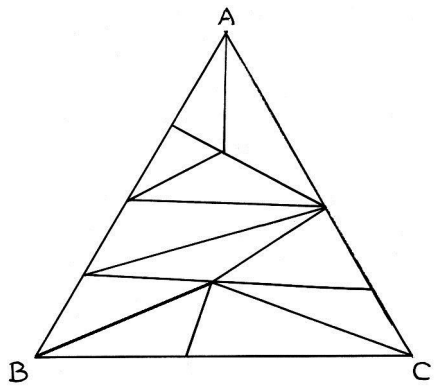
From that you can immediately name the two secret numbers.

How does the trick work? The answer is on page 30. □

SPERNER'S LEMMA

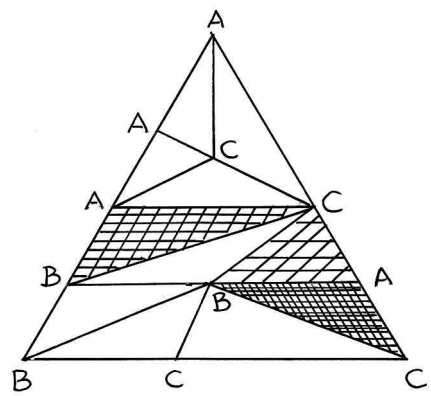
Here is a result that would make lovely IMO problem, except that it is a known result. It was discovered by the German mathematician Emanuel Sperner

Start with a triangle ABC, and subdivide it into smaller triangles in any way you like.



The new points are then labelled A, B or C, subject to the constraints:

points on the edge AB are labelled either A or B
points on the edge BC are labelled either B or C
points on the edge CA are labelled either C or A
points inside the triangle may be labelled either A, B or C.

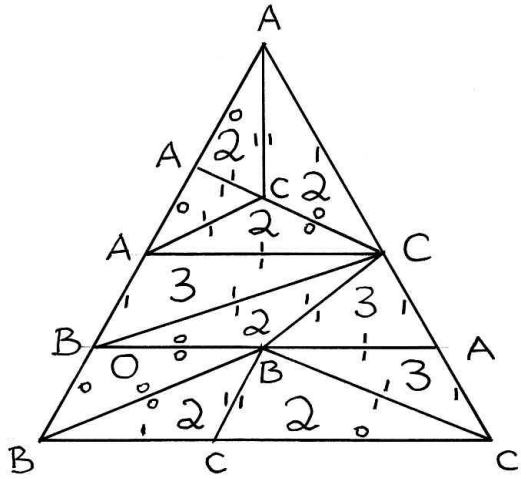


Now look at all the little triangles, and shade those that have vertices A, B, C. There are three in the example above.

Sperner's Lemma states that no matter how the triangle is divided, or how the vertices are labelled, the number of shaded triangles will always be odd.

The proof is by a clever piece of double counting.

Label the edge of each little triangle in the figure on the inside of each of its edges with 1, if the edge has two different labels, and 0 if the edge has two labels the same.



Note that each interior edge occurs in two triangles, so is labelled twice.

Now add up all the numbers. Call this sum N. Along each edge of the big triangle, the sum will be odd, since the edge starts and ends with different labels. So the total of all the outside edge numbers is odd. Inside the triangle, every number appears twice, so the sum of the inside edge numbers is even.

So N is the sum of an odd number (from the outside edges) and an even number (from the inside edges). Thus N is odd.

In each triangle, write the sum of the labels on its edges. That sum will be

0 if the three vertices are the same
2 if two edges are the same, and one different
3 if all three vertices are different.

Suppose now that there are m triangles labelled ABC, n triangles with just two vertices the same and p triangles with three vertices the same. Then the sum of all the triangle numbers is $3m + 2n + 0p = 3m + 2n$.

Now this sum must be equal to N: $3m + 2n = N$, ad odd number.

It follows that m must be odd.

This activity can be set as a classroom game. Two players take turns to label the vertices according to the rules stated, and when they form a triangle ABC they shade it and score a point.

Sperner's Lemma says that this game can never end in a draw. $\square$

CRYPTOGRAPHY CRISIS

Modern cryptography relies on what are colloquially known as "unbreakable codes". Recent developments have caused a major stir in the world of cryptography. A Swiss team has shown that many of the unbreakable codes can in fact be easily cracked.

In cryptography, a message is converted into a number (or string of numbers), usually in binary notation for easy computer use. Every cryptosystem relies on a function which converts an input message/number into an output message/number.

When Alice wants to send a secret message n to Bob, she applies a certain encryption function f to n, giving an output $f(n)$.

When Bob receives the message, he applies a decryption function, the inverse function f^{-1}, to it, to get $f^{-1}(f(n)) = n$, the original message.

This system relies on Alice and Bob agreeing to use a specific encryption function f. For most functions f, it is easy to work out the inverse function f^{-1}. For example, if f is the function which simply adds 3 to n (a method used over two thousand years ago by Julius Caesar), then the decryption function simply subtracts 3 from $f(n)$. So for Alice and Bob to communicate in secret, they must keep their encryption function f secret, because if the encryption function is known it may be easy to work out the decryption function and their secrecy is compromised.

A major revolution in cryptography occurred in the 1970s with the invention of "public key cryptosystems", using "trapdoor one-way functions". These cleverly-devised functions have the unexpected property that they can be made public, yet nobody can work out the inverse function.

So when Alice wants to send a message n to Bob, she looks up Bob's public key cryptosystem function f and sends him the encrypted message $f(n)$, knowing that even if her message is intercepted by an unauthorised person, it cannot be decrypted.

These special trapdoor one-way functions f depend on a very simple fact about prime numbers. It is easy to multiply two 100-digit prime numbers but, given their 200-digit product, it is very difficult to determine the two prime factors.

This is the basic idea underlying the RSA system (named after its inventors R. L. Rivest, A. Shamir, and L. Adleman).

Thousands of companies round the world use the RSA system, and their public keys (each consisting of the product of two huge prime numbers) are in the public domain.

In February this year a group of Swiss cryptographers published a paper highlighting a basic flaw in the system. Though it is indeed very difficult to find the prime factors of a large number, given two large numbers it is in fact very quick and easy to find the greatest common divisor (GCD) of the two numbers. The algorithm to do this was known two thousand years ago by the Greek mathematician Euclid (usually remembered today for his work in geometry).

The Swiss group collected all the public keys they could find on the internet and paired them off to find their GCDs. Usually the GCD turned out to be 1, which meant that the two systems had used four different prime numbers. But when the GCD was not 1, it was a prime number common to the two keys, and the other prime factors of the two keys could at once be found by simple division. Those two public keys were therefore insecure.

The Swiss team found that, on average, two out of every thousand public key systems was flawed. While this number is small, the Swiss announcement has put the cat among the pigeons. Security systems round the world are scrambling to find out whether their public keys are compromised by having a prime number in common with another key. □

KWAME NKRUMAH AWARD

The African Union Commission for Human Resources, Science and Technology has named Professor Oluwole Daniel Makinde, Director of Postgraduate Studies at the Cape Peninsula University of Technology in South Africa, as the winner of the 2011 African Union's Kwame Nkrumah Continental Scientific Award in the category Basic Science, Technology and Innovation.

The award was made during the 18th Heads of State and Government Summit on 29th January 2012 at the African Union Head Office in Addis Ababa, Ethiopia. The award was given in recognition of Professor Makinde's outstanding scientific achievements at continental level in the field of Science, Engineering, Technology and Innovation and for contributing through science to the socio-economic development of Africa. □

THE SUDOKU MINIMUM

Everybody knows Sudoku, the Japanese logic puzzle where the numbers from 1 to 9 have to be placed into a 9×9 square so that each column, each row, and each of the nine 3×3 sub-squares contain all nine numbers. The puzzle starts with a number of the squares filled in, and the challenge is to complete the square. A true Sudoku has only one solution.

Thousands of newspapers publish Sudoku puzzles, sometimes reassuring their mathephobic readers that no mathematics is required for their solution. While it is certainly true that algebra and trigonometry will not be helpful in cracking a Sudoku puzzle, long chains of careful logical reasoning will certainly come into play, and logic, after all, is the core of mathematics.

Apart from solving the puzzle, there are some interesting problems about setting Sudokus. How many complete Sudoku grids are there? The answer is 6×10^{21}. But if we exclude variations such as permuting the rows or columns, or permuting the digits, or transpositions, reflections and rotations, the number comes down to 5 472 730 538 essentially different complete Sudoku grids.

What is the greatest number of clues that can be provided in the grid without giving a unique solution? It's not difficult to see that as many as 77 squares can be filled, with two possible ways of filling in the last four.

A rather harder problem is the smallest number of clues that can be provided that will guarantee a unique solution.

Many puzzles with 17 clues and a unique solution have been published, and it is estimated that there are at least 50 000 of of them. But nobody has succeeded in devising a Sudoku with just 16 clues.

This last question has recently been answered. You can't do better than 17, says Gary McGuire, a maths professor at the University of Dublin. He and a team wrote a program to perform an exhaustive search for a 16-clue Sudoku puzzle. The search began in January 2011 on a Stokes machine and was completed, after 7.1 million core hours, in December 2011. No 16-clue Sudoku was found.

The Sudoku problem falls into an important category called "hitting set problems", which are NP-complete, i.e. computationally difficult. Hitting set problems have applications in several areas, including software testing and bioinformatics (gene expression analysis). □

THE MARRIAGE FORMULA

In 2006, New York Times writer John Tierney paired up with statistician Garth Sundem to create a formula that could predict the longevity of celebrity marriages. With five years of data at their disposal, the NYT team has revamped their formula and has published a revised version.

$$50\sqrt[15]{\frac{NYT}{ENQ}\cdot\frac{A_h+A_w}{Sc+5}\cdot M_d\left(\frac{M_d}{M_d+2}\right)^{T^2}}$$

The various components of the formula are:

NYT: number of times the wife's name has been mentioned in the *New York Times* since 1990

ENQ: number of times the wife's name has been mentioned in the *National Enquirer* since 1990

A_h: age of husband

A_w: age of wife

Sc: number of scantily-clad pictures of the wife in the top five images obtained via Google

M_d: number of months the couple dated before marriage

T: number of years for which you want to calculate the couple's chances of staying married

The formula will then give you the chance, as a percentage, that the couple will remain married for the next T years.

The formula is said to be surprisingly accurate. It correctly predicted the break-up of Demi Moore and Ashton Kutcher, Pamela Anderson and Kid Rock, and Britney Spears and Kevin Federline.

According to the formula, Prince William and Kate will stay married for some time, but Tom Cruise and Katie Holmes will not.

It's an interesting formula. The factor ENQ appears in the denominator, so if it is zero a couple has an expectation of an infinitely long marriage. However, if the woman's name has never appeared in the *National Enquirer*, they cannot be considered a celebrity couple, and the formula does not apply. □

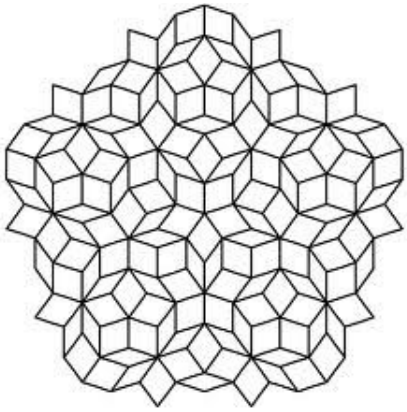
QUASICRYSTALS AND PENROSE PATTERNS

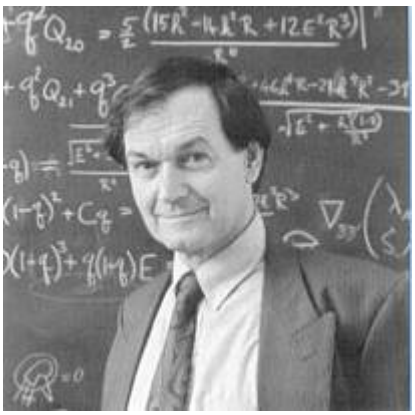
In 1982 the Israeli chemist Dan Schechtman was analysing the structure of metal alloys using X-ray crystallography. When beams of electrons are fired into the alloy, they are reflected off the layers of the alloy, forming interference patterns that allows one to determine the crystalline structure of the alloy.

Schechtman was examining an alloy of manganese and aluminium, which produced a beautiful structure. But it did not make sense: the crystals seemed to have pentagonal symmetry, in a non-repeating pattern. This was against all the accepted principles of crystallography. Equilateral triangles, squares and regular hexagons will tile a floor in a regular pattern without any breaks, but you can't do the same with pentagons. According to the International Union of Crystallography, a crystal has "a regularly ordered, repeating three-dimensional pattern".

Schechtman's alloy did not fit into the accepted theory. The fivefold symmetry did not repeat regularly across the plane. Schechtman labelled his discovery a "quasicrystal". This did not please the establishment. Schechtman's results were not believed by the hierarchy of academic chemists, and he battled to achieve recognition. He was thrown out of his research group, and found it difficult to get his results published. Nobel Prizewinner Linus Pauling was particularly rude: "There is no such thing as a quasicrystal, only quasi-scientists", he said.

Mathematics had predicted the possibility of quasicrystals before Schechtman found them. In the 1970s, the mathematician Roger Penrose had invented what are now called Penrose tiles: two rhombus-shaped figures which can be put together to tile the plane, without repetition. In 1981, the English crystallographer Alan Mackay found that atoms situated at the vertices of a Penrose tiling would have a diffraction pattern with pentagonal symmetry.

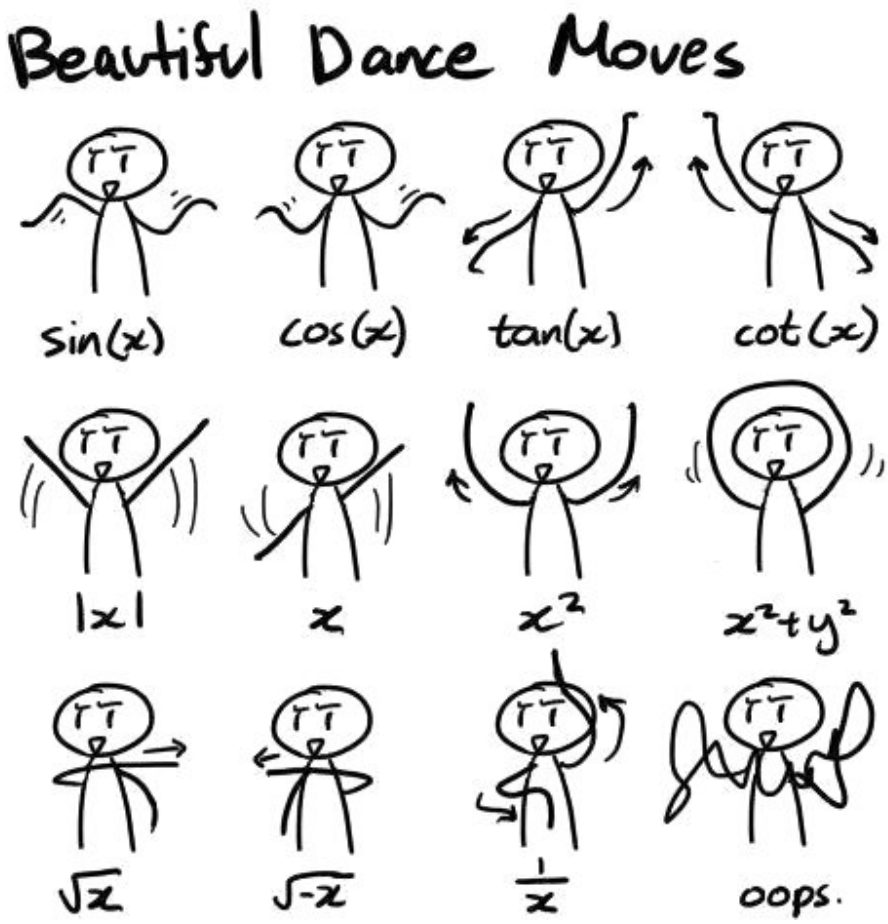




Eventually quasicrystals were accepted as real, and the International Union of Crystallography in 1992 changed its definition of a crystal to "a solid with a discrete diffraction diagram".

Since then, many quasicrystalline alloys have been found. Dan Schechtman's pioneer work was eventually recognised: at the end of last year he was awarded the 2011 Nobel Prize for Chemistry. □

DANCE MOVES



THE NOBEL AND ABEL PRIZES

When the Swedish multi-millionaire Alfred Nobel left money in his will to establish what are now known as the Nobel Prizes, he specified that they should be for Physics, Chemistry, Physiology/Medicine, Literature and Peace. They are awarded annually, and the first awards were in 1901. There is no Nobel Prize for Mathematics.

In 1936 the Fields Medal was established. For many years it has been regarded as the equivalent of a Nobel Prize in Mathematics.

Both awards are for work that is universally regarded as outstanding. But in other respects the two awards are very different.

The Nobel Prizes are awarded by Scandinavian committees. The Fields Medals are awarded by an international committee.

Nobel Prizes are awarded every year. Fields Medals (from two to four) are awarded only every four years, at the time of the four-yearly International Congress of Mathematicians.

Nobel Prizes are worth about a million dollars. A Fields Medal has a purse of $15 000.

Fields Medallists must be under the age of 40. There is no such restriction for the award of a Nobel Prize.

In 2003 a genuine equivalent to a Nobel Prize for Mathematics, the Abel Prize, was established. The establishment of the prize has a long history.

Back in 1899, when it was learnt that there would be no Nobel Prize for Mathematics, Norwegian mathematicians proposed the establishment of

a similar prize. It was to be called the Abel Prize, in memory of the Norwegian mathematician Nils Henrik Abel. The idea was to award the first Abel Prize on the occasion of the centenary of Abel's birth, in 1902. Nothing, however, came of it for the next hundred years.

In 2001, with the bicentenary of Abel's birth looming, the plans to establish an Abel Prize were revived, and this time everything came together. The Norwegian government announced that the new prize would be launched in 2002, but the Abel Prize was awarded only in 2003.

The Abel Prize is a Scandinavian award. It is worth about the same as a Nobel Prize. It has no age restriction. It is awarded every year. In every respect, it is much closer to the Nobel Prize than the Fields Medal.

The winner of the 2012 Abel Prize is the Hungarian mathematician Endre Szemerédi, who will receive the prize from King Harald of Norway in Oslo on 22 May. □

THE MONKEY AND THE COCONUTS

Five sailors are shipwrecked on a desert island, with nothing to eat except coconuts. They follow a monkey, who takes them to a grove of coconut palms, where they collect a large pile of coconuts. Because it is late, they agree to share them out the next morning.

During the night one of the sailors wakes up, and decides to take his share of the coconuts. He divides them into equal piles, and finds there is one left over. He gives the extra coconut to the monkey, hides his share, and goes back to sleep.

A short time later another sailor wakes up and does the same thing: he divides the pile into five equal shares, with one left over for the monkey, hides his share, and goes back to sleep.

The third, fourth and fifth sailors do the same, each dividing the pile they find, giving one to the monkey and hiding their share.

In the morning they all wake up, and without saying that they had each taken some of the coconuts during the night, they divide the remaining piles into equal shares, finding one left over, which they give to the monkey.

What is the smallest number of coconuts for which this is possible?

The answer can be found on page 30. □

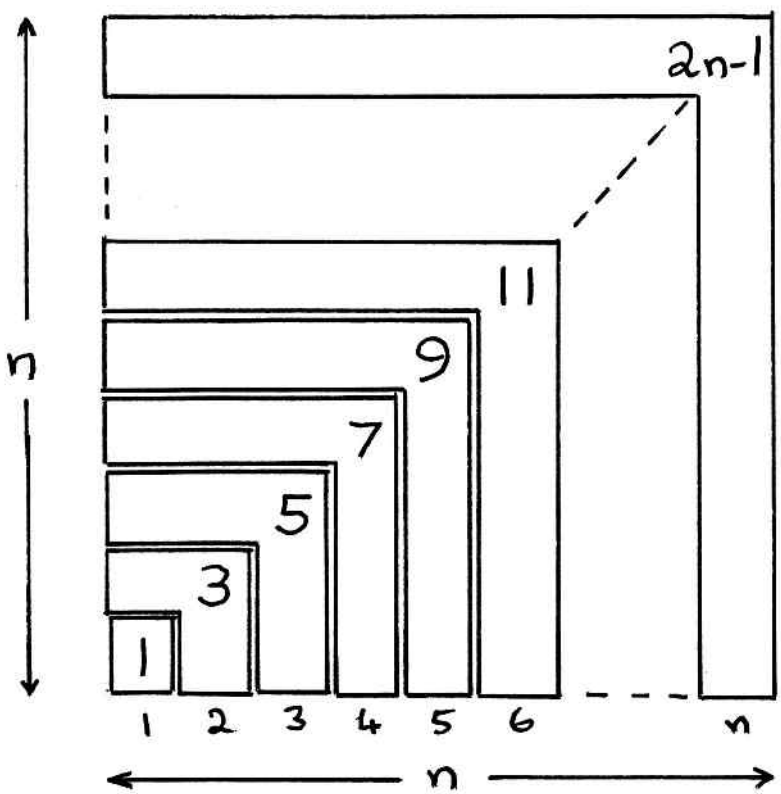
THE LEAP YEAR BUG

Microsoft's cloud (which hosts many thousands of servers on the internet) had a massive outage on 29 February because they forgot to take Leap Years into account.

Their platform included code that calculated a day one year in the future, and this was implemented by adding 1 to the year part of the date. That works fine for most days, but on 29 February of a leap year, it generates an invalid date in the next non-leap year.

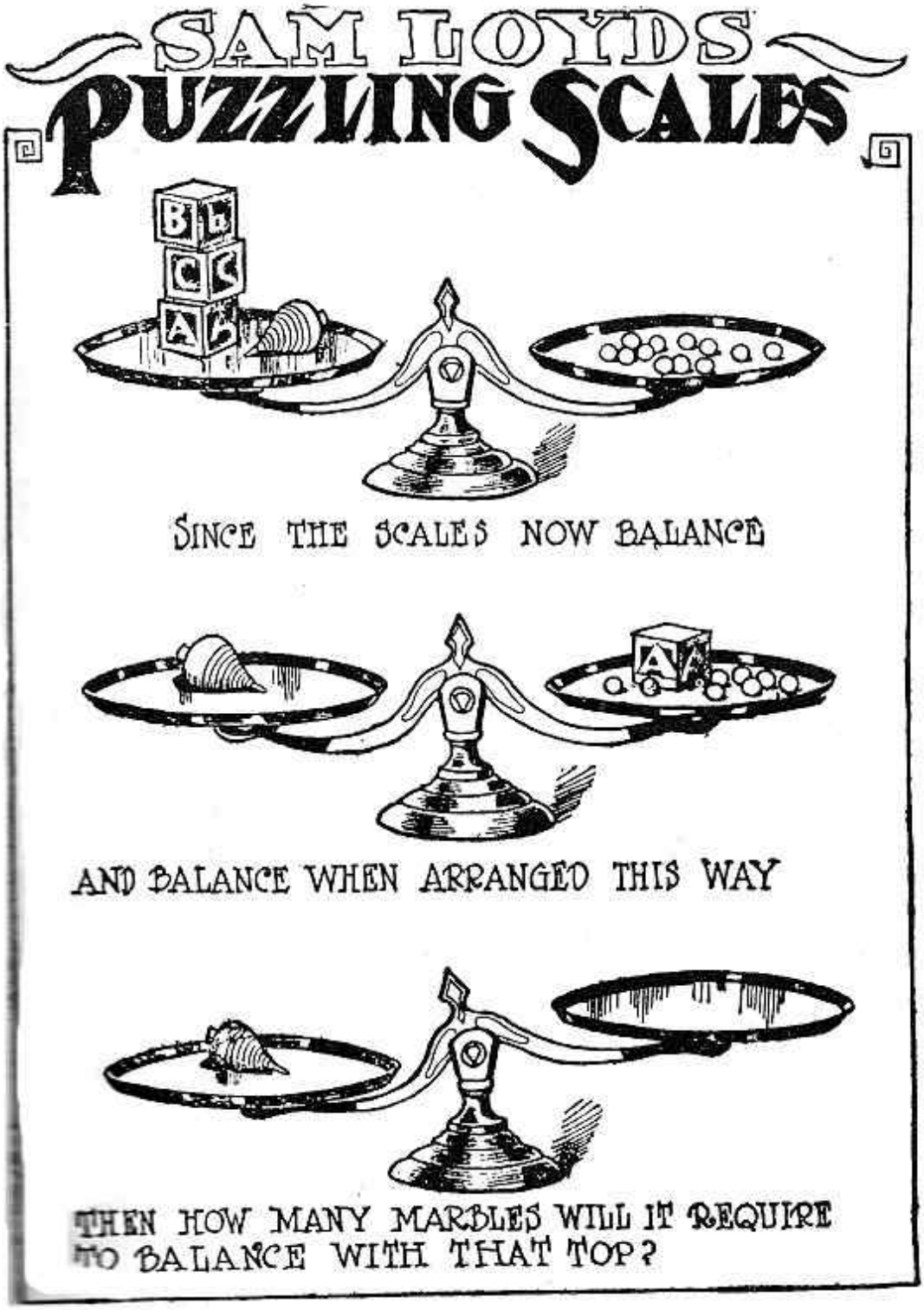
It's a good example of the sort of software bug that can be fixed at minimal cost if it is picked up early through good programming and good testing, but gets more expensive the later it is detected. In this case it was only detected after it caused a massive crash, and the disruption probably cost millions. □

PROOF WITHOUT WORDS



$$1 + 3 + 5 + 7 + \cdots + (2n - 1) = n^2$$

THE PUZZLING SCALES



This is one of the easier problems in Martin Gardner's *The mathematical puzzles of Sam Loyd*. It readily succumbs to a little bit of well-chosen algebra. But can you solve it in your head?

The answer is one page 31. □

BINARY CONVERSION

There are 10 types of people.
Those who understand binary and those who don't.

Our counting system is based on units of ten, because we have ten fingers. There are other counting systems. Time is measured in hours, with each hour divided into 60 minutes and each minute divided into sixty seconds. The same is true of angular measurement: a degree is divided into 60 minutes and a minute is divided into sixty seconds.

This base-sixty system was devised by the Babylonians of Biblical times, and is still in use today.

Monetary systems can be mixed. Until 1961 South African money was expressed in pounds, shillings and pence: the pound consisted of 20 shillings and the shilling consisted of 12 pence. To make things more complicated, the pound was also divided into four crowns, with a crown divided into two half-crowns (equal to two shillings and sixpence) while the penny was divided into two halfpennies or four farthings.

That was the arithmetic our grandparents had to handle.

Fifty years on, the binary system has become more relevant, since it's the way computers work. In order to communicate with a computer, we must know how to convert numbers expressed in our decimal system into binary numbers.

The method is simple; we shall illustrate it by converting 92 into binary.

Starting with 92, divide it by two (discarding the remainder) and divide that result by 2 (again discarding the remainder, and continue in this way until you land up with 1.

So you have the list of numbers 92 46 23 11 5 2 1

Under the odd numbers on this list write the digit 1, and under the even numbers write the digit 0:

92	46	23	11	5	2	1
0	0	1	1	1	0	1

Write this string of 0s and 1s in reverse order and you have the binary representation of 92: 1011100

Check it by putting in the powers of 2:

$$1 \cdot 2^6 + 0 \cdot 2^5 + 1 \cdot 2^4 + 1 \cdot 2^3 + 1 \cdot 2^2 + 0 \cdot 2^1 + 0 \cdot 2^0 = 64 + 16 + 8 + 4 = 92$$

Do the same with 59:

59	29	14	7	3	1
1	1	0	1	1	1

The binary representation of 59 is 111011.

Check:

$$1 \cdot 2^5 + 1 \cdot 2^4 + 0 \cdot 2^3 + 1 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0 = 32 + 16 + 8 + 2 + 1 = 59$$

Try 169:

169	84	42	21	10	5	2	1
1	0	0	1	0	1	0	1

The binary representation of 169 is 10101001. □

BEN TROVATO AND THE POINCARÉ CONJECTURE

> A Japanese study showed that taking 5g of chorella before drinking can prevent hangovers 96% of the time. From what I can make out, chorella seems to be some sort of algae capable of multiplying faster than that Russian maths freak who turned down a medal and a million-dollar prize after proving the Poincaré conjecture which states that in three dimensions you cannot transform a doughnut shape into a sphere without ripping it, although any shape without a hole can be stretched or shrunk into a sphere. How would you like to go up in front of a crowd and explain your thinking on that one? No wonder he still lives with his mother.
>
> (Ben Trovato, in the *Sunday Times*, 1 January 2012)

Who was the Russian mathematician, what was the name of the medal and what was the name of the million-dollar prize?

Answers can be found on page 30. □

BIRTHDAY CARD TRICK

Copy the five cards below and show them to a friend. Ask her to select those cards on which appear the day of the month on which she was born.

You can then immediately give the date.

1	3	5	7
9	11	13	15
17	19	21	23
25	27	29	31

2	3	6	7
10	11	14	15
18	19	22	23
26	27	30	31

4	5	6	7
12	13	14	15
20	21	22	23
28	29	30	31

8	9	10	11
12	13	14	15
24	25	26	27
28	29	30	31

16	17	18	19
20	21	22	23
24	25	26	27
28	29	30	31

The trick is very simple to do, When your friend has identified the cards on which her birthday date appears, all you have to do is add up the numbers that appear in the top left hand corners of those cards.

For example, suppose your friend's date was the 27th. Since 27 appears on Card 1, Card 2, Card 4 and Card 5, you simply add up 1 (from Card 1), 2 (from Card 2), 8 (from Card 4) and 16 (from Card 5):

$$1 + 2 + 8 + 16 = 27.$$

The secret of the trick lies in the binary system. By asking your friend to identify the cards on which the birthday date lies, you are asking five YES/NO questions, the answers to which are enough to determine $2^5 = 32$ outcomes. That's enough to identify on which date of the month the birthday falls.

The design of the cards is therefore based on the binary system. Each of the numbers from 1 to 31 is converted to binary, and then written on card n

if (and only if) a 1 appears in the nth position of the binary representation (counting from the right).

For example, the binary representation of 26 is 11010, since $26 = 16 + 8 + 2 = 2^4 + 2^3 + 2^1$, so 26 is put on cards 5, 4 and 2.

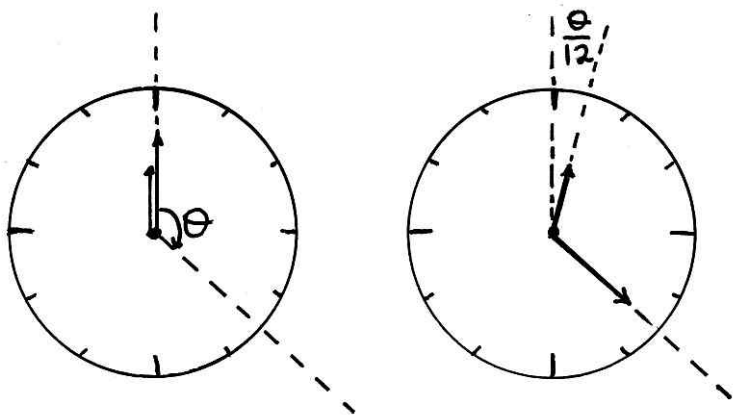
Another example: the binary representation of 11 is 1011, since $11 = 2^3 + 2^1 + 2^0$, so 11 is put on cards 4, 2 and 1.

This trick can baffle your audience in two ways. The first surpise is how you are able so quickly to work out the date when all you know is the cards on which the number appears. But even when you explain how you did it (by a quick addition of at most five numbers), it will not be at all obvious to the audience how the numbers were distributed among the cards to make the trick work. □

CLOCK TRISECTION

It is not possible to trisect an angle using only the Euclidean tools of ruler and compasses. There are, however, many simple gadgets that will enable one to trisect any angle. One of them is a clock.

Set the hands of the clock at 12. Position the angle on the face of the clock with one side at 12.00, and the other at an angle θ, measured clockwise.



Start the clock. When the minute hand reaches the other arm of the angle, it has rotated through θ, and the hour hand has rotated through $\theta/12$. Double this angle and double it again. It will be $\theta/3$, the required trisection.

This construction does not work with a digital clock. □

RUSSIAN PEASANT MULTIPLICATION

The following method for multiplying two numbers is said to have been widely used by peasants in Russia in pre-revolutionary times.

To multiply two large numbers, you only need to know how to multiply and divide by 2, and to add.

Let us suppose that you have to multiply 94 by 45.

Write them in two columns:

Multiply the number in the left hand column by 2 and divide the number on the right by 2. Ignore the remainder. Repeat this process until you have a 1 in the right hand column

94	45
188	22
376	11
752	5
1504	2
3008	1

Now add all the numbers in the left hand column that are opposite an odd number: $94 + 376 + 752 + 3008 = 4230$

That is the required answer: $94 \times 435 = 4230$

Try another example: 147×74.

147	74
294	37
588	18
1176	9
2352	4
4704	2
9408	1

Add the numbers on the left of odd numbers to get the required product:

$$294 + 1176 + 9408 = 10878$$

Why does the method work?

The clue lies in the binary system. On page 16 we showed how to convert a number from decimal to binary. The right hand column of numbers shows

how to express the number 74 as a sum of powers of 2:

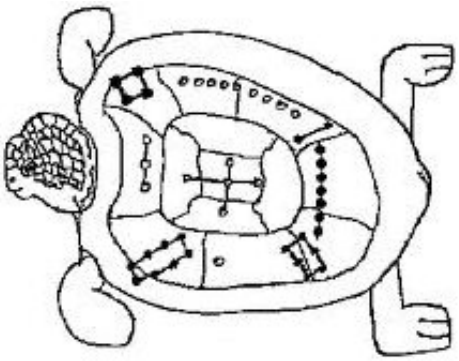
$$74 = 2^6 + 2^3 + 2^1 + 2^0$$

In binary notation 74 is written 1001011.

So $147 = 147 \times (2^6 + 2^3 + 2^1 + 2^0) = 9408 + 1176 + 294 + 1474$. □

THE CHINESE MAGIC SQUARE

The 3×3 Magic Square has been known for thousands of years, and occurs in many cultures. A Chinese legend of 650 B.C. tells of a huge flood, during which a turtle emerged from the water with a curious square pattern of dots on its back, representing the numbers from 1 to 9 as dots, with fifteen dots in each row, column and diagonal. Since 15 was a significant number in Chinese numerology (a year consists of 15 cycles of 24 days) the pattern was somehow used to control the flood.



4	9	2
3	5	7
8	1	6

Apart from reflections and rotations, the magic lo-shu is unique. This can be shown in a systematic way.

The sum of the numbers form 1 to 9 is 45, so the magic sum of each row, column and diagonal must be 15. Listing the ways of getting a sum of 15 from sets of three numbers

9+1+5 9+2+4 8+1+6 8+2+5 8+3+4 7+2+6 7+3+5 6+4+5

we see that there are exactly 8 possibilities, which is the number of rows, columns and diagonals of the square.

Since 5 occurs in four of these sums, it must be in the centre of the square, and since 8 occurs in three sums, it must be in a corner, with 2 in the opposite corner to make 15. The numbers 4 and 6 also appear in three sums, so must go into the other two corners. The rest of the numbers are then quickly put in place. □

NIM

Nim is an ancient Chinese game for two players, usually played with stone. The game starts with several piles of stones, not necessarily all containing the same number of stones. The players take turns in removing one or more stones from any *one* of the piles. (It is not permitted to remove stones from two different piles.) The winner is the player who takes the last stone.

The key to choosing the right move at any stage lies in thinking in binary. Three possible positions are shown. In position A, the piles consist of 18, 20 and 23 stones, in Position B the piles consist of 10, 21 and 30 stones and in Position C the piles consist of 18, 20 and 6 stones. Write the numbers in binary under each other.

A:		B:		C:	
18:	10010	10:	01010	18:	10010
20:	10100	21:	10101	20:	10100
23:	10111	30:	11110	6:	00110

The key to analysing a position is the number of 1's in each column.

In position A, the first and fifth columns contain an odd number of 1s. We call this position *odd*
Position B is also odd: the fifth column contains an *odd* number of 1s.
In position C, all columns have an even number of 1's. This is an *even* position.

There are now some key points to notice.

- A player can win only when faced with a position in which only one pile of stones is left: an odd position.
- The final winning move removes stones so that all three binary numbers are zero. This is an even position.
- A player cannot win when faced with an even position, for then there are stones left in at least two of the piles, and no move can clear all the piles.
- Any move from an even position leaves an odd position.
- If faced with an odd position, you can always remove stones to leave an even position. In example A above (an odd position), the first and

fifth columns have odd numbers of 1's. If you remove 17 stones (in binary, 10001) from the third pile you have the even position shown in Position C above.

The winning strategy in Nim is to ensure that your opponent always faces an even position, which has no winning move. Any move then leaves you facing an odd position, which you can convert to an even position. Carry on this way and eventually you will be faced with no stones left in two of the three piles. Taking the third pile, you win.

So to win at Nim you work out whether the original piles of stones are in an odd or even position. If odd, make sure you move first. If the starting position is even, courteously offer your opponent the first move. □.

SAMOA CHANGES SIDES

The Pacific island of Samoa axed a day from the end of last year when its government decided to move the country across the international date line. From being one of the last countries in the world to welcome the New Year, it is now among the first.

The reason was economic. When the international date line was drawn some 120 years ago, Samoa's trade was mainly with the United States and Europe, so it was advantageous to align the country's calendar with those of its trading partners. Today it is different. Samoa's important trading partners are Australia and New Zealand, and when those countries start a new working week they would find that their business partners in Samoa were still enjoying a Sunday break. And on Fridays in Samoa the Aussies and Kiwis could not be reached because they had started a weekend break.

Samoa therefore dropped Friday 30 December from its calendar, which is now the same as its ANZAC partners. There is still the time difference: Samoa is one hour ahead of New Zealand and three hours ahead of Sydney, Australia. But that's better than being twenty-something hours behind them.

This is the second time in recent years that Samoa has changed sides in a significant way. In 2009 Samoa switched from driving on the right to driving on the left, coming into line with driving practice in Australia and New Zealand. Another consideration was Japan, which drives on the left. Samoa is now able to import cheap second-hand cars from Japan. □

MEDICAL MATHS

Today there are many tests that are widely used to detect life-threatening diseases early. How effective are they? Should they be believed?

At a routine checkup, your doctor tells you that there is a simple and inexpensive blood test that can detect a rare but particularly nasty form of cancer. You agree to have the test done, and the doctor takes a blood sample and sends it off to the pathology laboratory.

Two days later the doctor calls to tell you that the test has come up positive. The good news is that the cancer can be cured since it has been caught at an early stage. The bad news is that the treatment, though effective, is very expensive and has a number of unpleasant side-effects.

Before agreeing to treatment you need to do a little bit of basic arithmetic.

The first thing to ask the doctor about is the accuracy of the cancer test when applied to people like you (of your age, gender, race, etc). The doctor tells you that when the test is administered on somebody like you with this form of cancer, it will detect the disease 99.9% of the time. That means that the test may miss the cancer in one out of a thousand patients.

On the other hand, the test can give a positive result when the patient does not have the cancer. This is very rare: it has been found to give a positive reading in 0.1% of all subjects who do not have the cancer.

That means the test registers a false positive result in one in a thousand cases.

So far, so good. The test sounds accurate. But there is one more important factor: the actual prevalence of this rare form of cancer, which is said to be 0.4%. That means that that two in every five hundred people have it.

The time has come to do a little arithmetic.

Suppose that the test is administered to a random sample of a million people like you. Of that group, 0.4% have the disease, which comes to $0.04 \times 1\ 000\ 000 = 4\ 000$ people.

But of the sample group of a million, 3 996 000 do not have the disease, and the test will register false positives on 0.1% of this group, i.e. $0.001 \times 3\ 990\ 000 = 3996$ people.

So when your test has come out positive, how do you know whether you are one of the 4000 who have the disease, or one of the 3994 who don't have the disease but have been incorrectly diagnosed?

You don't. All you can say is that the test suggests that you have a roughly 50-50 chance that you have the disease. □

SHOTGUN NUMBERS

Shotguns are classified by their *gauge*: 20-gauge, 16-gauge, 10-gauge etc. The lower the gauge, the bigger the gun.

While the *bore* of a shotgun barrel is its diameter, the gauge is defined as the number of solid lead spheres, with diameter equal to the internal diameter of the barrel, that would be required to make a pound of lead. So a 10-gauge shotgun has an barrel diameter equal to the diameter of a sphere made from one-tenth of a pound of lead. The bigger the bore, the smaller the gauge.

The table below shows the precise connection between these units, where the bore is given in inches.

Gauge	8	10	12	16	20	28
Bore	0.835	0.775	0.729	0.662	0.615	0.550

Hunters choose shotgun cartridges according to what they are hunting. Birdshot cartridges contain a large number of small pellets, while buckshot cartridges contain fewer, larger pellets. During manufacture, birdshot pellets are simply poured into the cartridges, but the larger buckshot pellets have to be packed in a specific geometric arrangement.

A spherical pellet, like the musket ball of old, is ballistically inefficient. As the shot leaves the barrel, it spreads into a circular pattern. The ideal pattern would be a circle with an even distribution, but in reality the distribution is Gaussian, with a higher concentration in the centre of the circle, tapering towards the circumference.

The end of a shotgun barrel can be fitted with a conical *choke*, designed to narrow the bore and reduce the spread of the pellets. □

VARSITY READINESS TEST

Are you ready for taking on university mathematics? The problems below are designed to test your skills in areas that are important for varsity maths, such as algebraic expertise, trigonometry skills, geometric insight, numerical facility and flexible but logical thinking.

Give yourself an hour or two to tackle the problems, preferably doing them in one continuous concentrated session. No calculators! No formula sheet! Then turn to page 31 to check your answers. There is no pass/fail mark. Every wrong answer, however, indicates a possible weakness in your mathematical background that needs to be fixed.

1. $(\sqrt{3}-\sqrt{2})^2(\sqrt{3}+\sqrt{2})^4$ is equal to
   (A) $5+2\sqrt{6}$ (B) $7-3\sqrt{6}$ (C) $4+3\sqrt{6}$ (D) $2\sqrt{3}+3\sqrt{2}$
   (E) $1+2\sqrt{2}+3\sqrt{3}$

2. In triangle ABC, points D and E lie on AB and AC, respectively, with DE parallel to BC. If $AD=p$, $DB=q$ and $AE=r$, then EC is equal to
   (A) $p+q-r$ (B) $\dfrac{pq}{r}$ (C) $q+r-p$ (D) $\dfrac{qr}{p}$ (E) $q+p-r$

3. $\sqrt{16^{4x^2}}$ is equal to
   (A) 4^{2x^2} (B) 4^{2x} (C) 16^{2x} (D) 16^{2x^2} (E) 16^{4x}

4. A bucket weighs F kg when full of water and H kg when it is half full. What is the weight of the bucket (in kilograms) when it is empty?
   (A) $F-H$ (B) $\frac{1}{2}(F+H)$ (C) $F-2H$ (D) $\frac{1}{2}(F-H)$ (E) $2H-F$

5. If $\tan 20^\circ = t$, then $\tan 50^\circ$ is equal to
   (A) $\dfrac{1-t^2}{2t}$ (B) $\dfrac{2-t^2}{t}$ (C) $\dfrac{1+2t^2}{2t}$ (D) $\dfrac{2+t^2}{t}$ (E) $\dfrac{1+t^2}{t}$

6. If $x+\dfrac{1}{x}=3$, the $x^2+\dfrac{1}{x^2}$ is equal to
   (A) 9 (B) 8 (C) 7 (D) $\dfrac{3\pm\sqrt{5}}{2}$ (E) $\dfrac{5\pm\sqrt{3}}{2}$

7. If $f(x)=\dfrac{x+1}{x-1}$ and $f(a)=5$, then $f(2a)$ is equal to
   (A) 2 (B) 4 (C) 6 (D) 8 (E) 10

8. $\dfrac{1}{a^{-1}+b^{-1}}$ is equal to

(A) $a+b$ (B) $\dfrac{1}{a+b}$ (C) $\dfrac{a+b}{ab}$ (D) $\dfrac{ab}{a+b}$ (E) $-a-b$

9. In quadrilateral $ABCD$, $AB = 4$, $BC = 3$, $CD = 2$ and $\angle ABC = \angle ACD = 90°$. Calculate $\tan \angle BAD$.

(A) $\dfrac{23}{14}$ (B) $\dfrac{21}{19}$ (C) $\dfrac{18}{13}$ (D) $\dfrac{25}{21}$ (E) $\dfrac{19}{16}$

10. $\sqrt{100a^4b^{16} + 36a^8b^{12}}$ is equal to

(A) $2a^2b^6\sqrt{9a^4 + 25b^4}$ (B) $10a^2b^8 + 6a^4b^6$ (C) $2a^4b^6(5a^2 + 3b^2)$

(D) $10a^2b^4 + 6a^4b^6$ (E) $2a^2b^6(3a^2 + 5b^2)$

11. The circumference of a circle is equal to the perimeter of a square. What is the ratio Area of square : Area of circle?

(A) $\pi/4$ (B) $2/\pi$ (C) 4 (D) $\pi/\sqrt{2}$ (E) $\sqrt{2}/\pi$

12. If $x = 2 + 3^t$ and $y = 2 + 3^{-t}$, then y expressed as a function of x is

(A) $\dfrac{3}{x-2}$ (B) $\dfrac{3x-5}{x-2}$ (C) $\dfrac{2x-3}{x-2}$ (D) $\dfrac{2x-1}{x-2}$ (E) $-x$

13. If $\tan x + \dfrac{1}{\cos x} = 2$, then $\tan x - \dfrac{1}{\cos x}$ equals

(A) -2 (B) 2 (C) $\frac{1}{2}$ (D) $-\frac{1}{2}$ (E) $\sqrt{2}$

14. $\log_3 9 \times \log_4 2$ is equal to

(A) $\log_7 18$ (B) $\log_{12} 7$ (C) $\log_7 11$ (D) $\log_{12} 18$ (E) $\log_7 7$

15. How many real solutions are there of the equation $2^x = x^2$?

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

16. In how many ways can 50 be expressed as the sum of two prime numbers?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

17. In triangle PQR, S is the midpoint of QR and $PS = QS = RS$. If $\angle PQS = x$ and $\angle PRS = y$, what is the relation between x and y?

(A) $x = y$ (B) $x = 2y$ (C) $2x = y$ (D) $2x + y = 180°$ (E) $x + y = 90°$

18. When the numbers below are arranged in increasing order, which number is in the middle?

(A) $5\sqrt{6}$ (B) $7\sqrt{3}$ (C) $5\sqrt{5}$ (D) $3\sqrt{7}$ (E) $8\sqrt{2}$

19. If $ABCD$ is a rhombus, with $\angle ABC = 46°$, then $\angle ACD$ is equal to

(A) 46° (B) 67° (C) 23° (D) 134° (E) 44°

20. Thabo buys p oranges. He squeezes $q\%$ of them to make fresh orange juice. How many oranges are left?

(A) $\frac{pq}{100}$ (B) $\frac{100p - pq}{100}$ (C) $\frac{100p - q}{100}$ (D) $\frac{p - q}{100}$ (E) $\frac{100q - pq}{100}$

21. If $p \circledast q$ means $\frac{1}{p+q}$, then $2 \circledast (3 \circledast 4)$ is equal to

(A) $\frac{1}{9}$ (B) 24 (C) $\frac{7}{15}$ (D) $\frac{15}{7}$ (E) $\frac{1}{24}$

22. A parabolic arch stands on level ground and has a span AB of 40 metres. The highest point (the centre of the arch) is 16 meters above the point M, the midpoint of AB. What is the height (in metres) of the arch above a point on AB 5 metres from M?

(A) 14.75 (B) 15 (C) 15.25 (D) 15.5 (D) 15.75

23. A vertical stick of height 84 cm casts a shadow 108 cm long. Next to it is another vertical stick of length 35 cm. How long is the shadow of the second stick?

(A) 59 cm (B) 55 cm (C) 54 cm (D) 48 cm (E) 45 cm

24. $\cos 25° \cos 35° + \cos 65° \cos 125°$ is equal to

(A) 1 (B) $\frac{1}{2}$ (C) $-\frac{1}{2}$ (D) -1 (E) $-\frac{\sqrt{3}}{2}$

25. If a and b are composite numbers such that $a + b = 31$, what is the biggest possible value of ab?

(A) 110 (B) 160 (C) 180 (D) 240 (E) 270

GEOMETRY QUOTE

Geometry enlightens the mind and sets one's mind straight. The mind that turns regularly to geometry is unlikely to fall into error.

(The Muqaddimah of Ibn Khaldun, 1377) □

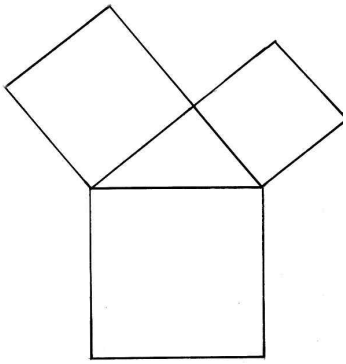
OVERPOWERED

"The earth is a six sextillion (6^{1021}) metric ton planet ... "

(Spotted by David Allison in the *Cape Times*, 19 March 2012) □

WHAT ARE CATHETI?

The longest side of a right-angled triangle is called the *hypotenuse*, where the Greek prefix *hypo* means "under". In old textbooks, the right-angled triangle was drawn with the hypotenuse horizontal: underneath the right angle.



The other sides of the right-angled triangle (the sides including the right angle) don't have a special name, although in days gone by they did: the were called the *catheti* (singular *cathetus*) derived from the Greek word for "letting down". Mathematically, a cathetus was defined as a straight line dropped perpendicularly onto another straight line. The word is not in the *Concise Oxford Dictionary* and is not in current mathematical use. Even professional mathematicians may never have heard the word.

The related word *catheter* is familiar today in its medical context: a catheter is a tube inserted (or "let down") into the body to extract or introduce fluid. □

IMO2014 IN CAPE TOWN

In *Mathematical Digest* **165** (October 2011) it was reported that South Africa's bid to host the International Mathematical Olympiad (IMO) in July 2014 had been successful.

The South African Mathematics Foundation has now announced that the 2014 IMO will be hosted by the University of Cape Town.

This will be the first time that the IMO has been held on the African continent. □

ANSWERS

Think of Two Numbers (p. 3):
Some simple algebra shows how the trick works

Suppose the secret numbers are m and n. Now follow the steps:

Multiply the first number by 5.	$5m$
Add 8.	$5m + 8$
Multiply your answer by 2.	$10m + 16$
Add the second number.	$10m + 16 + n$
Subtract 5.	$10m + n + 11$

When your friend gives the final number, simply subtract 11. The answer will be a three-digit number. The first two digits give the first secret number and the third digit is the second secret number.

The Monkey and the Coconuts (p. 13):
It seems clear that there is not one answer to this question. Since there are five sailors and six divisions, you can add $5^6 = 15\ 625$ coconuts to the pile at the beginning and the division process will still work out exactly. While it is seems that the smallest positive solution will be quite a large number, it turns out that there is an easy *negative* solution: -4 coconuts.

Think of it this way. Giving a coconut to the monkey could just as well be done before a division. So, starting with -4 coconuts, giving one to the monkey leaves a pile of -5 coconuts. A sailor taking his one-fifth share of -1 leaves a pile of -4 coconuts for the next division. In the last division, the monkey gets a coconut, and the pile of -5 coconuts is divided equally between the sailors (each getting -1).

So we have a solution in negative coconuts: -4. Increasing that number by $5^6 = 15625$ gives the smallest positive solution: 15 621 coconuts.

The Puzzling Scales (p. 15):
The top is balanced by 9 marbles.

Ben Trovato on Maths (p. 17):
The Russian mathematician was Grigorii Perelman, who in a sequence of papers published in 2002 and 2003 gave a proof of the Poincaré conjecture, first posed in 1904. The proof was confirmed correct in 2006 and Perelman was awarded a Fields Medal by the International Mathematical Union, which he refused to accept. The Poincaré Conjecture was one of the seven Millennium Mathematics Problems identified by the Clay Mathematics

Foundation of New York as the major unsolved problems of mathematics. They offered one million dollars for solutions. In 2010, having independently vetted the proof, the Clay Foundation pronounced it correct and offered Perelman their million-dollar prize. Perelman, by now a recluse in a St Petersburg apartment, turned it down and has since refused to have any further communication with the mathematical world.

Varsity Readiness Test: (p. 26):
ADDEA CADAA ACDED DEEBB CBEBC

SAFA TURNABOUT

The South African Football Association (SAFA) has made a 360-degree turnabout to change its policy to allow foreign players to play for local teams.

(*The Big Issue*, October-November 2011) □

LIGHT BULB RIDDLE

How many Heisenbergs does it take to change a light bulb?

If you knew that, you wouldn't know where the light bulb was. □

ERRATUM

Varsity Algebra Test (January 2012, p.25):
Question 10 should read:

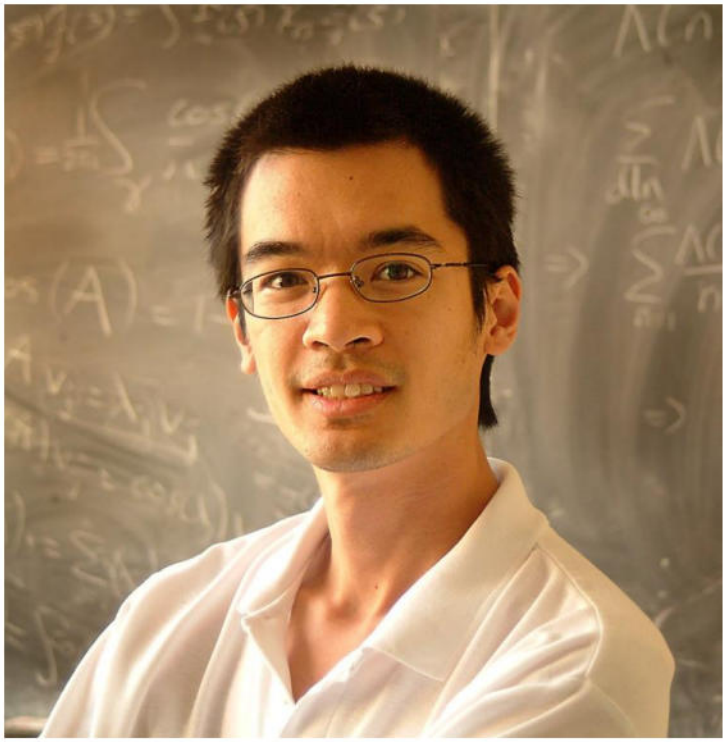
$(a+b)^{-1}(a^{-1}+b^{-1})$ is equal to

(1) $(a+b)^2$ $(a+b)^{-1}$ (3) $(ab)^{-1}$ (4) a^2+b^2 (5) ab

The error is regretted! □

mathematical digest

No. 168 July 2012



Terence Tao

mathematical digest

Number 168 **July 2012**

The popular image of the lone (and possibly slightly mad) genius — who ignores the literature and other conventional wisdom and manages by some inexplicable inspiration (enhanced, perhaps, with a liberal dash of suffering) to come up with a breathtakingly original solution to a problem that confounded all the experts – is a charming and romantic image, but also a wildly inaccurate one, at least in the world of modern mathematics. We do have spectacular, deep and remarkable results and insights in this subject, of course, but they are the hard-won and cumulative achievement of years, decades, or even centuries of steady work and progress of many good and great mathematicians; the advance from one stage of understanding to the next can be highly non-trivial, and sometimes rather unexpected, but still builds upon the foundation of earlier work rather than starting totally anew.

(From Terence Tao's blog on mathematical breakthroughs)

Department of Mathematics and Applied Mathematics
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Cover: Terence Tao was born in Adelaide, Australia in 1975. He was a mathematical prodigy, winning a Bronze Medal at the International Mathematical Olympiad in 1986 at the age of 11, and Silver and Gold Medals in the next two years. He earned his PhD at Princeton University at the age of 21. In 2004 he and Ben Green solved an old problem: they showed that it was possible to find arbitrarily long arithmetic progression of prime numbers (see MATHEMATICAL DIGEST **136**, July 2004). In 2006 Terence Tao was awarded a Fields Medal, the highest possible accolade for mathematical achievement. His latest achievement (see page 1) is making major progress in cracking a 250-year-old problem, the Goldbach Conjecture.

THE GOLDBACH CONJECTURE

The Goldbach Conjecture dates back to a discussion between two 18th Century mathematicians, Christian Goldbach and Leonard Euler. They noticed that even numbers larger than 2 always seemed to be expressible as the sum of two prime numbers ($4 = 2 + 2, 6 = 3 + 3, 8 = 5 + 3$, etc) and conjectured that this was always true. They were not able to prove the result, and nor has anybody else since then.

Goldbach also put forward a weaker form of the conjecture, namely that every odd number can be expressed as the sum of at most three prime numbers. If the original Goldbach Conjecture is true, then so is this "Weak Goldbach Conjecture", because if you are given an odd number, you can simply subtract 3 to get an even number, which is expressible as the sum of two primes. Putting the 3 back shows that the odd number is expressible as the sum of three primes.

Computers have been used to check the validity of both forms of the Goldbach Conjecture for numbers up to 19 digits long. No exceptions have been found, and as the numbers get larger and larger it seems that there are more and more ways of splitting them into the sum of two or three primes. So there is very good reason to believe that both conjectures are true. But a proof is still needed, and for a quarter of millennium it has eluded the best mathematical minds.

Even the Weak Goldbach Conjecture has proved intractable. Some progress was made in the 1930s, when it was proved that the result was true for "sufficiently large" odd numbers. In other words, there are only a finite number of odd numbers which are not expressible as the sum of three prime numbers. But nobody has been able to reduce that "finite number" to "zero".

Earlier this year the Australian-born mathematician Terence Tao, now at the University of California, Los Angeles, made a breakthrough, by proving that all odd numbers are expressible as the sum of at most five primes. He hopes that his proof can be beefed up to trim the number of primes down to three.

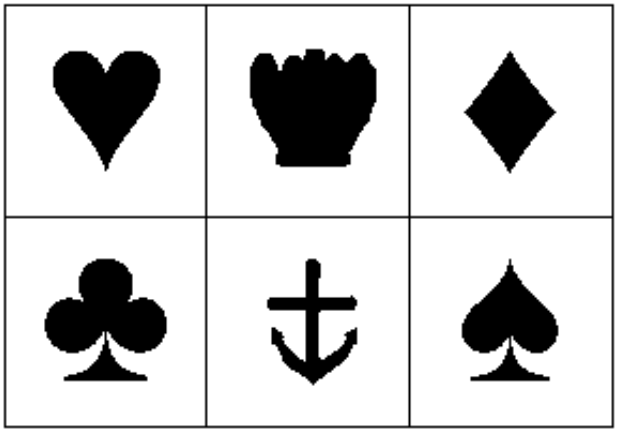
Once upon a time number theory was regarded as an academic pursuit with absolutely no applications. No longer. Terence Tao suggests that research on cracking the Goldbach Conjecture will lead to progress in important areas such as data encryption. $\square$

CROWN AND ANCHOR

Crown and Anchor is a dice game dating from the 18th century, when it was very popular among British seamen. It is still played today in fairgrounds and casinos.

Three special dice are used. Instead of six numbers, the faces show the six symbols: Crown, Anchor, Club, Diamond, Heart, Spade.

The game is played between a banker and one or more players. The players place bets on a mat showing the six symbols, and the three dice are thrown.



If there is a bet on any symbol which comes up on any of the dice, the banker pays the player the amount of the stake for each of the dice showing that symbol: even money if just one die shows that symbol, 2:1 or 3:1 if two or three dice show the symbol, respectively.

At first glance it looks like an attractive game for the player. If the bet is on the Crown, for example, there is a chance of one in six that a given die will come up with that symbol. Since there are three dice, that looks like an even chance of the Crown being shown on at least one of the dice. And since the payout is more than the original stake if more than one of the dice shows a Crown, it seems like the player has an advantage over the bank.

It is just that sort of naive (and incorrect) understanding of the basic laws of probability that makes the game attractive to players and a solid source

of income for the casino. The truth is different. In the long run, the banker wins, with a profit margin of about 8% on all bets cast.

To analyse the game, we need to calculate the probabilities that, on the throw of three dice, the Crown appears three times, twice, once, or not at all, and the expected winnings on a bet of one unit in each of these cases.

The Crown appears on all three dice with probability $(\frac{1}{6})^3 = \frac{1}{216}$. So the expected payout in this case is $3 \times \frac{1}{216} = \frac{3}{216}$.

The probability that the three dice show two Crowns and another symbol is $3 \times (\frac{1}{6})^2 \times (\frac{5}{6}) = \frac{15}{216}$. (The 3 is there because the other symbol could appear on any of the three dice.) In this case the expected payout is $2 \times \frac{15}{216} = \frac{30}{216}$.

The probability that the three dice show just one Crown and two other symbols is $3 \times (\frac{1}{6}) \times (\frac{5}{6})^2 = \frac{75}{216}$. (Once again, the 3 is there because the Crown could appear on any one of the three dice.) In this case the expected payout is $1 \times \frac{75}{216} = \frac{75}{216}$.

The probability that the three dice show no Crowns is $(\frac{5}{6})^3 = \frac{125}{216}$. In this case you lose your bet: the expected payout is $(-1) \times \frac{125}{216}$.

So on a bet of one unit that a Crown will appear the expected payout is

$$\frac{3}{216} + \frac{30}{216} + \frac{75}{216} - \frac{125}{216} = -\frac{17}{216}.$$

Since $\frac{17}{216} = 0.078$ (to three decimal places), on average you lose 7.8% of your bet every time you play.

Crown and Anchor is a game for losers. Unless you are the banker. □

FABIAN MATHEMATICS

Sidney Webb was a leading politician of the early 20th Century. He was at the centre of a group of socialist reformers (another was George Bernard Shaw) who believed in "evolutionary socialism": implementing reforms gradually. The called themselves Fabians, after the Roman general Quintus Fabius Maximus, renowned for his slow-but-sure military tactics.

Sidney Webb summed up the Fabian philosophy by quoting a simple piece of mathematics:

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = 2$$

"The inevitability of gradualness cannot fail to be appreciated." □

FOUR SQUARES AND NINE CUBES

Some numbers are squares: 1, 4, 9, 16, etc.

Some numbers can be expressed as the sum of at most two squares. They are the numbers above, and also

$$2(=1^2+1^2), 5(=1^2+2^2), 8(=2^2+2^2), 10(=1^2+3^2), 13(=2^2+3^2)\ldots.$$

If we are allowed to use three squares, then more of the gaps can be filled:

$$3(=1^2+1^2+1^2), 6(=1^2+1^2+2^2), 11(=1^2+1^2+3^2), \ldots.$$

With four squares, more gaps can be filled:

$$7(=1^2+1^2+1^2+3^2), 15(=1^2+1^2+2^2+3^2)\ldots.$$

In fact, all the gaps can be filled, a result stated and proved by Joseph Louis Lagrange, a French/Italian mathematician, in 1770.

Theorem: Every positive integer can be expressed as the sum of at most four squares.

The proof is too complicated for inclusion here.

In the same year that Lagrange established his four-square theorem, a Cambrige mathematician Edward Waring (who was also a practising doctor) stated that any number could be expressed as the sum of at most nine cubes. His conjecture was based on a study of specific examples and a search for patterns, and he provided no proof. Only in 1912 was Waring's conjecture proved to be true.

Most numbers require fewer than nine cubes: 23 is the first that needs the full complement: $23 = 7\times 1^3 + 2\times 2^3$. From there on fewer than nine cubes are needed, until you get to 239.

How can you express numbers as the sum of fourth powers, or fifth, or higher powers? It was only in 1986 that it was finally established that you need at most 19 fourth powers to make up any natural number. Not long before that, in 1964, the case of fifth powers had been wrapped up: you need 37 of them to make up any natural number.

Going back to 1906, the German mathematician David Hilbert proved that for every positive integer k there is a number, denoted by $g(k)$, such that any positive integer can be expressed as a sum of at most $g(k)$ kth powers. The surprising feature of this result is that the proof was not constructive: Hilbert established the existence of the number $g(k)$ for every k, but without showing how to find the number in any specific case.

Investigating sums of powers is today still a very active field of research in number theory. □

COPYRIGHTING PI

Last year a music video was released on YouTube, featuring the song *What Pi Sounds Like*. The composer, Michael Blake, had taken the first few score digits of pi and translated them into notes.

This brought a protest and a law suit from Lars Erikson, who claimed that the music was plagiarized from his 1992 piece *Pi Symphony*. Erikson had registered his music with the US copyright office.

The case came to court on March this year. In dismissing the complaint, Judge Michael H. Simon said: "Pi is a non-copyrightable fact, and the transcription of pi to music is a non-copyrightable idea.

It was significant that the case was heard on Pi Day: in American date notation March 14 is 3.14, the first three digits of pi. □

SCHOLARSHIP PROBLEM

A university was selecting a student for a sports scholarship, and the maths professor and the rugby coach were given the job of interviewing one of the candidates.

The maths prof asked the first question: "What is 7 times 8?"

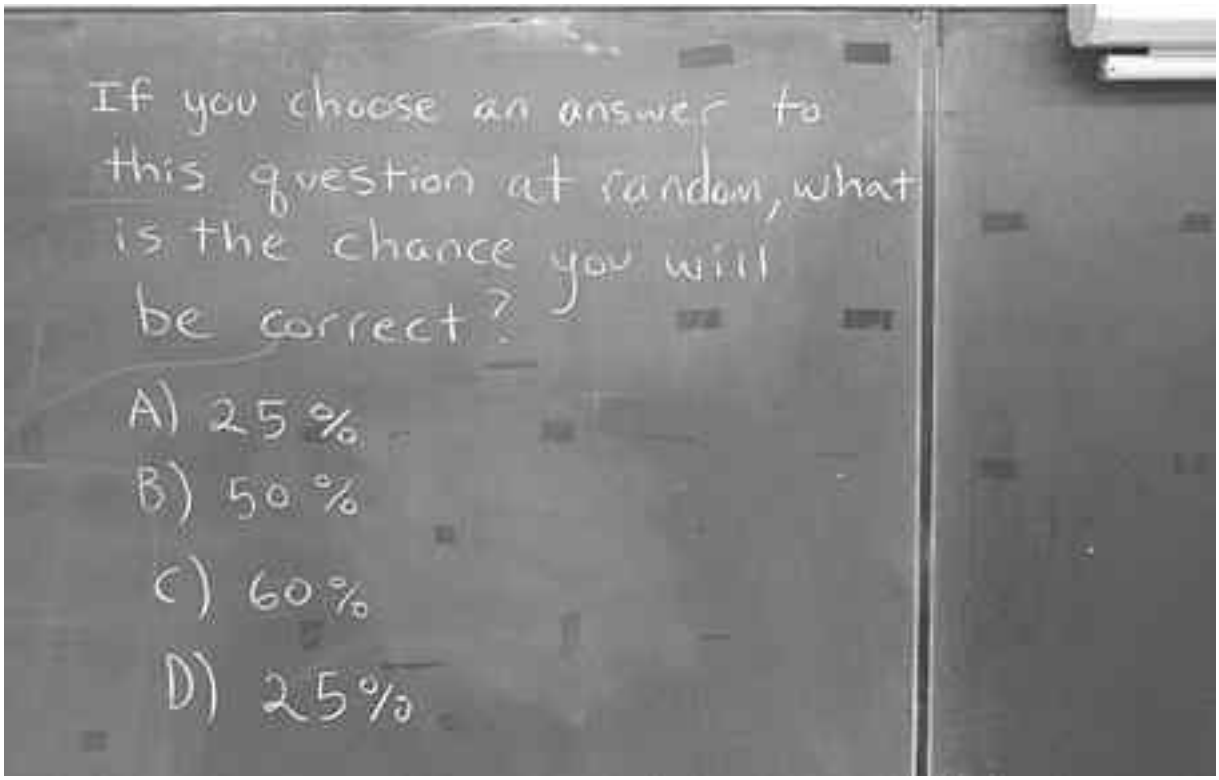
After a long and awkward pause, the student answered "Fifty-six."

The coach immediately jumped up and shouted "Aw, come on Prof! Give him another chance." □

NONE OF THE ABOVE

Ivan B. Holdsworth

In November 2011 Raymond Johnson posted a problem on his blackboard.



This trick question (reportedly devised by Ryan Grover) inspired the following variations.

What is the chance of randomly guessing the correct answer to this question?

(A) 20% (B) 35% (C) 40% (D) 60% (E) none of the above

The correct answer is A, and the chance of randomly selecting any answer is 20%, as they are unique, and the chance of randomly selecting any answer is 20%.

Another version is

What is the chance of randomly guessing the correct answer to this question?

(A) 40% (B) 20% (C) 20% (D) 40% (E) none of the above

The correct answer is A or D, as the chance of randomly selecting 40% (or 20%) is 40%, and the chance of selecting 'none of the above' is 20%, but the chance of selecting 20% is 40%.

Another much more challenging one is

What is the chance of randomly guessing the correct answer to this question?

(A) 20% (B) 20% (C) 20% (D) 60% (E) none of the above

None of the possible answers is correct.
If the answer to the question is 'none of the above', the chance of randomly selecting it is 20%.
If the answer to the question is 20% then the chance of randomly selecting 20% is 60%.
If the answer to the question is 60% the chance of randomly selecting 60% is 20%.
Thus a circular reference situation exists, and therefore none of the answers is correct. □

PYTHAGOREAN NUMBERS

A *Pythagorean triple* is any triplet of whole numbers a, b, c (with $a < b < c$) such that $a^2 + b^2 = c^2$.

Examples are $(3, 4, 5), (5, 12, 13), (7, 24, 25), (8, 15, 17), (9, 40, 41) \ldots$, and many more. These are called *primitive* Pythagorean triples: the three numbers have no common factor. We will use the abbreviation PPT.

The above list shows that 3, 4, 5, 7, 8, 9 all occur in at least one PPT, while 1, 2, 6 and 10 do not.

We also note that 5 appears in two different PPTs: (3, 4, 5) and (5, 12, 13), as does 13, in (5, 12, 13) and (13, 84, 85).

Questions to think about:

1. Find an even number that occurs in three different PPTs.
2. Find an odd number that occurs in three different PPTs.
3. Which even numbers do not appear in any PPT?
4. Find five prime numbers that each occur in two different PPTs. □

ISAAC NEWTON AND THE GUINEA

When currency was based on gold and silver coins, the golden *guinea* and the *pound sovereign* were common English coins. The pound was set at 20 shillings, but variations in the price of gold in the 17th century caused the guinea to fluctuate in value, at one time going up as high as 30 shillings.

This was just one of many coinage problems England was experiencing at the time. Clipping and counterfeiting were rife, and the English government appointed the great mathematician and physicist, Isaac Newton, to the position of Master of the Mint, to sort out the currency.

Newton carried out the job with energy and great success. One of his achievements was stabilising the guinea, which he fixed at 21 shillings.

A William and Mary Guinea
Minted under Isaac Newton

Newton's guinea remained a standard coin until 1816, when the gold sovereign was set as the unit in its place.

Though the guinea coin was no longer minted, it remained a unit of account. The name had an aristocratic ring, and prices for land, horses, artworks, Saville Row tailoring and Harley street medical fees were often quoted in guineas. It was traditional for auctioneers to sell an item in guineas but pay the seller in sovereigns, the odd shilling being the auctioneer's 5% commission. This practice continued even after Britain decimalised its currency in 1971.

Many major horse races in Britain, Ireland, Canada and Australia still have names such as "The 1000 Guineas" or "The 2000 Guineas", though the actual value of the prize money is today far higher.

The name of the coin came from the source of its gold content, the Gold Coast of colonial times, now Ghana. Arab traders picked up the term, and the Arabic name for today's Egyptian pound is *el-gineih*. □

RECURRING DECIMALS

Every fraction can be expressed as a recurring decimal. The six proper fractions with denominator 7 have decimal representations with a period of length 6, with an interesting pattern: the same six digits (142857) occur in each of the fractions, in all six possible cyclic rearrangements.

$$\frac{1}{7} = 0.\dot{1}4285\dot{7} \qquad \frac{2}{7} = 0.\dot{2}8571\dot{4} \qquad \frac{3}{7} = 0.\dot{4}2857\dot{1}$$

$$\frac{4}{7} = 0.\dot{5}7142\dot{8} \qquad \frac{5}{7} = 0.\dot{7}1428\dot{5} \qquad \frac{6}{7} = 0.\dot{8}5714\dot{2}$$

Something similar occurs with the decimal expansions of the proper fractions with denominator 13, which also have recurring patterns of six digits.

$$\frac{1}{13} = 0.\dot{0}7692\dot{3} \qquad \frac{2}{13} = 0.\dot{1}5384\dot{6} \qquad \frac{3}{13} = 0.\dot{2}3076\dot{9} \qquad \frac{4}{13} = 0.\dot{3}0769\dot{2}$$

$$\frac{5}{13} = 0.\dot{3}8461\dot{5} \qquad \frac{6}{13} = 0.\dot{4}6153\dot{8} \qquad \frac{7}{13} = 0.\dot{5}3846\dot{1} \qquad \frac{8}{13} = 0.\dot{6}1538\dot{4}$$

$$\frac{9}{13} = 0.\dot{6}9230\dot{7} \qquad \frac{10}{13} = 0.\dot{7}6923\dot{0} \qquad \frac{11}{13} = 0.\dot{8}4615\dot{3} \qquad \frac{12}{13} = 0.\dot{9}2307\dot{6}$$

This time there are two separate cyclical patterns of 6 digits: 076923 and 153846.

The numbers 7 and 13 are the only two prime numbers whose reciprocals have decimal representations with recurrence cycles of length 6. This can be easily proved.

If the decimal expansion is $0.abcdefabcdefabcdef...$, then

$$\frac{1}{p} = \frac{abcdef}{999999}.$$

This gives

$$10^6 - 1 = pk$$

where k is the six-digit number $abcdef$. So p is a divisor of

$$10^6 - 1 = (10^3 - 1)(10^3 + 1) = 999 \times 1001 = 3^3 \times 37 \times 7 \times 11 \times 13.$$

Of the prime numbers in this factorization,

$\frac{1}{3} = 0.\dot{3}$ has period 1, $\frac{1}{11} = 0.\dot{0}\dot{9}$ has period 2 and $\frac{1}{37} = 0.\dot{0}2\dot{7}$ has period 3

So only $\frac{1}{7}$ and $\frac{1}{13}$ have period 6. $\square$

UCT MATHEMATICS COMPETITION

The annual University of Cape Town Mathematics Competition took place on the UCT campus on 16 April this year, attracting 7251 participants from 150 Western Cape high schools. Each school could enter up to five individuals and five pairs, in each grade (8 to 12).

The question papers were set by a team of local teachers and staff of the UCT Department of Mathematics and Applied Mathematics. Each paper consisted of 30 questions, ranging from rather easy to quite difficult.

Gold Awards and Casio calculators and watches were awarded to the top ten individuals and top three pairs in each grade.

Grade 8: Individuals

1	Sanjiv Ranchod	Westerford High School
2	Matthew Booth	Rondebosch Boys' High School
2	Andrew McGregor	Rondebosch Boys' High School
2	Yaseen Mowzer	Fairbairn College
5	Mu-aath Upadhey	Westerford High School
6	Benna Marais	Paarl Gimnasium
7	Andrew Woolard	S A College High School
8	Nakiesha van Wyk	Spes Bona Secondary School
9	Bronson Rudner	S A College High School
10	Rebecca Stead	El Shaddai Christian School

Grade 8: Pairs

1	Jonathan Botha / Woo Duk Jin	Fairmont High School
2	James Sloane / Geonhee Wang	Brackenfell High School
3	Sumin Kim / Eunji Lee	Parel Vallei High School

Grade 9: Individuals

1	Soo-Min Lee	Bishops
2	Tae Jun Park	Rondebosch Boys' High School
3	Mark Doyle	Parel Vallei High School
4	Michael Thomson	Bishops
5	David Brooke	Rondebosch Boys' High School
6	Jonti Oehley	Bishops
7	Amy Chung	Herschel High School
8	Shandukani Mulaudzi	Herschel High School

9	Cameron Naidoo	Bergvliet High School
10	Christopher Hooper	Rondebosch Boys' High School

Grade 9: Pairs

1	Christian Cotchobos / Sam Jeffery	Bishops
2	Georg Janowski / Art van der Lingen	Constantia Waldorf
3	Matthew Blackbeard / Julian Loggenberg	Elkanah House

Grade 10: Individuals

1	Robin Visser	St George's Grammar School
2	Christopher Kim	Reddam House College
3	Warren Black	Bishops
3	Murray McKechnie	Bishops
5	Jonathan Alp	Parklands College
5	Ryan Broodryk	Westerford High School
7	Matthew Poulter	Rondebosch Boys' High School
7	Michelle van der Merwe	Herschel High School
9	Sungkuk Lee	Milnerton High School
9	Reuben Steenekamp	Reddam House Constantia

Grade 10: Pairs

1	Liam Byren / Ahmad Dutton	Rondebosch Boys' High School
2	Youn Suk Choi / Thomas Prins	Cannons Creek High School
2	Ye-eun Jung / Rachel Suttle	Wynberg Girls' High School

Grade 11: Individuals

1	Guy Paterson-Jones	Bishops
2	Lauren Denny	Rustenburg Girls' High School
3	Annemiek Meyer	Reddam House Constantia
4	Gerben Draaijer	Paul Roos Gimnasium
5	Musa Omar	Livingstone High School
6	Lauren Duckitt	Rustenburg Girls' High School
6	Altus van Bosch	Bellville High School
8	Martin Batěk	Reddam House Constantia
9	Sevi Steingaszner	Bishops
10	Dylan Carstens	Hoërskool Stellenberg
10	Dong-Eup Nam	Bishops

Grade 11: Pairs

1	Jaydon Farao / Ziyaad Adam	Bishops
2	Daniel Chung / Adam Smith	Bishops
3	Zack Bashala / Ju-an Delport	Fish Hoek High School

Grade 12: Individuals

1	Robert Spencer	Westerford High School
2	Mickey Chew	Elkanah House
3	Matthew Coulson	Reddam House Constantia
4	Hekkie Breytenbach	Paarl Gimnasium
5	Timothy Meier	Westerford High School
6	Jeong Tae Kim	S A College High School
6	Paul le Roux	Parel Vallei High School
8	Ihsaan Bassier	Rondebosoch Boys' High School
9	Johannes Schoeman	Parel Vallei High School
10	Khadija Brey	Wynberg Girls' High School
10	Seshadari Moodley	Heritage College

Grade 12: Pairs

1	Nimombe Namtchougli / Claude Piessou	Abbotts College Claremont
2	Shinyoung Kim / Ha Som Suh	Cannons Creek High School
3	Ingrid Kok / Hendrik van Schalkwyk	Jan van Riebeeck High School

School Awards

Every school that entered at least ten individuals, but did not win a Gold Award, qualified for a School Prize. The prize (a book donated by Oxford University Press) was awarded to the individual in that school who had achieved the highest ranking in any grade in the school.

Al-Azhar High School	Nadeema Smith (10)
Atlantis Sekondêre Skool	Teneal Bosman (10)
Bridge House	Nina Buys (8)
Camps Bay High School	Cullam Geyser (10)
Cape Academy of Maths & Science	Zakariyah Toyer (10)
CBC St John's Parklands	Seok-Young Jeong (11)
Chesterhouse	Richard Curran (10)
Claremont High School	Tala Ross (9)
Cosat	Odwa Madikane (12)
Cravenby Secondary School	Chanté Gordon (10)
Curro Private School	Tiaan Smit (8)
D F Malan High School	Tonia Schoeman (10)
Darul Arqam Islamic High School	Omar Larney (9)
Darul Islam High School	Saulegh Samaai (11)
De Kuilen High School	Janneman Gericke (12)
Diazville Sekondêre Skool	Jamy Stofberg (9)

Drostdy H T S	Jaco du Plessis (10)
Durbanville High School	Christel Kruger (9)
Esangweni Senior Secondary School	Anathi Mangqwengqwe (12)
French School	Jules van der Walt (8)
Gardens Commercial High School	Wardiah Mouden (10)
Garlandale High School	Treverson Dlamini (11)
Grassy Park High School	Luqmaan Daniels (9)
Groote Schuur High School	Kayla Harker (8)
Harold Cressy High School	Yahyaa Awaldien (8)
Herzlia Middle School	Olvia Bernstein (9)
Hoër Meisieskool Bloemhof	Lize Draaijer (12)
Hoërskool Bellville	Altus van Bosch (10)
Hoërskool Durbanville	Christa Wessels (11)
Hoërskool Hopefield	Ruan Brand (8)
Hoërskool President	Lelanie Hoffmann (11)
Hoërskool Stellenbosch	James-John Matthee (11)
Hoërskool Strand	Carl Theunissen (10)
Hoërskool Swartland	Tienie Bester (8)
Hottentots Holland High School	Jae-Seong Mun (12)
Hout Bay High School	Ncedo Jako (12)
Hugenote Hoërskool	Annalise Heydenrych (9)
Iqhyiya Senior Secondary School	Zimkhitha Twala (9)
Islamia Boys' College	Abdul-Mueen Gazgay (8)
Islamia Girls' College	Munira Hoosain (11)
J G Meiring	Jamie Ceasar (9)
Khanyolwethu Senior Secondary School	Yamkela Matolengwe (10)
Kleinvlei High School	Riaan Vermeulen (9)
Leap Science & Maths No.1	Onela Makabane (12)
Leap Science & Maths No.2	Andile Simanga (12)
Lentegeur Senior Secondary School	Malcolm van der Pos (10)
Luhlaza High School	Zimie Sigenu (10)
Malibu High School	Seth Strydom (8)
Manyano High School	Sanele Mangena (10)
Mondale Senior Secondary School	Ethan Buckinjohn (10)
Muizenberg High School	Henry Ker (10)
New Orleans S S School	Sharfa Abrahams (10)
Norman Henshilwood High School	Haneef Ebrahim (9)
Oval North Senior Secondary School	Michaela Lee Wyngaard (10)
Paarl Boys' High School	Graham Hockey (9)

Paarl Girls' High School	Marina Smuts (9)
Pinelands High School	Clinton Cloete (12)
Plumstead High School	Mikaeel Hendriks (9)
Reddam House Atlantic Seaboard	Lisa Zou (12)
Rhenish Girls' High School	Hyerim Kim (12)
Rhodes High School	Nikita Adams (8)
Rocklands Secondary School	Cameron Kleinsmith (8)
Sans Souci Girls' High School	Kirsten Kannemeyer (11)
Sarepta Sekondêre Skool	Shola Domingo (9)
Sea Point High School	Siya Kali (9)
Settlers High School	Ishrat Undre (10)
Simon's Town High School	Nosipho Manyase (10)
Somerset College	Adam Henman (10)
Sophumelela High School	Ndiphiwe Ncwadi (12)
South Peninsula High School	Ali Waggie (11)
Spine Road High School	Zaahid Ismail (10)
Steenberg High School	Jacog Wegener (12)
Strandfontein Secondary School	Ancois Jardine (9)
Table View High School	Marc Beauchamp (8)
The Oracle Academy	Abdul-Basiet Firfirey (10)
Weston High School	Jowellyn Afrikaner (12)
Westridge High School	Sharne Sandlana (8)
Windsor High School	Ammaarah Kamaldien (9)
Wittebome High School	Razaan Bester (12)
Woodlands High School	Shingai Gunha (12)
Worcester Gymnasium	Zeyu Sun (12)
Wynberg Boys' High School	Jason Bright (10)

Trophies

The Diane Tucker Trophy, awarded to the top girl in the competition, was won by Lauren Denny (Rustenburg Girls'High School).

The UCT Trophy, awarded for the best performance by a school which had competed at most three times in the competition, was won by Claremont High School.

The Moolla Trophy, for a notable performance by a school from a disadvantaged community, was awarded to Manyano High School.

The Mona Leeuwenburg Trophy, for the best overall performance, was awarded to Bishops.

School ranking

A school score is obtained by adding the top 30 scores (individual or pairs) in each school. By this measure the top 20 schools were:

1 Bishops
2 Rondebosch Boys' High School
3 Parel Vallei High School
4 Westerford High School
5 Reddam College Constantia
6 Rustenburg Girls' High School
7 Herschel High School
8 S A College High School
9 Stellenberg High School
10 Wynberg Boys' High School
11 Paul Roos Gymnasium
12 Elkanah House
13 Paarl Boys' High School
14 Paarl Gimnasium
15 Bridge House School
16 Wyjnberg Girls' High School
17 Herzlia
18 El Shaddai Christian School
19 Bergvliet High School
20 Pinelands High School

A free set of competition papers, with answers, may be obtained by sending a stamped self-addressed B5 envelope (220 mm by 160 mm) to

The Director, UCT Mathematics Competition
Department of Mathematics and Applied Mathematics
University of Cape Town
7700 RONDEBOSCH

Books of question papers of previous years, with full solutions, statistics and lists of award winners, are also available — see the order form at the back of the magazine. □

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UCT MATHS CHALLENGE AND OLYMPIAD

For a number of years the top performers in the UCT Mathematics Competition have been invited to take part in a second round, the UCT Mathematics Challenge. This year the Challenge took place on Saturday 19 May, and a total of 136 wrote the paper: 60 Juniors (grades 8 and 9) and 76 Seniors (grades 10, 11 and 12).

The Challenge papers (page 17) each consisted of ten questions to be completed in two hours. Only the answers (usually numerical) were required.

The top students in each division were invited to come back the next week to write the UCT Mathematics Olympiad. It was again a two hour paper, but this time the problems required full solutions. The question papers may be found on page 20.

In the Junior division, the top five were

1. Soo-min Lee (grade 9, Bishops)
2. Tae Jun Park (grade 9, Rondebosch Boys' High School)
3. Alec de Wet (grade 9, Paarl boys' High School)
3. Michael Thomson (grade 9, Bishops)
5. Yaseen Mowser (grade 8, Fairbairn College)
5. Andrew McGregor (grade 8, Rondebosch Boys' High School

In the Senior division, the top five were

1. Robert Spencer (grade 12, Westerford High School)
2. Robin Visser (grade 10, St George's Grammar School)
3. Khadija Brey (grade 12, Wynberg Girls' High School)
4. Lauren Denny (grade 11, Rusteburg Girls' High School)
5. Mickey Chew (grade 12, Elkanah House)

□

10 TYPES OF PEOPLE

There are 10 types of people.

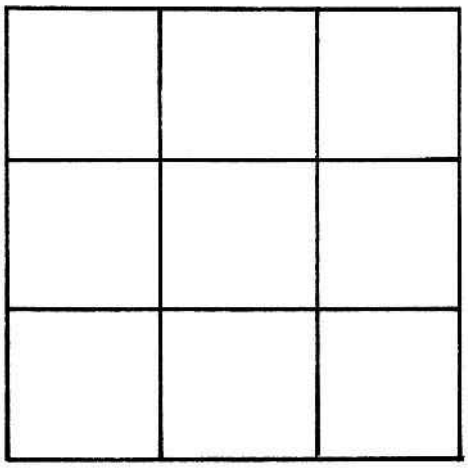
Those who understand ternary numbers, those who don't and those who expected this to be a joke about binary numbers □

UCT MATHEMATICS CHALLENGE

Junior paper

1. What are the last two digits of 3^{2012}?
2. Between 12:00 and 13:00 the hands of a clock are at right angles exactly twice. How many minutes elapse between these times?
   Give your answer in minutes, correct to two decimal places.
3. How many three-digit numbers can be formed which do not contain any zeros, and do not contain consecutive 7's?
4. What is the largest positive even number that cannot be expressed as the sum of two odd composite positive integers?
5. Find the sum of all positive integers n such that $\dfrac{7n-50}{n-8}$ is an integer.
6. If A, B, C and D stand for the digits 6, 7, 8 and 9, but not necessarily in that order, what is the largest possible value of $BAD+CAD+DAB$? (Here XYZ stands for a three-digit number with digits X, Y and Z.)
7. How many three-digit numbers have twice as many even factors as odd factors?
   (The *factors* of 24 (for example) are 1, 2, 3, 4, 6, 8, 12, and 24.)
8. Put the nine fractions below in the 3×3 grid shown so that every row, column and diagonal has the same total.

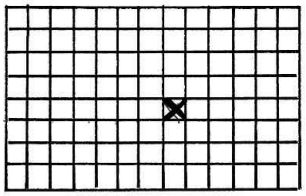
$$\frac{1}{6}, \quad \frac{1}{4}, \quad \frac{1}{3}, \quad \frac{5}{12}, \quad \frac{1}{2}, \quad \frac{7}{12}, \quad \frac{2}{3}, \quad \frac{3}{4}, \quad \frac{5}{6}$$



9. A group of 35 teenagers were asked to list their favourite takeaways. All of them named at least one. Some gave two favourites, but none named three. It was found that:
24 liked pizza
16 liked fish and chips
10 liked burgers.
Everybody who liked burgers also liked pizzas, and 9 of the pizza fans didn't like either burgers or fish and chips.

   How many liked only fish and chips?

10. Find the number of rectangles, of all sizes, in the diagram that do not contain the letter X.



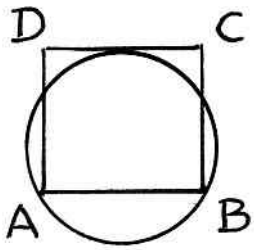
Senior paper

1. What are the last two digits of 3^{2012}?

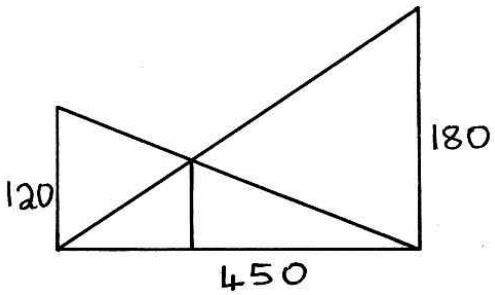
2. Find the sum of all positive integers n such that $\dfrac{7n-50}{n-8}$ is an integer.

3. If $xy = 2$, $yz = 3$ and $zx = 5$, what is the value of $x^2 + y^2 + z^2$? Give your answer as a fraction in lowest possible terms.

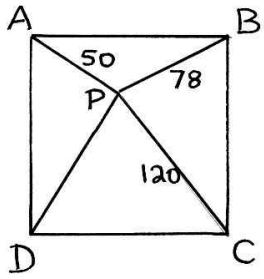
4. A circle of radius 15 goes through the vertices A and B of the square $ABCD$, and is tangent to the side CD.
What is the area of the square?



5. Two vertical poles of heights 120 cm and 180 cm stand on level ground 450 cm apart. Two ropes join the top of each pole to the bottom of the other pole.
At what height above the ground (in cm) do the two ropes cross?

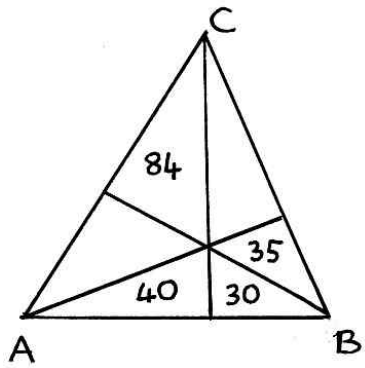


6. A point P lies inside a square such that $AP = 50$, $BP = 78$ and $CP = 120$. What is the length of DP?



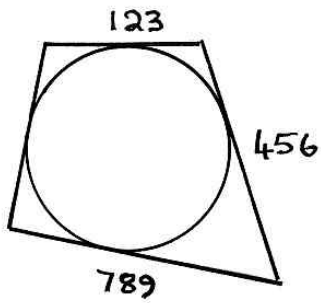
7. The numbers 1447, 1005 and 1231 are all four-digit numbers beginning with 1 that have exactly two identical digits. How many such numbers are there?

8. In the figure shown, triangle ABC is divided into six smaller triangles by lines drawn from the three vertices through a common point. The area of four of these triangles are as indicated. Find the area of triangle ABC.



9. If m and n are prime numbers such that $253m+114n = 2013$, determine $m^2 + n^2$.

10. A circle is inscribed in a quadrilateral with sides 123, 456 and 789. Find the length of the fourth side of the quadrilateral.



Answers are on p.30. □

READER'S PROBLEM

Submitted by Dr Sudan Hansraj (University of KwaZulu-Natal):

Prove that

$$\frac{1}{3} + \frac{2}{3}\cdot\frac{1}{5} + \frac{2}{3}\cdot\frac{4}{5}\cdot\frac{1}{7} + \frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\cdot\frac{1}{9} + \cdots + \frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\cdots\frac{98}{99}\cdot\frac{1}{101} < 1$$

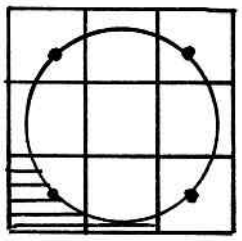
It looks complicated, but there is an amazingly simple proof. Can you find it, before turning to page 30? □

UCT MATHEMATICS OLYMPIAD 2012

Junior paper

1. At dinner on a camping expedition, each tin of soup was shared between 2 campers, each tin of meatballs was shared between 3 campers and each tin of chocolate pudding was shared between 4 campers. Each camper had all three courses and all tins were emptied. The camp leader opened 156 tins in total.
   How many campers were on the expedition?

2. The diagram shows a 3×3 square divided into nine 1×1 squares, and a circle. The circle passes through the centres of the four corner squares.
   What is the area of the shaded region (inside two adjoining squares but outside the circle)?

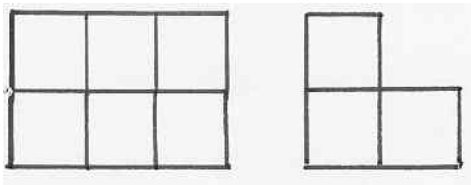


3. You have 1001 coins, of which 500 are counterfeit. A counterfeit coin differs in weight from a true coin by just one gram (either heavier or lighter). You are given one of the coins, and are asked to determine whether it is genuine or counterfeit. You have a pan balance (but no weights) which you can use to weigh one group of coins against another. The balance has a pointer which will show the difference between the weights of the objects in the two pans.
   What is the smallest number of weighings you need to carry out to determine whether or not the coin given to you is counterfeit?

4. Show how to express 2012 as a sum of positive integers so that the product of these integers is as large as possible.

5. Find all positive integers m and n such that $5m - 7n = 1$.

6. Prove that a 9×9 square cannot be tiled with 13 2×3 rectangles and one L-shaped piece consisting of three unit squares.



Senior paper

1. You have 1001 coins, of which 500 are counterfeit. A counterfeit coin differs in weight from a true coin by just one gram (either heavier or lighter). You are given one of the coins, and are asked to determine whether it is genuine or counterfeit. You have a pan balance (but no weights) which you can use to weigh one group of coins against another. The balance has a pointer which will show the difference between the weights of the objects in the two pans.
   What is the smallest number of weighings you need to carry out to determine whether or not the coin is counterfeit?

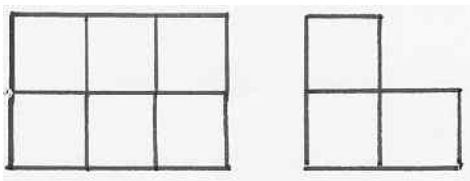
2. What is the largest number of regions into which the plane can be divided by n straight lines?

3. Show how to express 2012 as a sum of positive integers

$$a_1 + a_2 + a_3 + \cdots + a_n$$

   so that the product $a_1 a_2 a_3 \cdots a_n$ is as large as possible.

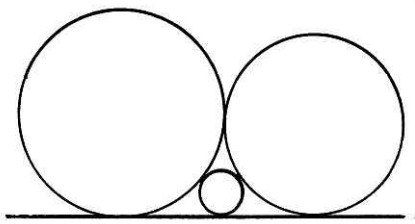
4. Prove that a 9×9 square cannot be tiled with 13 2×3 rectangles and one L-shaped piece consisting of three unit squares.



5. The numbers 1, 2, 3, ..., 100 are to be divided into disjoint subsets, such that for any two numbers in the same subset, neither is a multiple of the other. What is the smallest number of subsets needed?

6. Three circles, of radii $a > b > c$, are tangent to each other and to a common line as shown. Prove that

$$\frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}} = \frac{1}{\sqrt{c}}.$$



UCT MATHEMATICS OLYMPIAD SOLUTIONS

Junior paper

1. Each camper had half a tin of soup, one third of a tin of meatballs and a quarter of a tin of chocolate pudding. This is a total of $\frac{1}{2}+\frac{1}{3}+\frac{1}{4}=\frac{13}{12}$ tins per person. If there were N campers, they used $\frac{13}{12}N$ tins of food. So $\frac{13}{12}N = 156$, giving $N = 144$.

2. The square has area 9. The circle has radius $\sqrt{2}$ and area 2π. The four areas round the circle have the same area, namely $(9-2\pi)/4$.

   The first two problems featured in the 2011 UK Mathematical Challenge.

3. Just one weighing is enough.
   Set the specified coin aside. Divide the remaining 1000 coins into two equal piles and weigh them against each other on the pan balance. If the pans balance, or differ by an even number of grams, then the 1000 coins contain an even number of counterfeit coins, and the coin you have been given is genuine.

   If the pans differ by an odd number of grams, they contain an odd number of counterfeit coins, and the coin you have been given is counterfeit.

4. Suppose $a_1 + a_2 + \cdots + a_n = 2012$, where $a_1, a_2, \ldots a_n$ are positive integers.
   If $a_i = 1$ for any i, then adding a_i to any other term a_j will increase the product, since $a_j + 1 > a_j \cdot 1$.

   If $a_i \geq 5$ for any i, replacing a_i by $a_i - 3$ and 3 increases the product, since $3(a_i - 3) > a_i$.

   If $a_i = 4$, replacing it with $2 + 2$ leaves the product unaltered.

   So we need only consider products made of 2's and 3's.

   If there are three 2's in the product, they can be replaced by two 3's.

   So the biggest product is obtained by using as many 3's as possible. In the case of $2012 = 3 \times 670 + 2$, the biggest product is therefore 2×3^{670}.

5. A little experimentation shows that possible solutions pairs are

$$(3, 2), (10, 7), (17, 12) \ldots$$

and one can conjecture that the general form of the solution pairs is $(3+7k, 2+5k)$, where $k = 1, 2, 3, \ldots$.

This conjecture must be checked.

First of all, all such pairs satisfy the equations, since

$$5(3+7k) - 7(2+5k) = 15 + 35k - 14 - 35k = 1.$$

Next, if (a, b) is a solution pair, then $5a - 7b = 1$. (1)
Since $(3, 2)$ is also a solution pair, $5 \times 3 - 7 \times 2 = 1$. (2)
Subtracting equation (2) from equation (1) gives

$$5(a-3) = 7(b-2).$$

Since 5 and 7 are prime numbers, it follows that $a - 3$ is a multiple of 7 and $b - 2$ is the same multiple of 5, i.e. $a = 3 + 7k$ and $b = 2 + 5k$.

6. Colour the nine rows of the 9×9 board red, blue, green, red, blue, green, red, blue, green.

   The 2×3 rectangles that are placed vertically cover two red squares, two blue squares and two green squares, wherever and however they are placed.

   When the rectangles are placed horizontally, they cover three squares of each of two colours, and none of the third.

   Suppose that there are a horizontal rectangles covering red and blue squares, b horizontal rectangles covering blue and green squares and c horizontal rectangles covering green and red rows.

   There are therefore $a + b + c = n$ horizontal rectangles and $13 - n$ vertical rectangles. Together, the rectangles cover
   $3a + 3c + 2(13 - n)$ red squares
   $3a + 3b + 2(13 - n)$ blue squares, and
   $3b + 3c + 2(13 - n)$ green squares.

   The numbers of red, blue and green squares covered are all congruent to each other modulo 3 (i.e. they all leave the same remainder on division by 3). However, wherever it is placed, the L-shaped piece covers two squares of one colour, one of another and none of the third. So the numbers of squares of each colour leave remainders of 1, 2 and 3 on division by 3, and so cannot be equal. Together the 14 pieces cannot cover the same number of red, blue and green squares.

Senior paper

1. See Junior paper question 3.

2. Let R_n be the largest number of regions into which the plane is divided with n lines. By simple counting it can be found that

$$R_1 = 2, R_2 = 4, R_3 = 7, R_4 = 11.$$

By constructing a difference table, it can be conjectured that the formula is quadratic. With that assumption, it is not difficult to show that the formula is $\frac{1}{2}(n^2 + n + 2)$. This solution does not, however, deserve full credit.

Suppose you have a configuration of n lines, giving R_n regions. A new line can then be placed so that it cuts each of the first n lines, and does not pass through any of their intersections. This line cuts through $n + 1$ regions, and therefore increases the number of regions by $n + 1$.

So R_n
$= n + R_{n-1}$
$= n + (n - 1) + R_{n-2}$
$= n + (n - 1) + (n - 2) + R_{n-3}$
$= n + (n - 1) + (n - 2) + (n - 3) + R_{n-4}$
$= n + (n - 1) + (n - 2) + (n - 3) + \cdots + 3 + 2 + 1 + R_1$
$= \frac{1}{2}n(n + 1) + 1$
$= \frac{1}{2}(n^2 + n + 2)$

The result can also be proved by mathematical induction.

3. See Junior paper question 4.

4. See Junior paper question 6.

5. Clearly $\{1\}$ must be one of the sets, and the numbers 2, 4, 8, 16, 32 and 64 must all be in separate sets. So a minimum of 7 sets is needed.

In fact, 7 is sufficient, as shown by the 7 sets below.

$$\{1\}, \{2, 3\}, \{4, 5, 6, 7, \}, \{8, 9, \ldots, 15\}, \{16, 17, \ldots, 31\},$$

$$\{32, 33, \ldots, 63\}, \{64, 65, \ldots, 100\}$$

An alternative partition is to classify numbers by the number of its prime factors (counting repeated factors). All numbers from 1 to 100 have at most six prime factors (since $2^7 > 100$), so there are seven such sets:

$$\{1\}, \{2, 3, 5, \ldots\}, \{4, 6, 9, 10, \ldots\} \ldots \{32, 48\}, \{64, 96\}$$

6. In the diagram, PQ, TU and AS are parallel.

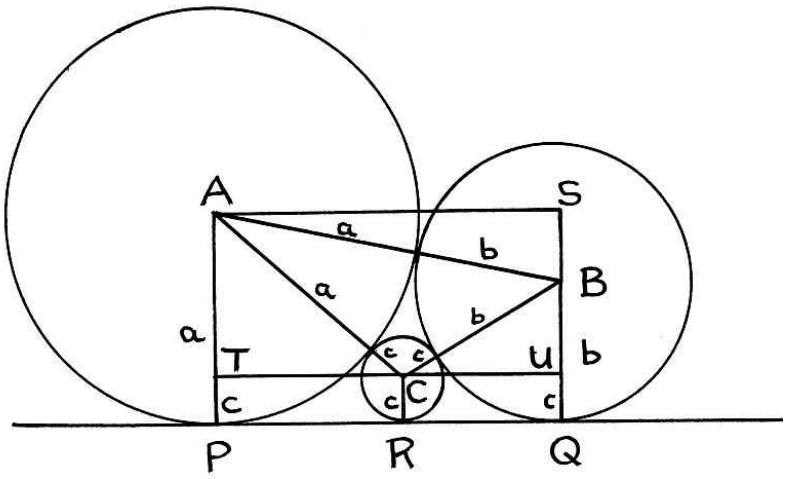
Since $AB = a + b$ and $BS = a - b$,
$PQ = AS = \sqrt{AB^2 - BS^2} = \sqrt{(a+b)^2 - (a-b)^2} = \sqrt{4ab}$.

Since $AC = a + c$ and $AT = a - c$,
$PR = TC = \sqrt{AC^2 - AT^2} = \sqrt{(a+c)^2 - (a-c)^2} = \sqrt{4ac}$.

Since $BC = b + c$ and $BU = b - c$,
$RQ = CU = \sqrt{BC^2 - BU^2} = \sqrt{(b+c)^2 - (b-c)^2} = \sqrt{4bc}$

From $PQ = PR + RQ$ it follows that $\sqrt{4ab} = \sqrt{4ac} + \sqrt{4bc}$, from which it follows (dividing by $\sqrt{4abc}$) that

$$\frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}} = \frac{1}{\sqrt{c}}.$$



AN EULER LINE PROBLEM

Thomas Hagspihl
St Andrew's College, Grahamstown

I was doing concurrency with my top grade 10 class and I showed them the Euler line on Sketchpad.

I decided to test them on it. Is it possible to find a triangle with all three vertices and all three points on the Euler line (circumcentre, centroid and orthocentre) having integer coordinates?

Then I can ask them to find the three points, prove that they lie on a straight line, and verify the 2 : 1 ratio.

An answer can be found on page 31. □

VARSITY READINESS TEST

Are you planning a university degree in science, engineering or commerce? If so, you will need to take one of more courses in mathematics. How ready are you for varsity maths?

The problems below are designed to test your skills in areas that are important for varsity maths, such as algebraic expertise, trigonometry skills, geometric insight, numerical facility and flexible logical reasoning.

If a problem looks complicated, look again. It may have a simple solution if viewed from the right angle.

No diagrams have been given for the geometry problems. That is deliberate. It is a useful skill to be be able to draw a figure from a written description.

No calculators! No formula sheet!

Give yourself two or more hours to tackle the problems, preferably doing them in one continuous concentrated session. Then turn to page 30 to check your answers. There is no pass/fail mark. Every wrong answer, however, indicates a possible weakness in your mathematical background that needs to be fixed.

1. If $5^p = 7$, $7^q = 9$, $9^r = 11$ and $11^s = 25$, what is the value of $pqrs$?

   (A) 1 (B) 2 (C) 3 (D) 4 (E) 5

2. The angles of a triangle are in the ratio $2 : 3 : 5$. What is the difference between the largest and the smallest angle?

   (A) $9°$ (B) $18°$ (C) $36°$ (D) $45°$ (E) $54°$

3. What is the sum of the first hundred digits of the decimal expansion of $\frac{6}{7}$?

   (A) 398 (B) 453 (C) 560 (D) 583 (E6) 601

4. $\sqrt[3]{27^{27x^{27}}}$ is equal to

   (A) 27^{3x^9} (B) 27^{3x^3} (C) 27^{27x^9} (D) $27^{9x^{27}}$ (E) $27^{3x^{27}}$

5. In triangle ABC, points P, Q and R lie on the sides AB, BC and CA, respectively. If $BP = BQ$, $CQ = CR$ and $\angle PQR = 40°$, determine $\angle PAR$.

   (A) $60°$ (B) $70°$ (C) $80°$ (D) $90°$ (E) $100°$

6. $\sin 15^\circ \cos 15^\circ$ is equal to

(A) $\tan 15^\circ$ (B) $\tan 30^\circ$ (C) $\dfrac{\sqrt{3}}{4}$ (D) $\dfrac{1}{\sqrt{2}}$ (E) $\dfrac{1}{4}$

7. A computer hard drive spins at 7200 rpm. What is its angular velocity in degrees per second?

(A) 43 200 (B) 54 400 (C) 36 600 (D) 48 480 (E) 24 360

8. The remainder when $2x^4+2x^3-x^2-15x+8$ is divided by $2x^2-4x+1$ is

(A) $4x+5$ (B) $3x-1$ (C) $2x+3$ (D) $-x+7$ (E) $x+4$

9. In triangle ABC, $\tan A = \frac{3}{4}$ and $\sin B = \frac{5}{13}$. The value of $\cos C$ is

(A) $\dfrac{33}{65}$ (B) $\dfrac{56}{65}$ (C) $\dfrac{17}{65}$ (D) $\dfrac{41}{65}$ (E) $\dfrac{8}{65}$

10. If $\tan A + \dfrac{1}{\cos A} = 2$, then $\cos A$ is equal to

(A) $\dfrac{6}{11}$ (B) $\dfrac{5}{9}$ (C) $\dfrac{3}{7}$ (D) $\dfrac{4}{5}$ (E) $\dfrac{7}{8}$

11. Triangle ABC is right-angled at C and D lies on BC. If $AB = 14$, $AD = 10$ and $CD = 5$, the perimeter of triangle ABD is

(A) $24+5\sqrt{2}$ (B) $24+3\sqrt{3}$ (C) 29 (D) 30 (E) 31

12. If $x^2-4x+3<0$ and $y = x^2+4x+3$, then

(A) $y<0$ (B) $0<y<3$ (C) $3<y<8$ (D) $8<y<24$ (E) $24<y<35$

13. The diagonals AC and BD of quadrilateral $ABCD$ intersect at the point E inside the quadrilateral. If $AD = AE$, $BE = CE$ and $\angle ADE = 2x^\circ$, then the measure of $\angle CBE$ is

(A) $45^\circ - x^\circ$ (B) $90^\circ + \frac{1}{2}x^\circ$ (C) $90^\circ - \frac{1}{2}x^\circ$ (D) $45^\circ + 2x^\circ$ (E) $90^\circ - x^\circ$

14. If $f(x)$ is a quadratic function such that $f(-1) = -3$, $f(1) = 1$ and $f(2) = 12$, find $f(3)$.

(A) 8 (B) 11 (C) 17 (D) 29 (E) 35

15. If $a-b=3$ and $ab=7$, then $\dfrac{1}{a}-\dfrac{1}{b}$ is equal to

(A) $\dfrac{1}{3}$ (B) $-\dfrac{3}{7}$ (C) $\dfrac{7}{3}$ (D) $-\dfrac{1}{3}$ (E) $\dfrac{1}{7}$

16. If $\sin x = \dfrac{7}{11}$, then $\cos 2x$ is equal to

(A) $\dfrac{\sqrt{73}}{121}$ (B) $\dfrac{8}{119}$ (C) $\dfrac{4}{11}$ (D) $\dfrac{\sqrt{77}}{11}$ (E) $\dfrac{23}{121}$

17. If $\log_{10} 2 = a$ then $\log_{10} 500$ is equal to

(A) $3 - a$ (B) $3 + a$ (C) $3a$ (D) $\dfrac{3}{a}$ (E) $\dfrac{a}{3}$

18. If $\dfrac{1}{n} + \dfrac{1}{n(n-1)} = \dfrac{1}{4}$, then n is equal to

(A) 3 (B) 4 (C) 5 (D) 6 (E) 7

19. A rectangle is made up of three rows of four squares each. The area of each square is 4. What is the perimeter of the rectangle?

(A) 28 (B) 30 (C) 36 (D) 40 (E) 48

20. A boy runs from his home to school at 9 km/h, then walks back home at 5 km/h. What was his average speed, to one decimal place, for the trip to school and back?

(A) 6.3 km/h (B) 6.4 km/h (C) 6.6 km/h (D) 6.9 km/h (E) 7.1 km/h

21. If $a - b - c = 2$, $b - c - a = -3$ and $c - a - b = 5$, then $a + b + c$ is equal to

(A) -1 (B) -2 (C) -3 (D) -4 (E) -5

22. Solve for x: $ax^2 + b(a-1)x = b^2$.

(A) $x = a$ or $\dfrac{1}{b}$ (B) $x = a$ or $\dfrac{1}{b}$ (C) $x = -b$ or $\dfrac{b}{a}$
(D) $x = a + b$ or $\dfrac{a}{b}$ (E) $x = b - a$ or $\dfrac{a}{b}$

23. Let P be the area of a triangle with sides of length 25, 25 and 30. Let Q be the area of a triangle with sides of length 25, 25 and 40. What is the relationship between P and Q?

(A) $P = \frac{9}{16}Q$ (B) $P = \frac{3}{4}Q$ (C) $P = Q$ (D) $P = \frac{4}{3}$ (E) $P = \frac{16}{9}Q$

24. Which of the following statements is true?

(A) $10^8 < 5^{12} < 2^{24}$
(B) $2^{24} < 10^8 < 5^{12}$
(C) $10^8 < 2^{24} < 5^{12}$
(D) $5^{12} < 10^8 < 2^{24}$
(E) $2^{24} < 5^{12} < 10^8$

25. In triangle ABC, $\angle A = 50°$. The bisectors of $\angle B$ and $\angle C$ meet in O, inside the triangle. The size of $\angle BOC$ is

(A) 100° (B) 105° (C) 110° (D) 115° (E)120° □

IMO TEAM SELECTION

The 2012 International Mathematical Olympiad (IMO) will be held in Mar del Plata, Argentina, from 4–16 July. The team to represent South Africa was announced in May. They are: Mickey Chew (Elkanah House, Cape Town), Dylan Nelson (Benoni High School), Junho Son (St Albans College, Pretoria), Robert Spencer (Westerford High School, Cape Town), Dalian Sunder (Star College, Durban), Robin Visser (St Georges Grammar School, Cape Town)

The Team Leader is Maciek Stankiewicz (University of Cape Town), with Deputy Team Leader Dr Koos van Zyl (Rhodes University)

The team was chosen after a long process that began early last year with a correspondence programme of reading and problem-solving assignments. At the end of 2011 a group of about 50 of the front-runners in this programme were invited to attend a training camp at the University of Stellenbosch.

Early in the New Year the pace quickened, with a series of monthly assignments, which had the effect of filtering out a group of 16 who were invited to the IMO Selection Camp at Rhodes University in April. At Rhodes they were given high-level training in solving IMO problems, and wrote a series of tough IMO-style tests. The IMO team was chosen from this group.

Meanwhile, the search for an IMO team for 2013 has already begun. The South African Mathematics Foundation has set up an Olympiad training programme, designed to develop Olympiad problem-solving skills. Participation is free. For details of how to take part in the programme, go to the website `www.samf.ac.za`. □

GRANDMA'S PROBLEM

Grandma has four grandchildren. The product of their ages is 67184. The oldest grandchild is under 40, and is 30 years older than the youngest grandchild. Only one of the grandchildren is a teenager.

How old are Grandma's grandchildren?

The answer is on page 31. □

ANSWERS

UCT Challenge Answers (p.17):
Junior paper

(1) 41 (2) 32.73 (3) 712 (4) 38 (5) 64

(6) 2606 (7) 225 (8)

$\frac{3}{4}$	$\frac{1}{6}$	$\frac{7}{12}$
$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$
$\frac{5}{12}$	$\frac{5}{6}$	$\frac{1}{4}$

(9) 11 (10) 2316

Senior paper

(1) 41 (2) 64 (3) $\dfrac{361}{30}$ (4) 576 (5) 72

(6) 104 (7) 432 (8) 315 (9) 130 (10) 456

Reader's Problem (p. 19):
Imagine a large carrot being nibbled by 50 rabbits, taking turns.
The first rabbit eats $\frac{1}{3}$ of the carrot.
The second rabbit eats $\frac{1}{5}$ of the remaining $\frac{2}{3}$ of the carrot.
The third rabbit eats $\frac{1}{7}$ of the remaining $\frac{2}{3} \cdot \frac{4}{5}$ of the carrot.
The fourth rabbit eats $\frac{1}{9}$ of the remaining $\frac{2}{3} \cdot \frac{4}{5} \cdot \frac{6}{7}$ of the carrot.
And so on.
The 50th rabbit eats $\frac{1}{101}$ of the remaining $\frac{2}{3} \cdot \frac{4}{5} \cdot \frac{6}{7} \cdots \frac{98}{99}$ of the carrot.

Each term in the sum represents the fraction of the carrot eaten by a rabbit. After all those nibbles, there is still some carrot left, so the sum of all the terms is less than 1.

An Euler Line Problem (p. 26):
To find a suitable example, it is easiest to work with right-angled triangles, because in a right-angled triangle the orthocentre is at the right angle and the circumcentre is the midpoint of the hypotenuse.

With $A(-3,0)$, $B(3,0)$ and $C(0,3)$, the orthocentre is $(0,3)$, the centroid is $(0,1)$ and the circumcentre is $(0,0)$. They clearly lie on a straight line (the y-axis) and are in the required the $2:1$ ratio.

Can you find a it non-right-angled triangle with its vertices, circumcentre, centroid and orthocentre all having integer coordinates?

What about looking for an equilateral triangle with the same property?

Varsity Readiness Test (p. 26):
BEBDE EACED EDEDB EACAB DCCBD

Grandma's Problem (p. 29):
The grandchildren's ages are 4, 19, 26 and 34.

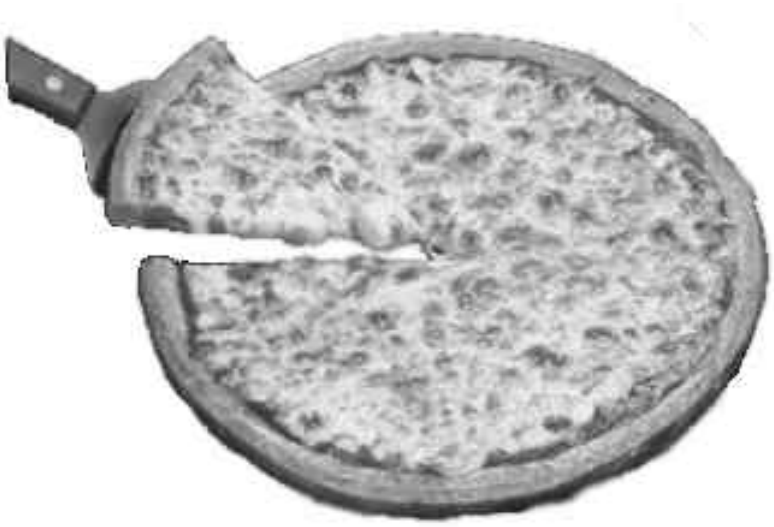
ON PROBLEM SOLVING

I find that "playing" with a problem, even after you have solved it, is very helpful for understanding the underlying mechanism of the solution better. For instance, one can try removing some hypotheses, or trying to prove a stronger conclusion.

Its also best to keep in mind that obtaining a solution is only the short-term goal of solving a mathematical problem. The long-term goal is to increase your understanding of a subject. A good rule of thumb is that if you cannot adequately explain the solution of a problem to a classmate, then you havent really understood the solution yourself, and you may need to think about the problem more (for instance, by covering up the solution and trying it again).

(Terence Tao, in his blog on problem solving) □

THE PIZZA FORMULA



The volume of a pizza of radius z and thickness a is

$$pi.z.z.a$$

□

mathematical digest

No. 169 October 2012

Imo Argentina

mathematical digest

Number 169 **October 2012**

Often I have considered the fact that most of the difficulties which block the progress of students trying to learn analysis stem from this: that although they understand little of ordinary algebra, still they attempt this more subtle art.

Leonard Euler, *Introductio in analysin infinitorum* (1748)

Department of Mathematics and Applied Mathematics
University of Cape Town

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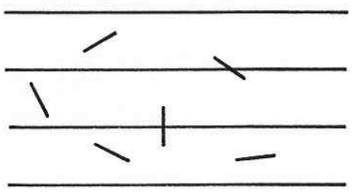
Cover: The 2012 International Mathematical Olympiad took place in Mar del Plata, Argentina in July. See page 16.

BUFFON'S NEEDLE PROBLEM

In 1777 a French nobleman, Georges-Louis Leclerc, Comte de Buffon, posed a simple problem:

> Suppose that you randomly drop a needle of length l onto a sheet of ruled paper, in which the ruled lines are a distance d apart, and $l < d$. What is the probability that the needle will cross one of the lines?



With the aid of integral calculus, one can prove that the required probability is $\dfrac{2l}{\pi d}$. The result means that if you drop a needle N times onto the ruled paper, and count P intersections, then $\dfrac{P}{N}$ will be approximately $\dfrac{2l}{\pi d}$. That means that π is approximately $\dfrac{2lN}{dP}$.

That suggests that you can work out the value of π by experiment: just drop the needle thousands of times, and count the intersections.

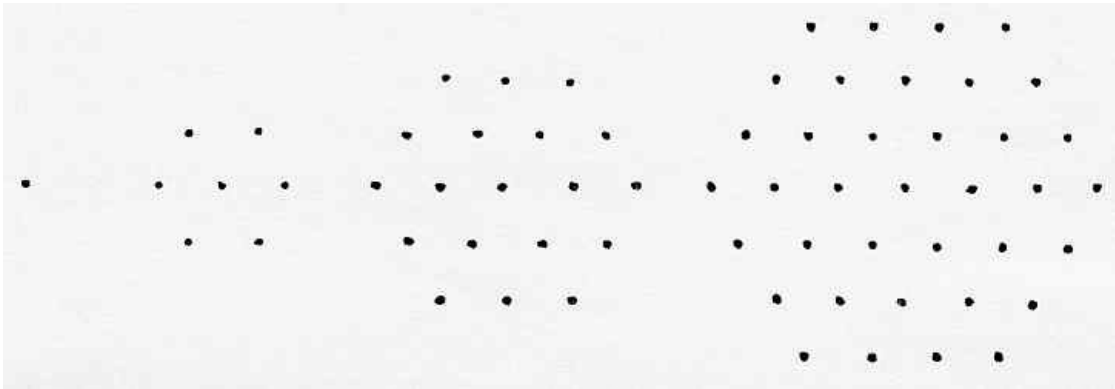
That is the theory. In practice, the experiment will not yield any decent accuracy. If you expect five-digit accuracy, then you will have to measure the needle, and the distance between the lines on the ruled paper, with that accuracy: in other words, if the needle is 5 cm long, and the lines are 6 cm apart, then those lengths must be measured correct to to an accuracy of one-thousandth of a millimetre.

Then you will have to ensure that the ruled lines are not done with a fat pencil, but are as thin as possible, so as to rule out ambiguity about whether the needle crosses the line or not.

There will also be difficult cases when the end of the needle is very near a line, and you will have to use a microscope to decide whether the needle crosses the line or not.

All in all, it's a hopeless way of calculating pi. $\square$

HEXAGONAL NUMBERS



Hexagonal numbers are given by the number of dots in regular hexagonal arrays, as illustrated above. Counting the dots, we see that the first four hexagonal numbers are 1, 7, 19 and 37. Work out the next two hexagonal numbers yourself. Can you, on the basis of the pattern revealed so far, guess a formula for H_n, the nth hexagonal number?

The numerical pattern for generating hexagonal numbers can be seen in the table below.

$$\begin{aligned}
H_1 &= 1 &&= 1\\
H_2 &= 2+3+2 &&= 7\\
H_3 &= 3+4+5+4+3 &&= 19\\
H_4 &= 4+5+6+7+6+5+4 &&= 37
\end{aligned}$$

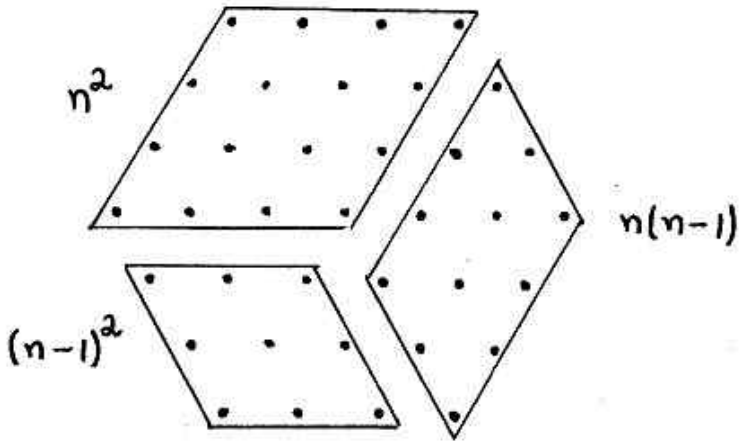
The nth hexagonal number is thus given as a sum of a series which rises and falls arithmetically. The sum starts with n, rises to $2n-1$ and falls back to n. There are $2n-1$ terms in the sum.

$$\text{So} \quad H_n = \begin{aligned}[n+(n+1)+\cdots+(2n-3)+(2n-2)]+(2n-1)+\\ [(2n-2)+(2n-3)+\cdots+(n+1)+n].\end{aligned}$$

There are $n-1$ terms in each square bracket. You will notice that if a term in the first bracket is added to the term in the second bracket below it, the sum is $3n-2$ every time. So

$$\begin{aligned}
H_n &= [(3n-2)+(3n-2)+\cdots+(3n-2)]+(2n-1)\\
&= (n-1)(3n-2)+(2n-1)\\
&= 3n^2-3n+1.
\end{aligned}$$

Here is a geometrial way of deriving the same formula. Simply divide the hexagon as shown below, for the case $n = 4$.



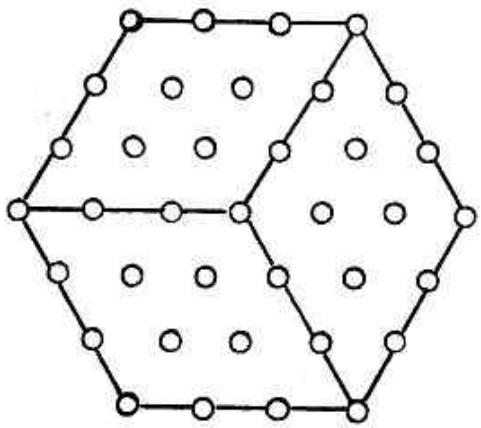
The nth hexagonal number is thus $n^2+n(n-1)+(n-1)^2$, which is readily shown to be $3n^2 - 3n + 1$, as before.

Have you noticed anything about the two expressions

$$n^2 + n(n-1) + (n-1)^2 \text{ and } 3n^2 - 3n + 1$$

(except that they are equal)? They are both equal to $n^3 - (n-1)^3$. That suggests another way of deriving the formula for the nth hexagonal number.

Consider an $n \times n \times n$ cube of spheres. Viewed obliquely, three sides will form a hexagon.



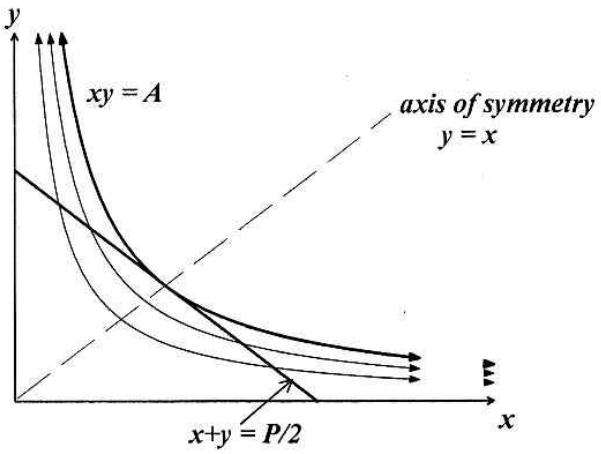
The number of spheres visible is the nth hexagonal number. If the three visible layers are removed, an $(n-1) \times (n-1) \times (n-1)$ cube will be revealed. So the nth hexagonal number is $n^3 - (n-1)^3$. $\square$

FENCING A FIELD

Marcus Bizony
Bishops

Commonly among the first examples in textbooks to illustrate the use of calculus to solve optimisation problems is the case of the farmer who wants, with a fixed amount of fencing, to enclose the largest possible rectangular field. The classic approach is to let x be one of the dimensions of the field, so that $\frac{1}{2}P - x$ is the other dimension (where P is the fixed perimeter determined by the amount of fencing available), and then the area of the field is $A = x(\frac{1}{2}P - x) = \frac{1}{2}Px - x^2$.
Now we can either use calculus to find the value of x for which A is a maximum, or we can use quadratic theory, noting that the minimum is obtained when x is midway between the two zeros of A, namely at $x = \frac{1}{4}P$.

A more elegant solution to the problem views it as one in which we have to maximize xy subject to the constraint $x + y = P/2$: the graphs alongside show clearly that the optimal situation occurs at the apex of the hyperbola, which is where $x = y$. Then x and y are both equal to $\sqrt{A}$ and also to $P/4$, showing that the optimal field is actually square, and that the maximum area achievable with P units of fencing is $P^2/16$.



This same diagram simultaneously solves the equivalent (but different sounding) problem of determining the minimum amount of fencing needed to achieve a particular area, if the field has to be rectangular.

Now suppose the farmer has an additional component to his problem, which is that the cost of fencing differs on the two pairs of opposite sides, being a per unit in the x-direction and b per unit in the y-direction, and that now he wishes to minimize his total cost rather than total length of fencing.

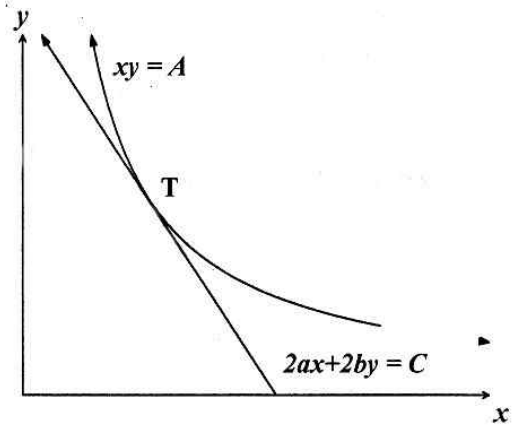
Thus for a given value of xy, say A, he wishes to minimize $C = 2ax + 2by$.

The classic approach would still be to find an expression for C in terms of just x (or just y), differentiate and then set the derivative to zero to find

the critical value of x. Again, in fact, the function involved turns out to be quadratic, so no calculus as such is needed.

Does a graphical approach offer any insights?

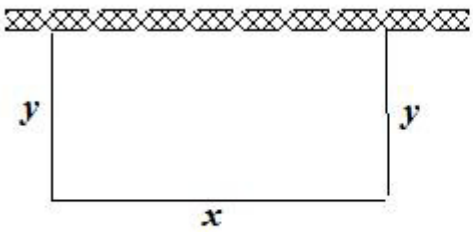
Clearly the minimum value of C is going to be achieved at the point T where the line $2ax + 2by = C$ is tangent to the hyperbola $xy = A$. The obvious continuation now is to find the equation of the tangent by using calculus, but quadratic theory can again help us more simply.



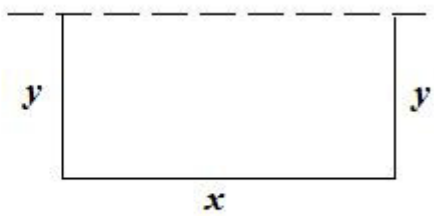
The two graphs meet where $y = \frac{A}{x}$ and $by = \frac{C}{2} - ax$. That means that $\frac{bA}{x} = \frac{C}{2} - ax$, and hence that $2ax^2 - Cx + 2bA = 0$, rewritten as $2a\left(x - \frac{C}{4a}\right)^2 + \left(2bA - \frac{C^2}{8a}\right) = 0$. But since the line is a tangent to the hyperbola, this equation has coincident roots: it follows that $2bA - \frac{C^2}{8a} = 0$ (so $C^2 = 16abA$) and the value of the two equal roots is $\frac{C}{4a}$.

So we have: if the field has predetermined area A, the cost C is minimized by putting $x = \frac{C}{4a} = \frac{\sqrt{16abA}}{4a} = \sqrt{\frac{bA}{a}}$, $y = \sqrt{\frac{aA}{b}}$ and getting $C = 4\sqrt{abA}$.

Alternatively, if the cost is to be predetermined as C, then the maximum area that can be achieved requires $x = \frac{C}{4a}$, $y = \frac{C}{4b}$ and this gives $A = \frac{C^2}{16ab}$.



$C = 2y + x$

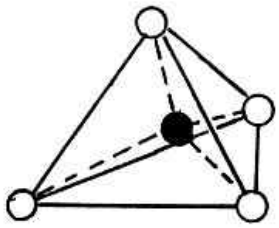


$C = 2y + 1{,}5x$

The result can be applied to cases other than those where the costs in one direction are different from those in the other: the first diagram shows how the cost function $y + 2x = C$ derives from a situation in which one side of the field is to be built against a wall, or river, so that fencing along that one edge is not required; and the second diagram has our farmer's field sharing a boundary with a neighbour's field, the neighbour having agreed to pay half the cost of the fencing along the common border. $\square$

THE TETRAHEDRAL ANGLE

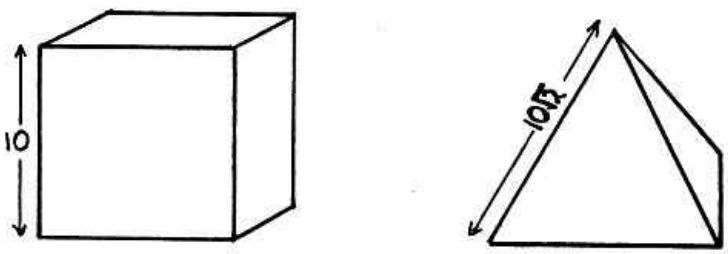
Methane (CH_4) is an example of a molecule in which four identical groups are bonded to a central atom. In methane, the outer level of the carbon atom contains four electron-pairs. Mutual repulsion of the four identical electron clouds of the surrounding hydrogen atoms directs them to the corners of a regular tetrahedron, with the carbon atom at the centre of the tetrahedron.



What is the angle that the electron-pairs make with the central carbon atom? Chemists refer to this as the *tetrahedral angle.*

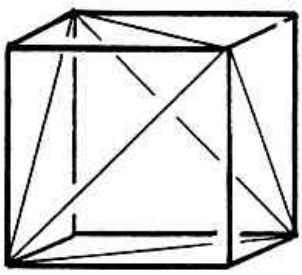
There are several ways of calculating the tetrahedral angle. We shall describe a quick method, which is inspired by a neat little geometrical puzzle with which you can baffle your friends.

First of all, then, the puzzle. Get some stiff cardboard and make a cubical box, open at the top with side length 10 cm. Then make a regular tetrahedron with side length 14 cm. (The theoretical side length of the tetrahedron is $10\sqrt{2}$ cm, but we have rounded the figure down for practical purposes.)

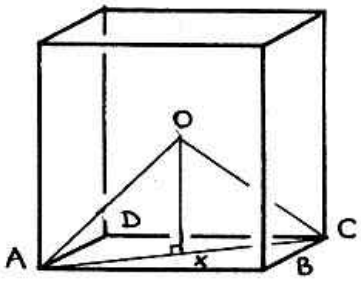


Now find an unsuspecting nonmathematical friend, and challenge him or her to put the tetrahedron into the box.

It's very satisfying to find that many people find this little puzzle quite difficult. They will usually try to drop the tetrahedron into the box with one of its faces flush with a face of the box. That way, it just won't fit. The trick is to drop the tetrahedron into the box with one edge along the diagonal of the base of the box.



Now let's get back to that tetrahedral angle. The centre of the tetrahedron is clearly the same as the centre of the cube. So the problem is to find $\angle AOC$ in the diagram below, where O is the centre of the cube.



If $AB = 2$ and X is the centre of $ABCD$, then $OX = 1$ and $AX = \sqrt{2}$. So $\tan \angle AOX = \sqrt{2}$, and $\angle AOX = 54°44'$. The tetrahedral angle is just double this angle, namely $109°28'$. □

SMART ANSWERS

A student was writing an exam in formal logic. The first question in the paper was

> "If this is a question, answer it. "

The student gave an answer which earned full marks. What was it?

The same student was required to attend an oral examination in philosophy. When he entered the professor's office he was invited to sit down. The professor pointed to the chair on which he was sitting and said

> "Prove that the chair exists."

The student's answer again earned full marks. What was it?

See page 30 for the student's smart answers. □

PAMO IN TUNISIA

The Pan African Mathematics Olympiad was held in Tunisia in September, and teams of four from eight countries took part: Burkina Faso, Egypt, Gambia, Ivory Coast, Nigeria, South Africa, Tanzania and Tunisia.

The South African team was Nicholas Cheng (Fairmont High School), Lauren Denny (Rustenburg Girls' High School), Chris Kim (Reddam House) and Shaylan Naidoo (Pearson High School). The Team Leader was Dirk Basson (University of Stellenbosch), with deputy Phil Labuschagne. It is a long-standing tradition that South African PAMO teams are chosen from those students who have not represented the country at an IMO.

The South African team

Lauren Denny, Shaylan Naidoo, Nicholas Cheng, Chris Kim

The PAMO papers (see page 26) are structured in exactly the same way as those of the IMO: two $4\frac{1}{2}$ hour papers on successive days, each containing three questions of seven points each.

Medals are awarded in IMO style: half the students win medals, and Gold, Silver and Bronze medals are awarded in the ratio as near as possible to

1 : 2 : 3. With 28 contestants this year, that meant two Gold, five Silver and eight Bronze.

The top student this year was Henry Aniobi (Nigeria), who obtained a perfect 42—the first time this has been achieved in PAMO history. He also won a special award for a particularly good solution. Close behind him, on 40, was Shaylan Naidoo, who won the second Gold Medal. Lauren Denny was the top girl, winning a Silver Medal and seventh place. Chris Kim and Nicholas Cheng were ranked ninth and tenth, respectively, and won Bronze Medals.

The host country, Tunisia, came overall first with a total of 116 points, followed by South Africa with 107 and Nigeria with 89.

Next year's PAMO will take place in Abuja, Nigeria, from 22 June to 2 July, linked to the Eighth Pan African Congress of Mathematicians. The PAMO closing ceremony and medal presentation will be part of the opening ceremony of the Congress. □

MATHS QUESTIONS

A company was hiring consultants. Three graduates were invited for an interview: a mathematician, an engineer and an accountant. All three were asked the same question: "What is one third plus two thirds?"

The mathematician answered: "One."

The engineer took out his calculator, punched in some numbers and said: "It's 0.999999999."

The accountant looked to left and right surreptitiously, and then whispered: "What would you like it to be?"

10 TYPES OF PEOPLE

There are 10 types of people.

Those who understand ternary numbers, those who don't, and those who expected this to be a joke about binary numbers. □

A DIFFICULT QUARTIC

Every quartic polynomial can be factorised into two quadratics with real coefficients (which may or may not be factorised into linear factors with real coefficients). This may be a surprise for those who believe that the quartic $x^4 + 1$ has no factors. Though it indeed has no rational factors, it has irrational ones: $x^4 + 1 = (x^2 - \sqrt{2}x + 1)(x^2 + \sqrt{2}x + 1)$.

The factorisation of any real quartic into a product of real quadratics is a consequence of the important *Fundamental Theorem of Algebra*, which states that

> Any polynomial with real coefficients can be factorized into a product of linear and quadratic factors, all with real coefficients.

The result was conjectured during the 18th century, but the proof turned out to be surprisingly elusive. So much so, that some mathematicians believed that the conjecture was not true. The Swiss mathematician Nicholas Bernoulli was convinced that the quartic polynomial

$$x^4 - 4x^3 + 2x^2 + 4x + 4$$

could not be factorized into a product of real linear and/or real quadratic factors, and sent it to his fellow countryman, Leonhard Euler.

Euler proved that Bernoulli was wrong, by simply finding two quadratic factors:

$$x^2 - \left(2 + \sqrt{4 + 2\sqrt{7}}\right)x + \left(1 + \sqrt{4 + 2\sqrt{7}} + \sqrt{7}\right)$$

and

$$x^2 - \left(2 - \sqrt{4 + 2\sqrt{7}}\right)x + \left(1 - \sqrt{4 + 2\sqrt{7}} + \sqrt{7}\right).$$

Energetic readers might like to test their algebraic skills by multiplying these two quadratics together to check Euler's work.

Euler (1707 - 1783), the greatest mathematician of the 18th century, tried to prove the Fundamental Theorem of Algebra, and failed.

Only right at the end of the century was the result finally proved, by the 22-year-old German mathematician Carl Friedrich Gauss. For this achievement, and many others, Gauss is now recognized as the greatest mathematician of the 19th century. □

THE CAGE CENTENARY

This year the centenary of the birth of the American experimental composer John Cage is being celebrated. His best-known work has the title $4'33''$, to be read as "four minutes and thirty-three seconds".

It was composed in 1952 and may be performed on any instrument or combination of instruments.
The score is very simple. The performer is instructed not to play the instrument during the entire three movements, the first lasting thirty seconds, the second two minutes and twenty-three seconds and the third one minute and forty seconds.

The idea is that the piece forces the listener to concentrate on the random background sounds which arise when any piece of music is performed.

After the first performance of the piece, John Cage wrote about the audience's reaction. "They missed the point. There's no such thing as silence. What they thought was silence, because they didn't know how to listen, was full of accidental sounds. You could hear the wind stirring outside during the first movement. During the second, raindrops began pattering the roof, and during the third the people themselves made all kinds of interesting sounds as they talked or walked out."

John Cage (who died in 1992) regarded $4'33''$ as his most important composition. The original score of the piece has been lost.

The work is performed surprisingly often, but seldom with serious intent. Articles discussing the composition range from the pretentiously learned to tongue-in-cheek musico-babble. It is strange that the question is seldom asked why the particular numbers were chosen as the title and length of the piece. The answer is that four minutes and thirty-three seconds amounts to 273 seconds, and 273 is the number of degrees below zero on the Kelvin scale at which all motion ceases. □

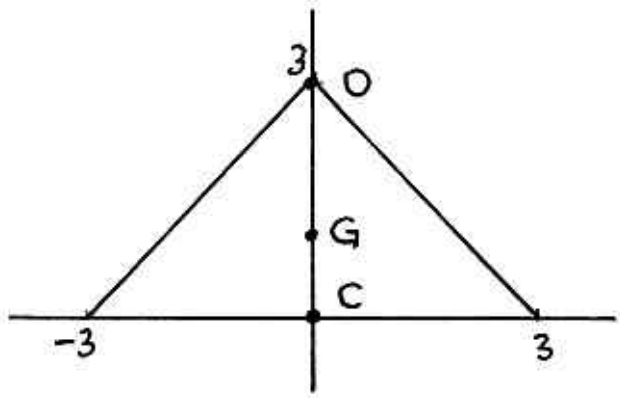
STATS QUOTE

An 20-year study has revealed that 73.407995% of all statistics are made up on the spot. □

LATTICE POINTS AND THE EULER LINE

In *Mathematical Digest* **168** (July 2012) Thomas Hagspihl (St Andrew's College, Grahamstown) posed the problem of finding a triangle with all three vertices and all three points on the Euler line (circumcentre, centroid and orthocentre) having integer coordinates. Points with integer coordinates are called *lattice points*.

A quick answer was given in the same edition of the magazine: the right-angled triangle with vertices $(-3, 0)$, $(3, 0)$ and $(0, 3)$. For this triangle the circumcentre C is $(0, 0)$, the centroid G is $(0, 1)$ and the orthocentre D is $(0, 3)$.



Professor Poobhalan Pillay (University of KwaZulu Natal) has now shown that there are infinitely many solutions.

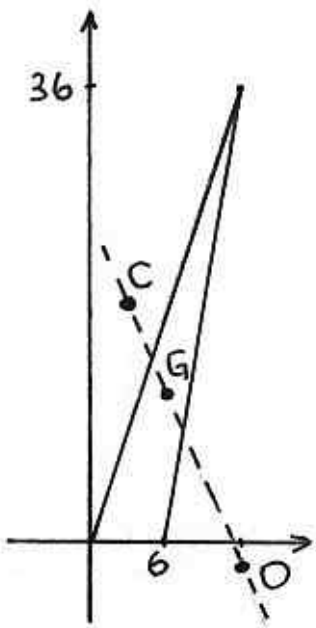
Without loss of generality set one of the vertices of the triangle at $(0, 0)$ and the second on the x-axis at $(6k, 0)$. With the third point at $(12m, 36)$ the coordinates of the three points on the Euler line all have integer coordinates:

Circumcentre: $(3k, 18mk + 2m^2)$

Centroid: $(4m + 2k, 12)$

Orthocentre: $(12m, 2mk - 4m^2)$.

If $m = k = 1$ the triangle has vertices with coordinates $(0, 0)$, $(6, 0)$ and $(12, 36)$, the circumcentre is $(3, 20)$, the centroid is at $(6, 12)$, and the orthocentre is $(12, -2)$.



It's a good exercise to check with a little algebra and coordinate geometry that these are the points on the Euler line as claimed. □

SUPERMARKET MATHS

In a supermarket with multiple checkout points, it is always irritating to find that the queue for the till next to your queue moves faster.

The way to minimise this type of irritation is to head for the first or last checkout points. Why?

It is reasonable to assume that all supermarket queues move at the same average speed. So if you join a queue, the chance that the queue on your right moves faster than yours is one half. But there is also the queue on your left, and the chance that your queue is the fastest of the three is one third.

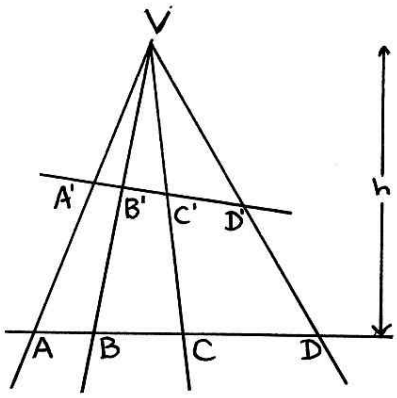
The fact is that whether your queue moves faster or slower than another queue is a random event. So if you are in a queue, the chance is two to one that one of the queues on either side of you moves faster.

However, if you queue up at the first or last checkout till, there is only one queue next to yours, and now it's an even chance that your queue will move faster than the queue next to yours. □

THE CROSS-RATIO OF PAPPUS

Pappus of Alexandria was the last of the great Greek mathematicians of antiquity. His principle work "The Collection" contains a systematic account of the most important works in ancient Greek mathematics, with a commentary. The book was intended as a guide to be used with the original works. When many of these works were lost, Pappus' *Collection* became especially important as the sole source of the original works.

But Pappus was not just a preserver and editor. He is also remembered as the mathematician who first formulated a theorem which turned out to be of fundamental importance in projective geometry. The figure shows one of the essential notions of projective geometry: the *central projection* of one line, $ABCD$, on another, $A'B'C'D'$, where AA', BB', CC' and DD' meet in a common point, V.



$$\frac{AC.BD}{AD.BC} = \frac{A'C'.B'D'}{A'.B'C'}.$$

Thus the *Cross-Ratio* of four points in a straight line is invariant under a central projection.

The proof depends on two standard formulae for finding the area of a triangle: the "half-base-times-height" rule, and the sine formula. We start with the left hand side of the equation. Suppose that the height of V above the line $ABCD$ is h.

Then
$$\begin{aligned}\frac{AC.BD}{AD.BC} &= \frac{\frac{1}{2}h.AC.\frac{1}{2}h.BD}{\frac{1}{2}h.AD.\frac{1}{2}h.BC}\\ &= \frac{\text{area } \triangle ACV.\text{area } \triangle BDV}{\text{area } \triangle ADV.\text{area } \triangle BCV}\\ &= \frac{\frac{1}{2}AV.CV.\sin\angle AVC.\frac{1}{2}BV.DV.\sin\angle BVD}{\frac{1}{2}AV.DV.\sin\angle AVD.\frac{1}{2}BV.CV\sin\angle BVC}\\ &= \frac{\sin\angle AVC.\sin\angle BVD}{\sin\angle AVD.\sin\angle BVD}.\end{aligned}$$

Similarly
$$\frac{A'C'.B'D'}{A'D'.B'C'} = \frac{\sin\angle A'VC'.\sin\angle B'VD'}{\sin\angle A'VD'.\sin\angle B'VC'},$$

and the Cross-Ratio Rule immediately follows. □

FRIDAY THE THIRTEENTH

Did you know that the thirteenth day of the month is more likely to be a Friday than any other day of the week? You may be excused for guessing that the probability that the thirteenth of the month falls on a Friday is 1/7. However, the peculiarities of our Gregorian calendar make the long range probability 0.143333, which is just a little higher than $1/7 = 0.142857$.

In the Gregorian calendar, years divisible by 4 are leap years unless they are divisible by 100. If they are divisible by 100, but not by 400, they are not leap years. Years divisible by 400 are, however, leap years. Thus 1988 was a leap year, 1900 was not, and 2000 was a leap year. This means that the leap year cycle repeats itself after 400 years.

In the 400-year cycle of the Gregorian calendar, therefore, there are 97 leap years, and the total number of days is $97 \times 366 + 303 \times 365 = 146097$. The important point to note is that 146097 is a multiple of 7:

$$146097 = 7 \times 20871.$$

thus the Gregorian calendar consists of an exact number of weeks, and we therefore know that if 1 January 1900 was a Monday, 1 January 2300 will be a Monday as well.

To work out the probability that the thirteenth day of a month will be a Friday, one needs to consider only one cycle of 400 years. The calculations are very tedious, so we will leave them out, and merely list the results. Of the 4800 occurrences of the thirteenth of the month

687	fall on a Sunday
685	fall on a Monday
685	fall on a Tuesday
687	fall on a Wednesday
684	fall on a Thursday
688	fall on a Friday
684	fall on a Saturday

The probability that the thirteenth is a Friday is therefore $\dfrac{688}{4800} = \dfrac{43}{300}$, which is just a little more than $\dfrac{1}{7}$. $\square$

IMO IN MAR DEL PLATA

The 53rd International Mathematical Olympiad took place in July in Mar del Plata, Argentina. Teams of up to six from one hundred countries took part, representing nearly 90% of the world's population.

The South African team was Mickey Chew (Elkanah House, Cape Town), Dylan Nelson (Benoni High School), Junho Son (St Albans College, Pretoria), Robert Spencer (Westerford High School, Cape Town), Dalian Sunder (Star College, Durban) and Robin Visser (St George's Grammar School, Cape Town). The Team Leader was Maciek Stankiewicz (University of Cape Town), with Deputy Team Leader Dr Jay van Zyl (Rhodes University).

The contest papers may be found on page 18.

IMO problems are selected by a long process, starting several months before the event. Participating countries submit proposals for problems, all of which must be original. An expert committee then draws up a short list of about 30, which are submitted to the Jury (the Team Leaders of the participating countries). Over a period of four days the Jury considers the problems, and eventually chooses six. When they are selected, the countries that proposed the problems are announced.

For the first time in the 21 years of South Africa's participation in the IMO, a South African problem was selected. The problem was devised by Liam Baker, himself a veteran of two IMOs, and a Bronze Medallist in Kazakhstan in 2010. Liam is currently a third-year student at UCT, majoring in Mathematics.

Liam's problem was very popular with the Jury, who set it as the first problem on the second contest day. (It can now be revealed that, had Liam's probem not been selected for the IMO, it would have featured as the final problem in this year's South African Mathematics Olympiad.)

It took the Jury three days to mark the papers and agree on the scores. Although the IMO is a contest for individuals, unofficial team scores are

always computed. Everybody expected China to top the rankings, but this year the South Korean team triumphed with a total score of 209 points, comfortably ahead of China's 195. Just one point behind was the USA (194), followed by Russia (177), Canada (159), Thailand (159), Singapore (154), Iran (151), Vietnam (148) and Romania (144).

For South Africa, Dylan Nelson and Dalian Sunder won Bronze Medals, with Robert Spencer, Junho Son and Mickey Chew earning Honourable Mentions (for solving at least one problem perfectly). South Africa's ranking was 56th.

The South African Team

Robert Spencer, Dylan Nelson, Junho Son, Mickey Chew, Dalian Sunder

Only six African countries took part: Ivory Coast, Morocco, Nigeria, South Africa, Tunisia and Uganda. Apart from South Africa's two Bronze Medals, Tunisia picked up a Silver Medal, Morocco two Bronze Medals and Nigeria one Bronze Medal.

Next year's IMO will take place in Colombia.

South Africa's participation in the IMO is organized by the South African Mathematics Foundation, and supported by Harmony Gold. □

IMO PROBLEMS

First day (Tuesday, 10 July 2012)

1. Given triangle ABC the point J is the centre of the excircle opposite the vertex A. This excircle is tangent to the side BC at M, and to the lines AB and AC at K and L, respectively. The lines LM and BJ meet at F, and the lines KM and CJ meet at G. Let S be the point of intersection of the lines AF and BC, and let T be the point of intersection of the lines AG and BC. Prove that M is the midpoint of ST.

   (The *excircle* of ABC opposite the vertex A is the circle that is tangent to the line segment BC, to the ray AB beyond B, and to the ray AC beyond C.)

2. Let $n \geq 3$ be an integer, and let $a_2,\ a_3,\ \ldots\ , a_n$ be positive real numbers such that $a_2 a_3 \ldots\ a_n = 1$. Prove that

$$(1 + a_2)^2 (1 + a_3)^3 \ldots (1 + a_n)^n > n^n.$$

3. The *liar's guessing game* is a game played between two players A and B. The rules of the game depend on two positive integers k and n which are known to both players.

   At the start of the game A chooses integers x and N with $1 \leq x \leq N$. Player A keeps x secret, and truthfully tells N to player B. Player B now tries to obtain information x by asking player A questions as follows: each question consists of B specifying an arbitrary set S of positive integers (possibly one specified in some previous question), and asking A whether x belongs to S. Player B may ask as many such questions as he wishes. After each question, player A must immediately answer it with *yes* or *no*, but is allowed to lie as many times as she wants; the only restriction is that, among any $k+1$ consecutive answers, at least one answer must be truthful.

   After B has asked as many questions as he wants, he must specify a set X of at most N positive integers. If x belongs to X, then B wins; otherwise, he loses. Prove that:

   1. If $n \geq 2^k$, then B can guarantee a win.

   2. For all sufficiently large k, there exists an integer $n \geq 1.99^k$ such that B cannot guarantee a win.

Second day (Wednesday, 11 July 2012

4. Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that, for all integers $a, b,\ c$ that satisfy $a + b + c = 0$, the following equality holds:

$$f(a)^2 + f(b)^2 + f(c)^2 = 2f(a)f(b) + 2f(b)f(c) + 2f(c)f(a).$$

(Here $\mathbb{Z}$ denotes the set of integers.)

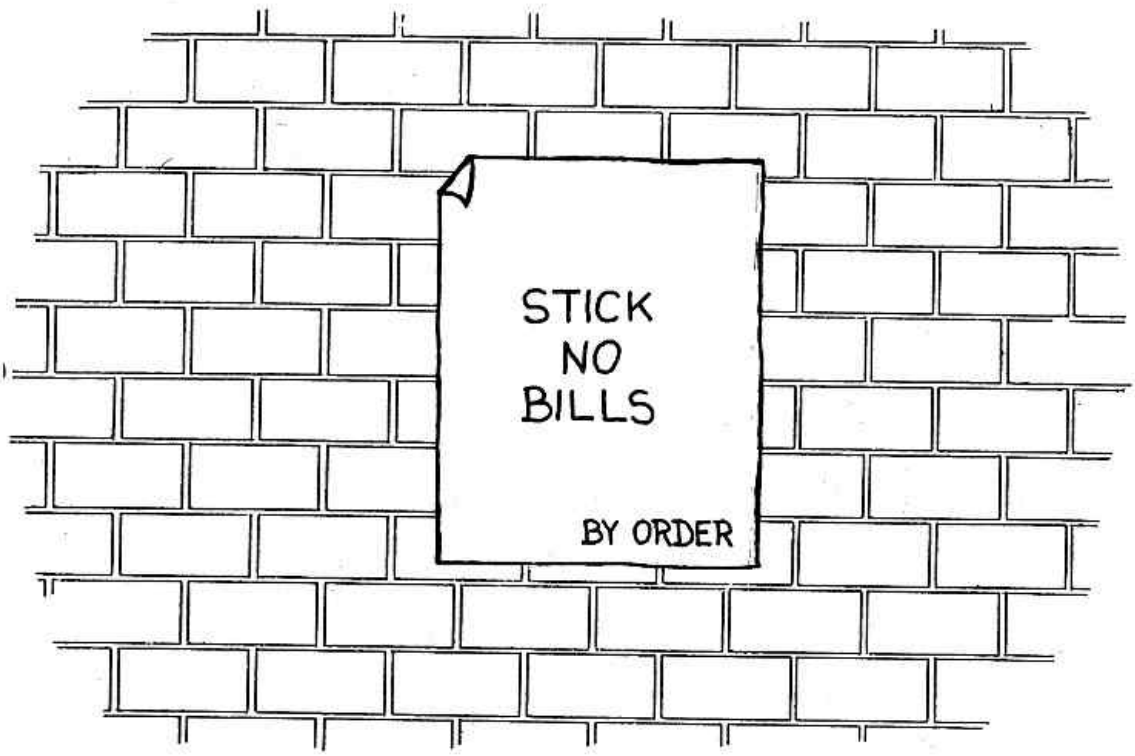
5. Let ABC be a triangle with $\angle BCA = 90°$, and let D be the foot of the altitude from C. Let X be a point in the interior of the segment CD. Let K be the point on the segment AX such that $BK = BC$. Similarly, let L be the point on the segment BX such that $AL = AC$. Let M be the point of intersection of AL and BK.
Show that $MK = ML$.

6. Find all positive integers n for which there exist non-negative integers $a_1,\ a_2,\ \ldots\ ,\ a_n$ such that

$$\frac{1}{2^{a_1}} + \frac{1}{2^{a_2}} + \ldots + \frac{1}{2^{a_n}} = \frac{1}{3^{a_1}} + \frac{2}{3^{a_2}} + \ldots + \frac{n}{3^{a_n}} = 1.$$

□

THE LAW OF SIGNS



□

THE LEAP SECOND BUG

Some of the world's most popular websites crashed on the morning of Sunday 1 July as a result of a "Leap Second" that was added to the world's clocks.

The basic unit of time, the second, was once defined in terms of the Earth's rotation: 24 hours, or 86 400 seconds. Because the Earth wobbles a little, and its spin is gradually slowed down by tidal friction, days are never exactly the same length, and since 1971 the second has been defined in terms of the vibration of the caesium atom: the duration of 9 192 631 770 periods of the radiation corresponding to the transition between the two hyperfine levels of the ground state of the caesium 133 atom. Coordinated Universal Time (UTC) has been the internationally agreed benchmark time since the Seventies.

Every few years an extra second has to be added to the world's clocks to keep them in sync with the Sun. Leap seconds are added at the end of December or June, as needed, and since 1971 there have been 24 of them. The extra pip on the time signal at midnight (GMT) on Saturday 30 June was the latest, and this year it played havoc with the source code of some of the web's most popular sites: Reddit, Meetup, FourSquare, Yelp, LinkedIn, Gawker and StumbleUpon came crashing down. Amadeus, the airline booking system, also crashed, and many Qantas flights were delayed. It was all rather reminiscent of the infamous Y2K Bug, and the more recent embarrassment when Microsoft overlooked the extra day at the end of February this year, and thousands of servers crashed. (See: The Leap Year Bug, *Mathematical Digest* **167**, April 2012.)

One site that did not crash at midnight on June 30 this year was Google, whose engineers foresaw the problem back in 2005 and found an elegant solution. They tweaked their computers to lengthen the second by just a few milliseconds each throughout the last Saturday of June until the milliseconds added up at midnight to an extra second, thus keeping their systems in sync with UTC. Google called their solutions a "Leap Smear".

There have been calls for a different system that will avoid having to make manual changes for computer networks whenever a Leap Second has to be added, but a meeting of the International Telecommunications Union held in January this year failed to agree on a way forward. □

THE GENERALIZED RABBIT AND CARROT PROBLEM

In *Mathematical Digest* **168** (July 2012) Dr Sudan Hansraj (University of KwaZulu-Natal) proposed the following problem:

Prove that

$$\frac{1}{3}+\frac{2}{3}\cdot\frac{1}{5}+\frac{2}{3}\cdot\frac{4}{5}\cdot\frac{1}{7}+\frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\cdot\frac{1}{9}+\cdots+\frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\cdots\frac{98}{99}\cdot\frac{1}{101}<1$$

A wonderfully simple proof, expressed in terms of rabbits nibbling a carrot, was given in the same issue.

Professor Poobhalan Pillay, also of the University of KwaZulu-Natal, notes that the problem is just a particular case of a general result:

Let $\{a_n\}$ be a sequence of real numbers satisfying $0 < a_n < 1$ for all n. Then

$$S_n = a_1 + (1-a_1)a_2 + (1-a_1)(1-a_2)a_3 + (1-a_1)(1-a_2)(1-a_3)a_4 + \ldots \\ + (1-a_1)(1-a_2)\ldots(1-a_{n-1})a_n < 1.$$

The result discussed was the special case $a_n = \dfrac{1}{2n+1}$.

The proof is simple.
Let $c_n = (1-a_1)(1-a_2)(1-a_3)\ldots(1-a_n)$. Then for all n, $S_n + c_n = 1$. The proof follows easily by induction. It is clear that c_n added to the last term of S_n is just c_{n-1}, etc. □

HEISENBERG IN A FERRARI

In a *Top Gear* programme of 2010, Jeremy Clarkson was extolling the joys of driving the Ferrari 458.

His only criticism of the car was a feature of the instrument panel: the speedometer and the satnav system shared the same screen. He complained that at any moment of time you either knew exactly where you are, but not how fast you are going, or how fast you are going without knowing exactly where you are.

Heisenberg would have understood the dilemma. □

VARSITY READINESS TEST

How ready are you for university maths? This test focuses on the maths skills your lecturers at varsity will expect you to have. And that's not just the maths profs: courses in the sciences, engineering, commerce and medicine require a sound mathematical background and all expect a high level of logical thinking, algebraic expertise, geometric insight and numerical skills.

Give yourself two or more hours to tackle the problems, without a calculator or formula sheet. Then turn to page 30 to check your answers. There is no pass/fail mark. Every wrong answer, however, indicates a possible weakness in your mathematical background that needs to be fixed before you arrive on your chosen university campus.

1. The difference between the squares of two consecutive integers is equal to

   (A) the average of the two integers
   (B) the sum of the two integers
   (C) the product of the two integers
   (D) 1
   (E) It depends on the integers.

2. If $f(x)$ is a quadratic function of x such that $f(1) = -4$, $f(2) = 5$ and $f(x) = f(-x)$ for all x, what is the value of $f(3)$?

   (A) 1 (B) 2 (C) 6 (D) 20 (E) 34

3. Two numbers a and b are such that $a^4 < b^3$ and $b^6 < a^5$.
   Which of the following is true?

   (A) $a < 1$ and $0 < b < 1$
   (B) $0 < a < 1$ and $b > 1$
   (C) $a > 1$ and $b > 1$
   (D) $0 < a < 1$ and $0 < b < 1$
   (E) $ab > 1$

4. What is the value of $\dfrac{2\cos 40^\circ - \cos 20^\circ}{\sin 20^\circ}$?

   (A) 2 (B) $\sqrt{2}$ (C) $2\sqrt{3}$ (D) $\sqrt{5}$ (E) 3

5. In a convex quadrilateral $ABCD$, $\angle BAD = 80^\circ$, $\angle ABC = 75^\circ$, $\angle ADC = 65^\circ$ and $AB = AC$. What is the size of $\angle BDC$?

   (A) 10° (B) 15° (C) 20° (D) 25° (E) 30°

6. If a, b, c and d are positive integers and $ac + bd + ad + bc = 91$, what is the value of $a + b + c + d$?

   (A) 20 (B) 36 (C) 68 (D) 75 (E) 92

7. Jacob and Julius throw dice (one each). What is the chance that Jacob will throw a higher number than Julius?

   (A) $\frac{1}{2}$ (B) $\frac{1}{3}$ (C) $\frac{1}{4}$ (D) $\frac{5}{12}$ (E) $\frac{1}{6}$

8. The sequence of natural numbers is divided into groups of three:
   (1, 2, 3), (4, 5, 6), (7, 8, 9) ... .
   What is the sum of the numbers in the 100th group?

   (A) 843 (B) 897 (C) 922 (D) 946 (E) 999

9. The five-digit number 2013A is divisible by 9. What is the remainder when it is divided by 7?

   (A) 3 (B) 4 (C) 0 (D) 5 (E) 1

10. $1 + \dfrac{2}{3 + \dfrac{4}{5 + \frac{6}{7}}}$ is equal to

    (A) $\frac{233}{151}$ (B) $\frac{232}{153}$ (C) $\frac{231}{155}$ (D) $\frac{234}{157}$ (E) $\frac{235}{159}$

11. The area of a circle is decreased by 51%. By what amount is the radius decreased?

    (A) 30% (B) 45% (C) 49% (D) 51% (E) 25.5%

12. The angles of a triangle are in the ratio 2 : 3 : 5. What is the difference between the largest and the smallest angle?

    (A) 9° (B) 18° (C) 36° (D) 45° (E) 54°

13. What is the sum of the first hundred digits of the decimal expansion of $\frac{6}{7}$?

    (A) 398 (B) 453 (C) 560 (D) 583 (E) 601

14. In triangle ABC, points P, Q and R lie on the sides AB, BC and CA, respectively. If $BP = BQ$, $CQ = CR$ and $\angle PQR = 45°$, determine $\angle PAR$.

    (A) 60° (B) 70° (C) 80° (D) 90° (E) 100°

15. Fifty numbers have an average of 76. Forty of these numbers have an average of 80. The average of the other ten numbers is

    (A) 60 (B) 4 (C) 72 (D) 40 (E) 78

16. If $a - b = 3$ and $ab = 7$, then $\frac{1}{a} - \frac{1}{b}$ is equal to

(A) $\frac{1}{3}$ (B) $-\frac{3}{7}$ (C) $\frac{7}{3}$ (D) $-\frac{1}{3}$ (E) $\frac{1}{7}$

17. If $\sin x = \frac{3}{7}$, then $\cos 2x$ is equal to

(A) $\frac{\sqrt{40}}{7}$ (B) $\frac{24}{49}$ (C) $\frac{4}{7}$ (D) $\frac{4\sqrt{10}}{49}$ (E) $\frac{31}{49}$

18. The diagonals AC and BD of quadrilateral $ABCD$ intersect at the point E inside the quadrilateral. If $AD = AE$, $BE = CE$ and $\angle ADE = 30°$, then the measure of $\angle CBE$ is

(A) 50° (B) 65° (C) 70° (D) 75° (E) 80°

19. Water from Tap A will fill a bath in 4 minutes, and water from Tap B will fill the same bath in 6 minutes. How long (in seconds) will it take to fill the bath if both taps are running together?

(A) 120 (B) 125 (C) 128 (D) 132 (E) 144

20. If $3x - 5y = 15$ and $7x - 4y = 23$, then $4x + y$ is equal to

(A) 12 (B) 11 (C) 10 (D) 9 (E) 8

21. Solve for x: $ax^2 + b(a-1)x = b^2$.

(A) $x = a$ or $\frac{1}{b}$ (B) $x = a$ or $\frac{1}{b}$ (C) $x = -b$ or $\frac{b}{a}$
(D) $x = a + b$ or $\frac{a}{b}$ (E) $x = b - a$ or $\frac{a}{b}$

22. In triangle ABC, $\angle A = 40°$. The sides AB and AC are produced to D and E, respectively. The bisectors of $\angle DBC$ and $\angle ECB$ meet in O, outside the triangle. The size of $\angle BOC$ is

(A) 60° (B) 65° (C) 70° (D) 75° (E)80°

23. The solution of the inequality $3^{2x+1} + 3 < 3^{x+2} + 3^x$ is the interval

(A) $(-1, 1)$ (B) $(1, \frac{3}{2})$ (C) $(-\frac{1}{2}, 3)$ (D) $(\frac{1}{2}, 3)$ (E) $(1, 2)$

24. If $\log_{10} 5 = a$ then $\log_{10} 200$ is equal to

(A) $3 - a$ (B) $3 + a$ (C) $3a$ (D) $\frac{3}{a}$ (E) $\frac{a}{3}$

25. If $(3x+1)(4-x)(2+7x)$ is multiplied out to the form ax^3+bx^2+cx+d, then $a + b + c + d$ is equal to

(A) 108 (B) 124 (C) 99 (D) 119 (E) 131

THE MAYAN APOCALYPSE

If you believe in the Infallibility of the Internet, the world will end in December this year. Such predictions are not new, but the latest is based on the calendar of the Mayas, a civilization that dominated Central America until the arrival of the Spanish *conquistadores* in the 16th century.

The Mayans had a complicated calendar, in cycles of both 260 and 365 days, and a 400-year cycle, the *baktun*. On 21 December 2012 the 13th *baktun* of the Mayan calendar will come to an end. Astrologers and numerologists and various flaky New Agers have seized the opportunity to predict the end of the world on that date. For the Mayans such predictions would have been nonsense: in their tradition the end of one *baktun* is simply followed by the beginning of the next.

Why the Mayan calendar cycle should have precedence over others in predicting the End of the World As We Know It is unclear. The most widespread calendar in use today is the Gregorian, which also goes in 400-year cycles. Pope Gregory XIII set it going on Friday 15 October 1582, but nobody predicted gloom and destruction when its first cycle ended on Thursday 14 October 1982.

Mathematical Digest will continue publication into the next *baktun*, so don't forget to renew your subscription. □

THE PUZZLE OF THE SIX SCIENTISTS

Six scientists are on a train trip together. They sit in a compartment facing each other, three to a seat. Each has written a textbook in his or her field, and to pass the time each is reading a textbook by one of the others. No two of them are reading books by the same author.

Professor Adams is reading mathematics in his corner, but strangely enough he detests physics. Dr Croft is reading a book by the person opposite her. Professor Bright is sitting between the mathematician and the chemist, and is reading a biology book. Dr Evans is sitting next to the geologist and reading a chemistry book. Professor Davies is sitting opposite the biologist and reading a geology book. The mathematician is sitting opposite the physicist. Dr Frank is sitting next to the astronomer.

Where were the scientists sitting, and whose book were they reading? The answer is on page 30. □

PAMO PROBLEMS

First day:

1. AB is a chord (not a diameter) of a circle with centre O. Let T be a point on segment OB. The line through T perpendicular to OB meets AB at C and the circle at D and E. Denote by S the orthogonal projection of T onto AB. Prove that $AS \cdot BC = TE \cdot TD$.
2. Find all positive integers m and n such that $n^m - m$ divides $m^2 + 2m$.
3. Find all real solutions x to the equation $[x^2 - 2x] + 2[x] = [x]^2$.
   (Here $[a]$ denotes the largest integer less than or equal to a. For example $[7] = 7$, $[7.3] = 7$ and $[-4.2] = -5$.

AFRICAN·MATHEMATICAL·UNION
Commission for Pan African Mathematics Olympiads

Second day:

1. The numbers
$$\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{2012}$$
   are written on the blackboard. Aicha chooses any two numbers from the blackboard, say x and y, erases them and writes instead the number $x + y + xy$. She continues to do this until only one number is left on the board. What are the possible values of the final number?
2. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that
$$f(x^2 - y^2) = (x + y)(f(x) - f(y))$$
   for all real numbers x and y.
3. (i) Find the angles of $\triangle ABC$ if the length of the altitude through B is equal to the length of the median through C and the length of the altitude through C is equal to the length of the median through B.

   (ii) Find all possible values of $\angle ABC$ of $\triangle ABC$ if the length of the altitude through A is equal to the length of the median through C and the length of the altitude through C is equal to the length of the median through B. $\square$

CALENDAR QUICKIES

1. A certain month has five Thursdays and the third Friday is the 17th. On what day does the 1st of the month fall?

2. An old tombstone in an English churchyard has the inscription

   In Loving Memory of
   Mabel Edith Smith
   Born 24 September 1653
   Died 24 January 1653
   Aged 4 months

   How is that possible?

3. Christmas Day this year falls on a Tuesday. On what day of the week will Christmas day fall in the year 2040?

4. Which of the years

   1800 1850 1900 1950 2000

   was a Leap Year?

5. A biography of the fabled mathematician Giovanni Battista Bombelli says that he was born in Rome on Friday 10 October 1582. Why is that not possible?

Answers are on page 30.

PUNCTURED EXCUSE

Four students missed a maths test one morning, and later that day came to see their lecturer. They said they were on the way to the university, but the car had a puncture and they were not able to arrive in time for the test.

The lecturer told them that he would set them a supplementary test to write the next day.

The first question in the supplementary test was "Which tyre had the puncture?" □

POCKET PROBLEMS

Can you solve these problems with your hands in your pockets?

1. A cube has its height reduced by one-half and its width reduced by two-thirds. Its length is trebled. The volume of the resulting rectangular block is 1372 cubic centimetres. What was the length of the sides of the original cube?
2. If the fraction $\frac{1}{7}$ has the same integer added to both the numerator and the denominator, the fraction is changed to $\frac{3}{5}$. What integer was added?
3. The difference between the square of two consecutive integers is 57. What are the two integers?
4. A triangle has sides of lengths 8, 15 and 17. A circle is drawn through its vertices. What is the radius of the circle?
5. What is the sum of the numbers $1 + 2 + 3 + \cdots + 99 + 100$?

Answers are on page 30.

MATHEMATICAL WORDS

Many mathematical words are in general use in English, with very different meanings.

Which mathematical words are described by the following?

- a type of lettuce
- a conservative old Englishman
- a large piece of timber
- an anaesthetist
- an immoral act
- a party or reception
- first class
- a final bringing together
- a factory output item
- a mixture of liquid and a solid
- dry
- a type of caterpillar
- a draft copy of a manuscript
- a brown colour
- trim or spruce
- crazy
- not original
- bringing races together

Answers are on page 30. □

MATHEMATICS AND COFFEE

Coffee and mathematics have often been said to go together.

The tradition goes back a long way. Coffee houses in London were often meeting places for mathematicians and scientists such as Isaac Newton, Abraham de Moivre, Edmund Halley and Robert Hooke.

In the 1660s a play ran in London called *Tarugo's Wiles,* or, *The Coffee House: A Comedy*, by Thomas Sydserf. In Act 3 there is a conversation between two coffee house customers:-

> **Customer 1**: I'm told Sir, that coffee inspires a man in the mathematics.
>
> **Customer 2**: So far as it keeps one from sleep, which you know is the ready way to distract, consequently the improvement of the mathematics.

The 20th Century Hungarian mathematician Paul Erdös put it more pithily:

"A mathematician is a device for turning coffee into theorems."

Coffee shops are still frequented by mathematicians. Today the coffee is not as important as the availability of free wi-fi and whether there are enough power points for their laptops. □

PARABOLA PROBLEM

The parabola $y = x^2$ is rotated clockwise until its left hand arm cuts the y-axis. Through what angle must the parabola be rotated?

The answer (see page 31) may surprise you. □

DID YOU KNOW?

Charles Dickens based the "How Not To Do It" office in his novel "Little Dorrit" on the experiences of his friend Charles Babbage, the computer pioneer. Babbage had a very frustrating time obtaining government support for his "Analytical Engine". □

ANSWERS

Smart Answers (page 7):

Logic exam answer: "If this is an answer, mark it."

Philosophy exam answer: "What chair?"

Varsity Readiness Test (page 22):
BDDCB ADBEA AEBDD BEDEE CCAAA

Six Scientists (page 25):

Name	Davies	Bright	Frank
Discipline	Chemist	Astronomer	Mathematician
Book	Geology	Biology	Physics

Name	Adams	Croft	Evans
Discipline	Biologist	Geologist	Physicist
Book	Mathematics	Astronomy	Chemistry

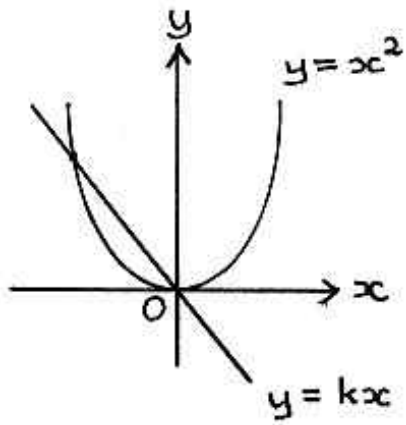
Calendar Quickies (page 27):

1. Wednesday
2. At that time New Year's Day was celebrated on 25 March.
3. Tuesday. The calendar repeats itself exactly every 28 years.
4. Only 2000. 1750 and 1850 are not divisible by 4, and 1800 and 1900 are not divisible by 400.
5. In 1582 the Julian calendar was reformed by Pope Gregory XIII. To correct the errors that had arisen over the centuries, ten days were dropped from the calendar, and October 4th was followed by October 15th.

Pocket Problems (page 28):
(1) 14 cm (2) 8 (3) 28 and 29 (4) 8.5 (5) 5050

Mathematical Words (page 28):
cos, right angle, log, number, sin, function, prime, summation, product, solution, sec, geometer, proof, tan, trig, irrational, derivative, integration.

Parabola Problem (page 29):
You might have thought that the parabola will have to be rotated through at least 45° before its left arm cuts the y-axis, but the surprising fact is that when the parabola is rotated clockwise through *any* angle, however small, its left arm will always cut the y-axis.



This is best seen by thinking about the rotation a little differently. Instead of rotating the parabola clockwise, think of rotating the y-axis anticlockwise, to the position $y = kx$.

The line $y = kx$ and the parabola $y = x^2$ intersect when $kx = x^2$, and this equation has two solutions: $x = 0$ and $x = k$. The second solution is what matters. For a small rotation, the value of k will be large and negative. But no matter how small the rotation, there will always be an intersection.

An alternative argument that any non-zero clockwise rotation will lead to an intersection is that, if a rotation gives an intersection, a slightly smaller rotation will also give an intersection, albeit further up the y-axis. So there is no smallest positive rotation: any positive rotation works. □

THE 2012 UCT MATHEMATICS COMPETITION

The annual report of the University of Cape Town Mathematics Competition has been published. It includes the five competition papers (grades 8 to 12), full solutions, statistical analyses and lists of the prizewinners and top schools.

Copies may be obtained from UCT at R45 each. For details of payment, see the order form at the back of the magazine. qed

ERRATUM

Varsity Readiness Test (July 2012, page 27:
The correct answer to question 9 is (A) $-\dfrac{33}{65}$. □

mathematical digest

No. 170 January 2013

Afghan Abacus

mathematical digest

Number 170 **January 2013**

Physicists like to think that they're dealing with reality. Some of them are quite arrogant about it and talk as if they were the only ones with a finger in the belly of the real. They think mathematicians are just playing games, making up their own rules and playing our own games. But with all their physical theories the possibility still exists that space and time are simply Kant's categories of apperception. ... But in math things are exactly the way they seem. I don't have to consider the possibility that maybe seven isn't really a prime, that my mind conditions seven to appear a prime. One doesn't—can't—make the distinction between mathematical appearance and reality, as one can—must—make the distinction between physical appearance and reality.

(Rebecca Goldstein, *The Mind-Body Problem: A Novel*)

Department of Mathematics and Applied Mathematics
University of Cape Town

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Cover: An Afghan shopkeeper using an abacus.
Photograph: Kevin Kelly

IMO 2014 AT UCT

The 55th International Mathematical Olympiad (IMO) will be hosted by the University of Cape Town in July 2014.

The IMO is the oldest and biggest of all the international science olympiads for high schools. It began in Romania in 1959, and has been held on every continent except Africa.

Over one hundred countries, representing over 90% of the world's population, send teams of six high school students to this prestigious annual event. During the years of apartheid South Africa would not have been welcome at the IMO, but in 1991, with the country moving towards democracy, South Africa was invited to send an Observer to the IMO in Sweden — significantly, a staunch opponent of apartheid. In 1992 the first South African team took part in the IMO in Russia.

The IMO tradition is that the host country pays all the expenses of visiting teams, except for their international travel. So for the last 21 years South African teams have been guests in countries all over the world, from Argentina to Vietnam. It is time to reciprocate the rest of the world's hospitality, and in 2011 an offer to host the IMO in South Africa in 2014 was made and accepted by the IMO Advisory Board.

The South African Mathematics Foundation, the national body responsible for the South African Mathematics Olympiad, then called for bids from cities and universities around the country to host the 2014 IMO, and subsequently selected Cape Town as the host city. The IMO will take place on the campus of the University of Cape Town from 3 to 14 July 2014.

IMO 2014
Cape Town - South Africa

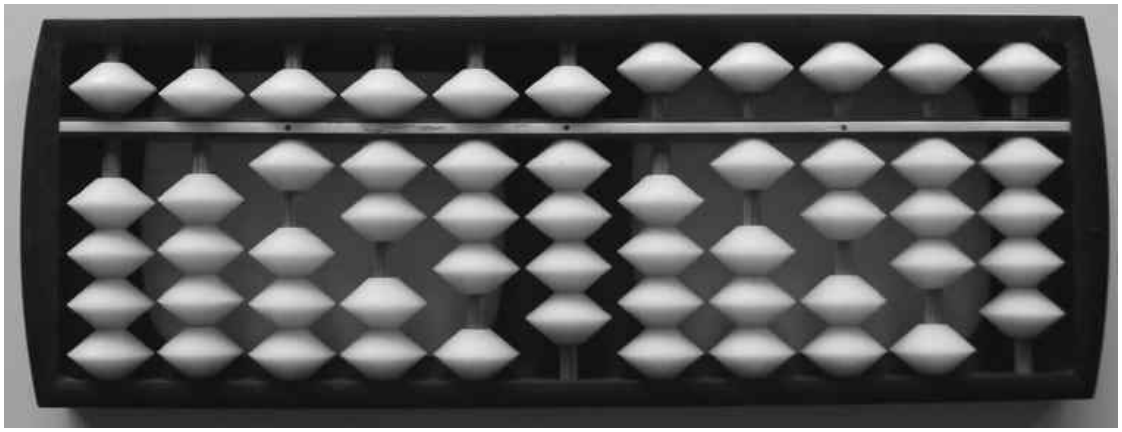
Hosting the IMO is an expensive exercise, costing between ten and fifteen million rands. The usual practice is for governments to cover about 80% of the costs, with the balance from private sources.

At the most recent IMO (in Argentina in July 2012) there were only six African countries: Ivory Coast, Morocco, Nigeria, South Africa and Uganda. One of the important thrusts of IMO2014 will be encouraging more African countries to take part. □

FLASH ANZAN

Japan is the home of Flash Anzan, the world's fastest number game.

Japanese children learn how to do basic arithmetic on the soroban, the Japanese version of the abacus. The picture below shows a soroban, on which the number 00123456789 has been entered.



It does not take much thought to see how addition or subtraction can be performed on a soroban. Beads are simply slid up or down, with the obvious rules of "carrying" or "borrowing" translated into sliding a bead up or down in the next row on the left.

Soroban operations are carried out from left to right, giving children a better appreciation of number size than paper-and-pencil arithmetic.

The soroban is an important part of Japanese culture, and Japanese children attend extra soroban classes after school. Soroban competitions are popular, culminating in an annual *All Japan Soroban Championship*. The high point of the event is Flash Anzan, in which the contestants must add up fifteen three-digit numbers flashed on a screen at high speed. They then have five seconds to write down the answer

Flash Anzan was invented by an abacus teacher Yoji Miyamoto. The twist is that the calculation must be done without an abacus; *anzan* means "imaginary soroban". The contestants visualise adding the numbers as they appear. At they end, they cannot remember any of the numbers, or the partial sums. But they have found the right answer.

That sounds straightforward enough, but the astonishing feature of Flash Anzan is the speed at which the numbers are flashed on the screen: fifteen numbers in under two seconds. That's so fast that the normal person barely has time to read them.

Flash Anzan contests are played in public, and make mathematics a spectator sport. The Flash Anzan World Record for adding fifteen three-digit numbers at speed is a stunning 1.70 seconds, held by Takeo Sasano, a school clerk in his 30s.

Performing arithmetic using an imaginary abacus is the fastest way to perform mental calculations. Naofumi Ogasawara of Japan won last year's Mental Calculation World Cup using this technique, as did the 2010 winner, the 11-year-old Indian calculation genius, Priyanshi Somani.

Neuropsychologists have found that doing abacus calculations uses a different part of the brain from pencil-and-paper arithmetic, which involves memory: knowing the basic number bonds ($3 + 5 = 8$, $9 - 2 = 7$, etc) and being able to recall and use them fluently. Sliding beads on an abacus, whether a real one or just an mental image, is an easier mechanical or visual operation. It has been found that Japanese children can play word games at the same time as doing soroban calculations. Understanding how the human brain does mathematics is a fascinating area of study. □

COSMIC GALL

John Updike

Neutrinos, they are very small.
They have no charge and have no mass
And do not interact at all.
The earth is just a silly ball
To them, through which they simply pass
Like dustmaids down a drafty hall
Or photons through a sheet of glass.
They snub the most exquisite gas,
Ignore the most substantial wall,
Cold-shoulder steel and sounding brass,
Infiltrate you and me! Like tall
And painless guillotines, they fall
Down through our heads into the grass.
At night, they enter at Nepal
And pierce the lover and his lass
From underneath the bed—you call
It wonderful; I call it crass. □

ANOTHER NOBEL PRIZE FOR MATHEMATICS

Alfred Nobel did not consider Mathematics as a suitable category for a Nobel Prize. However, it sometimes happens that a Nobel Prize is awarded for a scientific achievement that is essential mathematical.

A notable example is the award of the Nobel Prize in Medicine in 1979 to the University of Cape Town physicist Alan Cormack (right) for his pioneering work in computer-assisted tomography, also known as CAT scans. Cormack's insight was based on the mathematics of the Radon transform.

For a physicist to win a Nobel Prize in Medicine for a piece of mathematics is most unusual. However, it is not uncommon for the Nobel Prize in Economics to be awarded for an important contribution to the development of mathematical economics. Perhaps the most famous example is the award of the Nobel Prize in Economics in 1994 to John Nash for his work in Game Theory. The story of John Nash's achievement was movingly told in Sylvia Nasar's biography *A Beautiful Mind*, which was later made into a movie of the same name, starring Russell Crowe.

Last year mathematics featured among the Nobel Prizes yet again. The *Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel 2012* was awarded jointly to economist Alvin E. Roth (Harvard University) and mathematician Lloyd S. Shapley (University of California, Los Angeles) for their work on market design and matching theory, which relate to how people and companies find and select one another in everything from marriage to school choice to jobs to organ donations.

According to the Nobel Prize offical press release, the "prize rewards two scholars who have answered these questions on a journey from abstract theory on stable allocations to practical design of market institutions."

Shapley first worked on the theoretical framework for analyzing resource allocation in a paper co-authored with David Gale, which was published under the unlikely title "College Admissions and the Stability of Marriage", in the January 1962 issue of *The American Mathematical Monthly.* Its importance was immediately noticed, and to this day it is one of the journal's most frequently cited articles. □

THE CRICKET THERMOMETER

In 1897 the American physicist Amos Dolbear formulated a law that enables one to work out the temperature by counting the chirps of crickets.

Crickets are cold-blooded creatures, and their physical activity speeds up as the outside temperature rises. The males chirp to attract females by rubbing the ridges of their wings together. Females don't chirp.

The rate at which a cricket's metabolism works is governed by the Arrhenius equation, a basic formula in chemistry which gives the rate of chemical reactions as a function of temperature. In the cricket, chemical reactions that make the chirping muscles contract increase in speed as the outside temperature rises, and the chirping rate increases accordingly.

Dolber studied the snowy tree cricket *Oecanthus fultoni* and came up with a formula by which you can work out temperature by listening to the chirps of crickets. If N is the number of chirps per minute, then the temperature T_F in degrees Fahrenheit is given by the formula

$$T_F = 50 + \frac{N - 40}{4}$$

The formula is accurate to within one degree.

In degrees Celsius (T_C) the corresponding formula is

$$T_C = 18 + \frac{N - 40}{7},$$

which is not quite the same, but is good enough for practical purposes.

Dolbear's formula is specifically for the snowy tree cricket. Other crickets chirp at different rates, each with its own chirp rate formula.

A similar phenomenon has been observed in ants: the rate at which they scurry about is determined by the temperature.

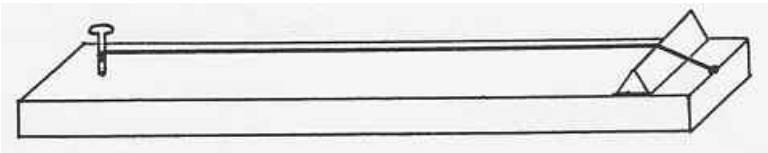
Dolbear's cricket thermometer is today better known than his major pioneering work: he invented the telephone 11 years before Alexander Graham Bell. Unfortunately, he could not prove his claim, and lost his challenge against Bell's patent in the U.S. Supreme Court. □

WHO WAS PYTHAGORAS?

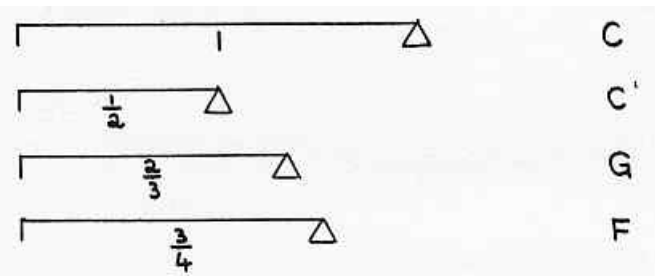
Pythagoras was born in about 580 B.C. on the island of Samos, just off the coast of Asia Minor. To escape the tyrannical rule of Polycrates, Pythagoras migrated to Croton, in the foot of Italy, and there established a semi-secret brotherhood whose ethical and religious principles were based on mathematical ideas.

It is impossible today to separate the teachings of Pythagoras from those of his followers. Nothing he wrote himself has survived, and it was customary for later Pythagoreans to attribute their discoveries to their founder. What seems certain is that the main achievement of Pythagoras was his theory that numbers are of central importance in explaining the world around us. Indeed, it was Pythagoras who formulated and investigated the first exact numerical law of physics.

A *Pythagorean monochord* is a simple musical instrument consisting of a piece of string stretched between two points. One point is fixed, while the second is a movable bridge which enables the length of the string to be altered without altering its tension. When the string is plucked, it emits a musical note.



Suppose that the monochord emits middle C when plucked with the bridge at the extreme right. Slide the bridge in to halve the length of the vibrating string and the monochord will produce top C, an *octave* above. Now move the bridge out again to make the string two-thirds of its original length and the plucked string will produce G, a *fifth* above C. A little further out, with the string three-quarters of its original length, we obtain F, a *fourth* above C.



To the Greek ear these were the fundamental harmonies, and to this day they form the building blocks of Western music. To the Pythagoreans the numbers had more than musical significance, for they formed the *tetraktys*, the holy number 10 ($= 1 + 2 + 3 + 4$). The harmony between nature and number had a mystic force which convinced the Pythagoreans of the truth of their motto "Everything is number".

The mathematical theory of the music was further developed by the Pythagoreans. From the basic note on the monochord a shortening of the string by $\frac{2}{3}$ produces a note a fifth higher. A further shortening by $\frac{3}{4}$ produces a note a fourth higher than that. But $\frac{2}{3} \times \frac{3}{4} = \frac{1}{2}$, and so we see that a fifth plus a fourth gives an octave. On the other had, the difference between a fifth and a fourth will entail dividing the ratios: $\frac{2}{3} \div \frac{3}{4} = \frac{8}{9}$, and shortening the string by $\frac{8}{9}$ produced *a tone* above the base note.

What about the famous theorem which bears Pythagoras' name? Alas, it seems to have been discovered only by later Pythagoreans when mathematical ideas had reached a more advanced stage in Greece. Even this was only a rediscovery, for Babylonian mathematicians had known the result over a thousand years earlier. Nor could the Pythagoreans lay claim to have proved the theorem, for the first general proof was produced in 300 B.C. by Euclid, who was not a Pythagorean. □

WHICH HAND?

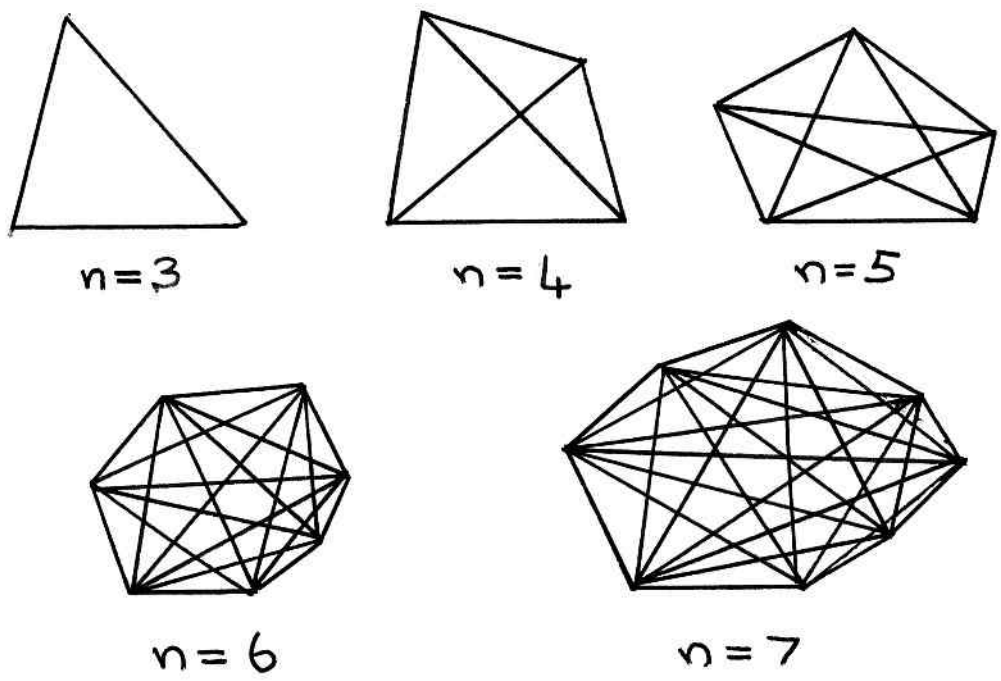
Here is a neat trick to baffle your non-mathematical friends. Ask a friend to conceal a 5c coin in one hand and a 10c coin in the other, without showing you which coin is in which hand. You then ask a simple question:

> "Multiply the amount of money in your
> left hand by 2 or 4 (whichever you
> like) and the amount in your right
> hand by 3 or 5 (again, whichever
> you like), add the two results
> and tell me the answer."

From the answer, you will be able to tell at once which coin was in which hand, by a very simple rule. If the total your friend gives is even, the 5c coin is in his left hand. If the total is odd, the 5c coin is in his right hand. Can your see why the trick works? □

COUNTING TRIANGLES

Draw an n-gon (a polygon with n sides) with all its diagonals. Assume that the polygon is convex and that no three diagonals meet in the same point. How may triangles can be counted in the figure?



If $n = 3$ there is just one triangle.

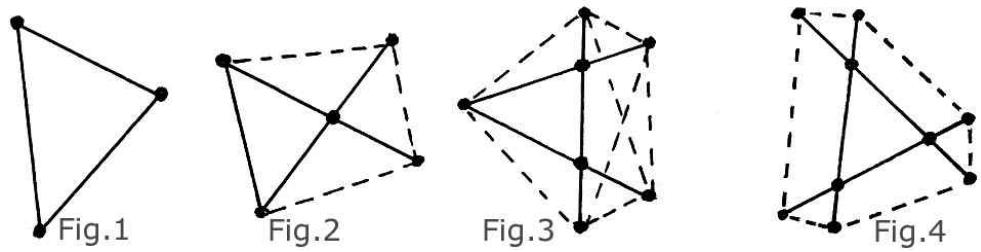
If $n = 4$ we have a quadrilateral and its two diagonals, forming 8 different triangles.

If $n = 5$ the figure is a pentagon with a star inside. Now it's more troublesome to count the triangles, but if you work consistently and carefully, you will find 35.

With $n = 6$ you have a hexagon, and the number of triangles starts getting hectic: there are 111.

With $n = 7$ you have a heptagon, and there are 287 triangles. There is no way they can be counted by hand. A better approach is needed.

Can one find a formula that gives the number of triangles as a function of n? The solution is given by four diagrams.



It's an interesting exercise to stop at this point and try to work out why these four diagrams give the answer. The figures suggest classifying the triangles according to the number of vertices they share with the original polygon.

In figure 1, the triangles being counted share all three vertices with the original polygon. For every choice of three vertices from the polygon, you get one triangle of this type. The number of triangles is therefore $\binom{n}{3}$.

In figure 2 we are looking for triangles that share two vertices with the original polygon. To construct such a triangle, you start with any two vertices of the polygon (joined by an edge or a diagonal of the polygon). The third vertex will be inside the polygon, at the intersection of two diagonals coming from the first two vertices and terminating at two other vertices of the polygon. The number of ways of choosing four such points is $\binom{n}{4}$. Each such choice produces four triangles. So the number of triangles of this type is $4\times\binom{n}{4}$.

In figure 3 we are looking for triangles with one vertex on the polygon, and two vertices inside. To construct such a triangle, start with just one vertex on the polygon, and two diagonals from it. The third side of the triangle is given by another diagonal.

Such a triangle therefore arises from a choice of five vertices, and there are $\binom{n}{5}$ ways of selecting them. Note that each choice generates 5 such triangles. So there are $5\times\binom{n}{5}$ triangles of this type.

In figure 4 we are looking for interior triangles lying inside the original polygon. These arise from three diagonals, with no common endpoints. There are $\binom{n}{6}$ ways of choosing the six points on the polygon, and each choice produces just one interior triangle.

So the total number of triangles is

$$\binom{n}{3} + 4\times\binom{n}{4} + 5\times\binom{n}{5} + \binom{n}{6}$$

$$= \frac{n(n-1)(n-2)(n^3+18n^2-43n+60)}{720}. \qquad \square$$

THE KNOT

The unit of speed of a ship is the *knot*. In the days of sail, a weighted wooden buoy was thrown from the stern of the ship, attached to a rope with knots tied in it every 47 feet 3 inches. A sailor would count the number of knots passing through his fingers during a 28-second interval, as timed with an hour glass.

So a speed of 1 knot was $\dfrac{47.285}{28} = 1.6875$ feet per second.

Since there are 3600 seconds in an hour, and 5280 feet in a mile, a speed of 1 knot is equal to $\dfrac{1.6875 \times 3600}{5280} = 1.15$ miles per hour (to two decimal places).

A *nautical mile* is 1.15 land miles (precisely 6080 feet).

Putting that in metric terms requires one to know that a foot is exactly 30.48 cm, which makes a knot equal to 0.51435 metres per second, or 1.85 km/hr (again, to two decimal places).

Knot are still used today in navigation and in measuring wind speed. □

FIBONACCI UNITS

The Fibonacci numbers

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, \cdots$$

increase without limit, so certainly cannot recur.

But suppose we just look at the units digits:

$$1, 1, 2, 3, 5, 8, 3, 1, 4, 5, 9, 4, 3, 7, 0, \cdots$$

Since there are only ten possibilities for single digits, and therefore only 100 possibilities for successive pairs of digits, at least before 200 the sequence of digits must repeat itself. How long is the sequence of Fibonacci unit digits before they start to recur?

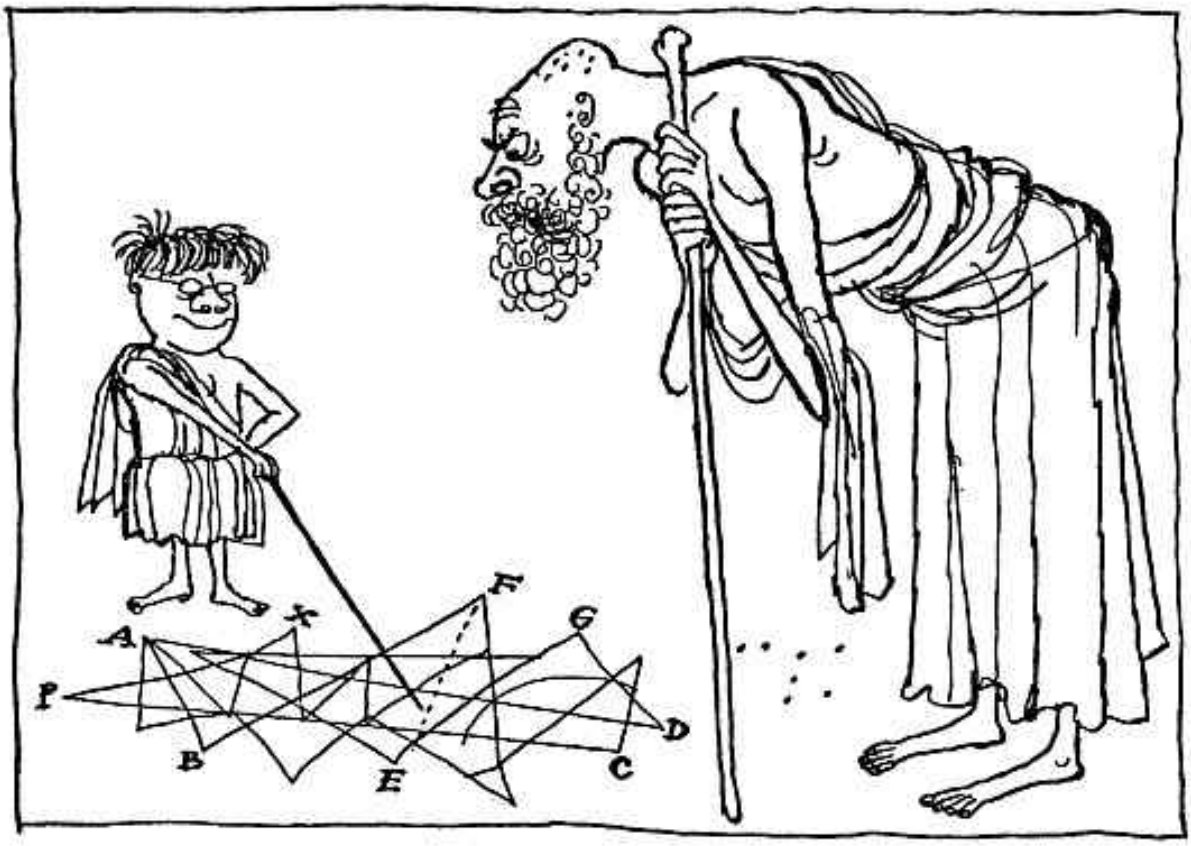
Which numbers are more likely to occur as Fibonacci units?

Answers are on page 28. □

LITERARY QUIZ

A classic account of school life was published in 1953 — just 60 years ago. This is what the hero of the book had to say about Pythagoras.

> Whenever he found a new thing about a triangle Pythagoras who had no shame jumped out of his bath and shouted 'Q.E.D.' through the streets of athens its a wonder they never locked him up.



Pythagoras puzzled by one of my theorems.

- What is the name of the book?
- What is the name of its hero?
- Who wrote the book, and who drew the cartoons?
- How many historical mistakes can you find in the brief paragraph above about Pythagoras?

Answers may be found on page 28. □

BOOK REVIEW

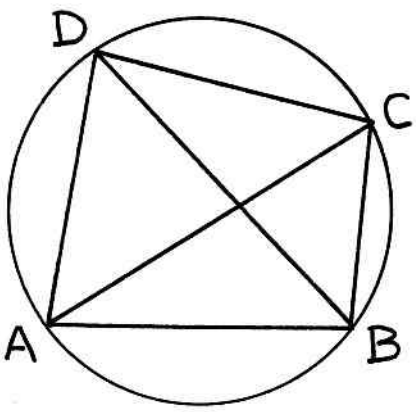
Marcus Bizony

Bishops

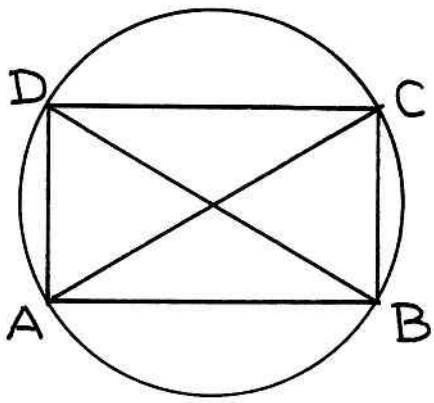
Ptolemy's Theorem through the Looking-Glass, by William Wynne Willson.

William Wynne Willson was a well-known figure among South African maths teachers, visiting this country often to provide training and encouragement, although his main occupation was at Birmingham University. When he died in 2010 he left behind an incomplete manuscript of this book, but with enough notes about the rest to enable his lifelong friend and colleague, Geoff Wain, to finish the work.

Ptolemy's Theorem says that for a cyclic quadrilateral, the product of the diagonals is equal to the sum of the products of the opposite sides. In the special case where the quadrilateral is a rectangle, Ptolemy's Theorem is equivalent to Pythagoras's, so that Ptolemy's result can be seen as a more general form of Pythagoras's, which on its own would make the theorem worthy of respect and study. Indeed, it is only a relatively modern geometry syllabus that fails to include it.



$$AC.BD = AB.CD + BC.DA$$



$$AC^2 = AB^2 + BC^2$$

Willson's aim was to arrive at a proof of the theorem using only transformations, and not the least of the fascinations of this book is the way in which he shows how the standard transformations can all be understood using reflections only. For instance, reflection in successive parallel lines

is equivalent to a translation, successive reflection in a pair of intersecting lines becomes a rotation.

The diagrams are all beautifully clear, large and in full colour, and are frequent enough to keep any reader satisfied. Chapters are short, so that in one sitting one can easily gain an insight and feel one's knowledge and understanding growing. Despite this, the book is slim, but because the material is presented simply and clearly, not because there are gaps left for the reader to fill in on his or her own. As a mathematics teacher I finished each chapter itching to talk it through with a class, because I thought it had something to offer them and a new way of looking at things that they would enjoy. The derivation of the standard angle theorems of circle geometry using purely reflections was certainly new to me, and in these days when so many learners have access to Geometer's Sketchpad or other dynamic software like Cabri (all the diagrams in this book have been produced using Cabri), the book is a wonderful source of ideas for classroom investigation.

In his introduction, Willson says he assumes no knowledge of mathematics other than what one needs to appreciate symmetry, or such as one might use in cookery calculations: one often comes across books that are said to be easily appreciated by layman and professional alike, but this time I found the promise to be true. But if the ideas used are simple, the results obtained are not trivial. A proof of Ptolemy's theorem using a standard Euclidean approach is not difficult, but this work is a fine demonstration of what de Villiers and others call the explicative function of proof, i.e. a way of understanding not only that something true but also what it really means and what implications it might carry. When a proof comes from an unexpected direction, as this one does, so much more does it add to our understanding of maths, rather than just tick the QED box.

Intriguingly, Willson's notes hinted at a further chapter to be called "The rest of trigonometry", but so sketchily that Geoff Wain could not determine what it would have contained. On the basis of what has been completed and made available to us, that is a chapter I am very sad we shall never see.

If your personal mathematics library includes more than just the textbooks you teach from, you should have a copy of this book in it. Certainly your school library should have it, and any students with an interest in geometry should be directed to it. It is published by Garland Publications, finely produced and at a very reasonable price. □

MATHEMATICAL DATES FOR 2013

UNIVERSITY OF CAPE TOWN

University of Cape Town Mathematics Competition
(for schools in the Western Cape)
Deadline for entries: Friday 8 March (by mail)/Friday 15 March (online)
Competition: Wednesday 17 April
Prize Giving: Wednesday 5 June

UCT Mathematics Series
Saturday 9 March
Saturday 17 August

Contact:
Department of Mathematics and Applied Mathematics
University of Cape Town
RONDEBOSCH 7701
Tel: (021) 651 3193 Fax (021) 650 5112
Email `john.webb@uct.ac.za`
Website `www.mth.uct.ac.za/competition`

NATIONAL

South African Mathematics Olympiad
First round: Thursday 14 March
Second Round: Tuesday 14 May
Third Round: Monday 9 September

Inter-Provincial Mathematics Olympiad
Saturday 7 September

Contact:
South African Mathematics Foundation
Private Bag X173
PRETORIA 0001
Tel: 012 392 9372 Fax 012 392 9312
Email: `info@samf.ac.za`
Website: `www.samf.ac.za`

South African Computer Applications Olympiad
First round: Friday 10 May (in local schools)
Final Round: Tuesday 25 June (in Cape Town)
Awards Function: Wednesday 26 June (in Cape Town)

South African Computer Programming Olympiad
First Round: Friday 26 July (in local schools)
Final Round: Saturday and Sunday 28 and 29 September (in Cape Town)
Awards Function: Monday 30 September (in Cape Town)

Contact:
Computer Olympiad
P.O Box 13013
MOWBRAY 7705

Tel: 021 448 7864 Fax: 021 447 8410
Website `www.olympiad.org.za`

INTERNATIONAL

Pan African Mathematics Olympiad
Abuja, Nigeria 23 June–2 July
Website: `www.pamo-official.org`

International Olympiad in Informatics
Brisbane, Australia 6–13 July 2013
Website: `www.ioi2013.org`

International Mathematical Olympiad
Santa Marta, Colombia, 17–28 July
Website: `www.imo-official.org/imo2013`

Dates and contact details are correct at the time of going to the press. □

BIRD BRAINS

A group of scientists went birdwatching. They went into a hide and watched a nest for some time. No birds emerged from the nest and they concluded it was empty.

Just then two birds flew into the next. A little later the watchers were very surprised when three birds flew out of the nest.

The experimental physicist said "This is a case of observational error."

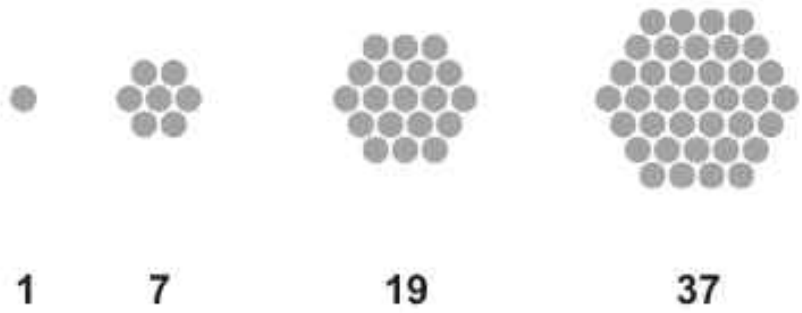
The theoretical physicist said "There has clearly been a quantum state change."

The biologist said "They could have been breeding inside."

The mathematician said: "If a bird flies in now, there will be no birds in the nest." □

HEXAGONAL NUMBERS REVISITED

Hexagonal numbers give the number of dots in a hexagonal array.



In *Mathematical Digest* **169** (October 2012) we gave three ways of deriving the formula

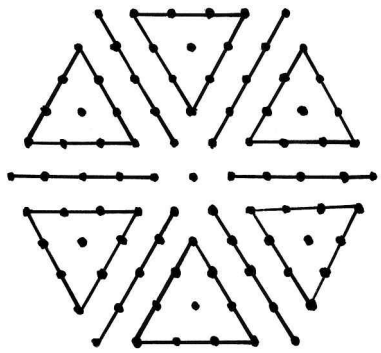
$$3n^2 - 3n + 1$$

for the nth hexagonal number.

In a discussion of hexagonal numbers in the UCT Mathematical Circle, Sanjiv Ranchod (Westerford High School) suggested another way of deriving the same formula.

Start with a hexagon with n dots along each side (in the diagram $n = 6$), and divide it into

- six equilateral triangles with $n - 2$ dots along each side: a total of $6 \times \frac{1}{2}(n-2)(n-1) = 3n^2 - 9n + 6$ dots
- six radial lines of $n - 1$ dots each: a total of $6(n-1)$ dots
- one central dot.



The total number of dots in the hexagon is

$$3n^2 - 9n + 6 + 6n - 6 + 1 = 3n^2 - 3n + 1.$$

□

FIND FOUR CUBES

Find four perfect cubes such that when 24 is added to each of them the results are perfect squares. In other words, find integers m and n such that

$$m^2 = n^3 + 24.$$

Three of the cubes are easy to find. The fourth might require some serious computer work. The full solution is given on page 28. □

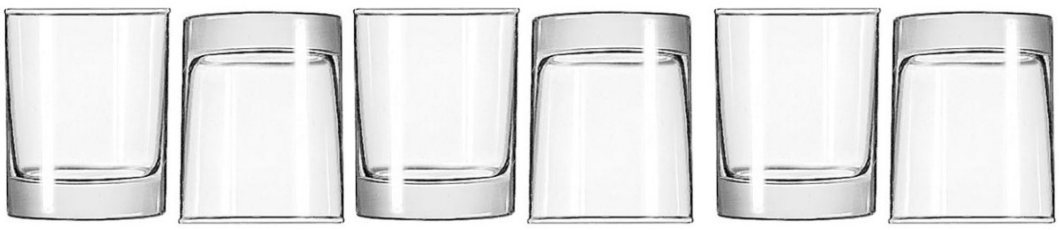
WASHINGTON QUOTE

The science of figures, to a certain degree, is not only indispensably requisite in every walk of civilised life, but the investigation of mathematical truths accustoms the mind to method and correctness in reasoning, and is an employment truly worth of rational beings. In a clouded state of existence, where so many things appear precarious to the bewildered research, it is here that the rational facilities find a firm foundation to rest upon. From the high ground of mathematical and philosophical demonstration, we are insensibly led to far nobler speculations and sublimer meditations.

(George Washington, in a letter dated 20 June 1788) □

TUMBLING TUMBLERS

Place six drinking tumblers in a row, three up and three down.



Take two tumblers, one in each hand, and turn them over. Then do the same with another pair of tumblers. Continue turning pairs of tumblers over as long as you like. How can you arrange that all six tumblers are the same way up?

The answer is on page 29. □

GEOMETRY READINESS TEST

How ready are you for university maths? This test focuses on the geometrical skills your lecturers at varsity will expect you to have. And that's not just the maths profs: courses in physics, chemistry and engineering all expect a high level of spatial insight, geometric knowledge and a sound grasp of the techniques of trigonometry.

Give yourself two or more hours to tackle the problems below, without a calculator or formula sheet. Then turn to page 30 to check your answers. There is no pass/fail mark. Every wrong answer, however, highlights a weakness in your mathematical background that needs to be fixed before you arrive on your chosen university campus.

1. In triangle ABC, D lies on BC such that AD is perpendicular to BC. If $AB = 19$, $BC = 15$ and $CA = 16$, then $BD - DC$ is equal to

   (A) $5\sqrt{2}$ (B) $4\sqrt{3}$ (C) $\sqrt{47}$ (D) $4\sqrt{51}$ (E) 7

2. In a convex quadrilateral $ABCD$, $\angle BAD = 80°$, $\angle ABC = 75°$, $\angle ADC = 65°$ and $AB = AC$. What is the size of $\angle BDC$?

   (A) 10° (B) 15° (C) 20° (D) 25° (E) 30°

3. $\sin 15°$ is equal to

   (A) $\frac{1}{4}$ (B) $\frac{1}{\sqrt{2}} - \frac{1}{2}$ (C) $\frac{\sqrt{6}-\sqrt{2}}{4}$ (D) $\frac{\sqrt{3}}{8}$ (E) $\sqrt{3} - 1$

4. A pyramid has a square base with sides of length a and the four sloping edges have length b. What is the height of the pyramid?

   (A) $\sqrt{b^2 - a^2}$ (B) $\sqrt{b^2 - a^2/2}$ (C) $\sqrt{b^2 - a^2/3}$ (D) $\sqrt{b^2 - a^2/5}$
   (E) $b - a/\sqrt{2}$

5. Triangle ABC is inscribed in a circle. If $\angle BAC = 30°$, BC $= 10$ and $AB = 12$, what is the diameter of the circle?

   (A) 15 (B) $10\sqrt{2}$ (C) 20 (D)$10\sqrt{3}$ (E) 25

6. If A and B are two points in the plane, the set of all points P such that AP is perpendicular to BP lie

   (A) on a straight line
   (B) on a parabola
   (C) on a hyperbola
   (D) on a circle
   (E) on two straight lines

7. In triangle ABC, $\tan A = \frac{3}{4}$ and $\sin B = \frac{12}{13}$. The value of $\cos C$ is

(A) $\dfrac{33}{65}$ (B) $\dfrac{16}{65}$ (C) $\dfrac{17}{65}$ (D) $\dfrac{41}{65}$ (E) $\dfrac{28}{65}$

8. If $\cos(A+B) = \frac{3}{5}$ and $\cos(A-B) = \frac{4}{5}$, then $\tan A + \tan B$ is equal to

(A) $\frac{15}{13}$ (B) $\frac{16}{15}$ (C) $\frac{19}{11}$ (D) $\frac{4}{3}$ (E) $\frac{8}{7}$

9. The area of a circle is decreased by 51%. By what percentage is the radius decreased?

(A) 30% (B) 45% (C) 49% (D) 51% (E) 25.5%

10. The angles of a triangle are in the ratio $2:3:5$. What is the difference between the largest and the smallest angle?

(A) 9° (B) 18° (C) 36° (D) 45° (E) 54°

11. In triangle ABC, points P, Q and R lie on the sides AB, BC and CA, respectively. If $BP = BQ$, $CQ = CR$ and $\angle PQR = 40^\circ$, determine $\angle PAR$.

(A) 60° (B) 70° (C) 80° (D) 90° (E) 100°

12. $\sin 15^\circ \cos 15^\circ$ is equal to

(A) $\tan 15^\circ$ (B) $\tan 30^\circ$ (C) $\dfrac{\sqrt{3}}{4}$ (D) $\dfrac{1}{\sqrt{2}}$ (E) $\dfrac{1}{4}$

13. A computer hard drive spins at 7200 rpm. What is its angular velocity in degrees per second?

(A) 43 200 (B) 54 400 (C) 36 600 (D) 48 480 (E) 24 360

14. If $\tan A + \dfrac{1}{\cos A} = 2$, then $\cos A$ is equal to

(A) $\dfrac{6}{11}$ (B) $\dfrac{5}{9}$ (C) $\dfrac{3}{7}$ (D) $\dfrac{4}{5}$ (E) $\dfrac{7}{8}$

15. Triangle ABC is right-angled at C and D lies on BC. If $AB = 14$, $AD = 10$ and $CD = 5$, the perimeter of triangle ABD is

(A) $24 + 5\sqrt{2}$ (B) $24 + 3\sqrt{3}$ (C) 29 (D) 30 (E) 31

16. The diagonals AC and BD of quadrilateral $ABCD$ intersect at the point E inside the quadrilateral. If $AD = AE$, $BE = CE$ and $\angle ADE = 2x^\circ$, then the measure of $\angle CBE$ is

(A) $45^\circ - x^\circ$ (B) $90^\circ + \frac{1}{2}x^\circ$ (C) $90^\circ - \frac{1}{2}x^\circ$ (D) $45^\circ + 2x^\circ$ (E) $90^\circ - x^\circ$

17. If $\cos A = \frac{3}{4}$, then $\cos 3A$ is equal to

(A) $-\frac{9}{16}$ (B) $\frac{9}{4}$ (C) $-\frac{7}{12}$ (D) $\frac{2}{15}$ (E) $\frac{9}{16}$

18. In triangle ABC, D lies on AC and $\angle ADB = \angle ABC$. If $AD = 4$ and $CD = 5$, then AB is equal to

(A) $3\sqrt{3}$ (B) 7 (C) $4\sqrt{2}$ (D) 6 (E) $2\sqrt{10}$

19. If $\sin x = \dfrac{7}{11}$, then $\cos 2x$ is equal to

(A) $\dfrac{\sqrt{73}}{121}$ (B) $\dfrac{8}{119}$ (C) $\dfrac{4}{11}$ (D) $\dfrac{\sqrt{77}}{11}$ (E) $\dfrac{23}{121}$

20. Town A is 13 km west of Town B. Town B is 4 km south of Town C. Town D is 3 km west of Town C.

The distance from A to D is

(A) between 9 and 10 km
(B) between 10 and 11 km
(C) between 11 and 12km
(D) between 12 and 13 km
(E) between 13 and 14 km

21. A rectangle is made up of three rows of four squares each. The area of each square is 4. What is the perimeter of the rectangle?

(A) 28 (B) 30 (C) 36 (D) 40 (E) 48

22. Let P be the area of a triangle with sides of length 25, 25 and 30. Let Q be the area of a triangle with sides of length 25, 25 and 40. What is the relationship between P and Q?

(A) $P = \frac{9}{16}Q$ (B) $P = \frac{3}{4}Q$ (C) $P = Q$ (D) $P = \frac{4}{3}$ (E) $P = \frac{16}{9}Q$

23. In triangle ABC, $\angle A = 50°$. The bisectors of $\angle B$ and $\angle C$ meet in O, inside the triangle. The size of $\angle BOC$ is

(A) 100° (B) 105° (C) 110° (D) 115° (E)120°

24. A right-angled isosceles triangle has an area of 8. The perimeter of the triangle is

(A) $8 + 4\sqrt{2}$ (B) $4 + 4\sqrt{2}$ (C) 16 (D) 12 (E) $8\sqrt{2}$

25. Triangle ABC is right-angled at B. If $AB = 5n$ and $BC = 12n$, where n is a positive integer, which of the following is not a possible length for AC?

(A) 39 (B) 69 (C) 91 (D) 104 (E) 143

Answers are on page 29. □

SAMO MEDALLISTS

The results of the 2012 South African Mathematics Olympiad were announced at a gala function in Durban on 13 October 2012.

JUNIOR WINNERS

Dawie du Toit Gold Medal:
Lourens van Niekerk (grade 9, Hoërskool Diamentveld)

Runners-up (in alphabetical order):
Ulrik de Muelenaere (grade 9, Hoërskool Waterkloof)
Sam Jeffrey (grade 9, Bishops)
Nic Kroon (grade 8, St Andrew's College)
Soo Min Lee (grade 9, Bishops)
Simon Maenaut (grade 9, Crawford College, Pretoria)
Tae Jun Park (grade 9, Rondebosch Boys' High School)
Louis Schoonwinkel (grade 9, Grey College)
Roger Song (grade 9, Reddam House, Bedfordview)
Michael Thompson (grade 9, Bishops)
David Wright (grade 8, St Stithian's College)

SENIOR WINNERS

Dirk van Rooy Gold Medal:
Dylan Nelson (grade 12, Benoni High School)

Runners-up (in alphabetical order):
Nicholas Cheng (grade 12, Fairmont High School)
Mickey Chew (grade 12, Elkanah House)
Nashlen Govindasamy (grade 10, Star College)
Shaylan Lalloo (grade 11, Pearson High School)
Lloyd Mahadeo (grade 11, Star College)
Martin Rencken (grade 12, Wartburg Kirchdorf High School)
Robert Spencer (grade 12, Westerford High School)
Dalian Sunder (grade 12, Star College)
Robin Visser (grade 10, St George's Grammar School)

See page 14 for information about the 2013 South African Mathematics Olympiad □

COMPUTER OLYMPIAD RESULTS

The final round of the 2012 Computer Programming Olympiad was written in Cape Town over the weekend of 6/7 October, and the results were announced at a gala dinner on 8 October.

Gold Medal:
Robert Spencer (grade 12, Westerford High School).

Silver Medals:
Stephen Barnes (grade 12, Pearson High School)
Paul le Roux (grade 12, Parel Vallei High School).

Bronze Medals:
Johannes Gericke (grade 12, De Kuilen High School)
Guy Paterson-Jones (grade 11, Bishops)
Shaylan Lalloo (grade 11, Pearson High School)

Runners-up:
Mohammed Ikram (grade 12, Orient Islamic School)
Dawood Kamdar (grade 12, Orient Islamic School)
Sean Markham (grade 12, Michaelhouse)
Darren Roos (grade 10, Pretoria Chinese School)
Renette Ros (grade 12, Oos-Moot High School)
Michael-Jon Rosslee (grade 12, St Andrew's College, Grahamstown)
Reuben Steenekamp (grade 10, Reddam House, Constantia)
Grant Zietsman (grade 12, Pretoria Boys' High School)

Information about the 2013 Olympiad can be found on page 15. □

THE EDDINGTON NUMBER

Arthur Eddington (1882–1944) was a British astrophysicist of the early 20th century, famous for his work on the theory of relativity. He led an expedition to observe the Solar eclipse of 29 May 1919, which provided one of the earliest confirmations of relativity, and became known for his popular expositions and interpretations of the theory.

Eddington was a keen cyclist who devised the Eddington number, which records the number of days in your life on which you have cycled at least the same number of miles. His own E number the year before he died was 77. (In other words, on 77 days he had cycled more than 77 miles). □

THE 2012 INTER-PROVINCIAL MATHEMATICS OLYMPIAD

Western Province teams were impressive in the 2012 Inter-Provincial Mathematics Olympiad, which took place on Saturday 29 September. Fifty teams in different parts of the country battled it out for top honours.

The Olympiad is in two divisions: Junior (up to grade 9) and Senior (grades 10, 11 and 12).

Provincial teams of ten are chosen on the basis of their performance in local maths competitions, the first two rounds of the unational Mathematics Olympiad and other local selection tests.

Two papers are set. The first is individual, with fifteen short but challenging problems to be solved in one hour. Then the team gets together to tackle a set of ten tough brainstrainers. Here teamwork is essential, as no single team member would be able to crack all ten problems in just one hour.

While the teams enjoy refreshments and compare notes, their scores are totted up and emailed to the Olympiad coordinator. Within minutes the final rankings come back and are announced to the eagerly waiting teams.

In the Junior division, the top team was Free State/Northern Cape, with 1478 points, followed by Western Province (1377) and Gauteng North (1336).

In the Senior division, the top team was KwaZulu-Natal, with 1398 points, followed by Western Province (1354) and Gauteng South (1130).

Adding the Junior and Senior scores saw Western Province come out top with 2731 points, comfortably ahead of KwaZulu-Natal (2537), Free State/Northern Cape (2472), Gauteng North (2452) and Gauteng South (2334).

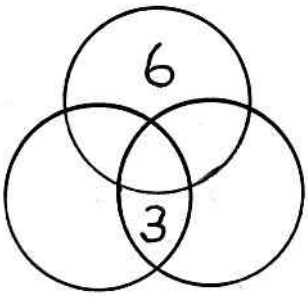
The Western Province teams were selected and trained as part of the Mathematics Outreach programme of the University of Cape Town.

The 2013 Inter-Provincial Mathematics Olympiad will take place on Saturday 7 September.

The Inter-Provincial Mathematics Olympiad is organized by the South African Mathematics Foundation and sponsored by the Actuarial Society of South Africa. □

SEVENN PROBLEM

Place the seven numbers 1, 2, 3, 4, 5, 6, and 7 in the seven parts of the Venn diagram shown, so that the four numbers in each circle add up to 7.



The solution is on page 29.

□

BISTROMATHICS

Bistromathics itself is simply a revolutionary new way of understanding the behaviour of numbers. Just as Einstein observed that space was not an absolute but depended on the observer's movement in space, and that time was not an absolute, but depended on the observer's movement in time, so it is now realized that numbers are not absolute, but depend on the observer's movement in restaurants.

(Douglas Adams) □

RELATIVITY IN VERSE

Arthur Eddington was the astrophysicist who in 1919 verified Einstein's Theory of Relativity by measuring the bending of the light of a star by the sun during an eclipse. Later he wrote the following quatrain, in imitation of Omar Khayyam.

Oh leave the Wise our measures to collate
One thing at least is certain, LIGHT has WEIGHT
One thing is certain, and the rest debate -
Light-rays, when near the Sun, DO NOT GO STRAIGHT. □

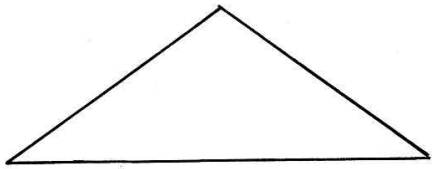
QI QUOTE

Stephen Fry: How may grains of sand, then, in the Sahara, d'you reckon?

Alan Davies: I lost count. It's quite a few. I got up to 17 and it's definitely more. □

OBTUSE DISSECTION

How can you cut up an obtuse-angled triangle into acute-angled triangles?



The problem is surprisingly tricky. What is the smallest number of acute-angled triangles needed?

See page 29 for a solution. □

MAD MEASURES

The *bananosecond* is the unit used for measuring accidents. It is the time between stepping on a peel and hitting the ground.

The unit of courage is the *colon*. A true hero has courage rated at one colon. Somebody with only half that amount of guts is rated a *semicolon*.

The amount of beauty which would launch a thousand ships is a *helen*. A *millihelen* would be enough to launch one ship.

The impact of an evangelist is measured in *grahams*. A *billigraham* would save a billion souls.

The *hoarsepower* is the basic unit of laryngitis.

The unit of pain is the *hurtz*.

A thousand cc (copies) of leather trousers is one *literhosen*. □

DOES MATHEMATICS EXIST?

It has been sufficiently pointed out that the objects of mathematics are not substances in a higher sense than bodies are, and that they are not prior to sensibles in being, but only in formula, and that they cannot in any way exist separately. But since they could not exist *in* sensibles either, it is plain that they either do not exist at all or exist in a special way and therefore do not exist without qualification. For 'exist' has many senses.

(Aristotle, in *The Metaphysics*) □

DIAMOND IN THE SKY

On 12 October 2012 the *Pretoria News* published a story about a new planet "made largely out of diamond", orbiting a star 40 light years away. The report said that the planet's "radius is twice that of Earth's but it is much more dense, with a mass eight times greater". The *Cape Times* carried the same story on its front page on the same date.

Henry Thackeray alerted us to the report, and commented "If the planet's radius is twice that of Earth and its mass is eight times that of Earth, then the planet's density and Earth's density are the same (assuming the planet and Earth are spherical) – so the planet is not "much more dense" than Earth."

Perhaps the reporter confused density and mass. But even then there was more dodgy science in the report. Since the specific gravity of diamond is 3.52 and that of the Earth is 5.5, the planet cannot be "made largely out diamond" and also have a "mass eight times greater." □

MARXIST MATHEMATICS

At last we see clearly what mathematicians have claimed for a long time, without being able to present rational grounds, that the differential-*quotient* is the original, the differentials dx and dy are derived: the derivation of the formulae demands that both so-called irrational factors stand at the same time on one side of the equation, and only if you put the equation back into this its first form $dy/dx = f'(x)$, as you can see, are you free of the irrationals and instead have their rational expression.

The thing has taken such a hold of me that it not only goes round my head all day, but last week in a dream I gave a chap my shirt - buttons to differentiate, and he ran off with them.

(Friedrich Engels, in a letter to Karl Marx,10 August 1881) □

LASER POWER

In the early days of the development of the laser, its power was measured in *Gillettes*: the number of razor blades the laser could burn through. Early lasers were in the 2–4 Gillette range. □

VARSITY MATHS OLYMPIAD

The first edition of a new maths Olympiad for university students, the South African Tertiary Mathematics Olympiad, was held on the afternoon of Saturday 6 October last year. A total of 142 undergraduates from 11 universities took part: University of Cape Town, University of the Free State, University of Johannesburg, University of KwaZulu-Natal, University of Limpopo, University of the North-West, Rhodes University, University of South Africa, University of Stellenbosch, University of Pretoria and the University of the Witwatersrand.

Meeting in their respective universities, the students tackled a 90-minute paper consisting of 20 problems. Though none of the problems required more than first-year university maths, most were quite challenging. Only answers were required, which meant that the papers could be quickly marked and the marks collated.

At the end of the afternoon a clear winner emerged: Henry Thackeray (University of Pretoria), with an excellent score of 18 correct answers. The next three places were taken by University of Cape Town students: Liam Baker (13), Sean Wentzel (12) and Ashraf Moolla (11). Then came a three-way tie on 9 points: Greg Jackson (University of Cape Town), Desi Nikolov (University of the Free State) and François van Niekerk (University of Stellenbosch). The top ten was completed with another three-way tie: Mark Chimes (University of Stellenbosch), A Vlasov (University of Pretoria) and Thomas Weighill (University of Stellenbosch) scored 7 points.

The Olympiad was organized by Professor Stephan Wagner (University of Stellenbosch).

The question paper, full solutions and results can be found at
`http://math.sun.ac.za/ swagner/TMO/scores.html`. □

DON'T UNDERESTIMATE THE ABACUS

"Oh, now, don't underestimate the abacus," said Reg. "In skilled hands it's a very sophisticated calculating device. Furthermore it requires no power, can be made with any materials you have to hand, and never goes bing in the middle of an important piece of work."

(Douglas Adams, in *Dirk Gently's Holistic Detective Agency*) □

ANSWERS

Fibonacci Units (p. 10):
The first 60 Fibonacci units are given below.

1	1	2	3	5	8	3	1	4	5
9	4	3	7	0	7	7	4	1	5
6	1	7	8	5	3	8	1	9	0
9	9	8	7	5	2	7	9	6	5
1	6	7	3	0	3	3	6	9	5
4	9	3	2	5	7	2	9	1	0

Since the 59th and 60th numbers in the list are 1 and 0, the sequence carries on with 1, 1, 2, 3, ... , showing recurrence.

Several interesting features can be seen in this sequence.

- Every fifth digit is five, which means that ever fifth Fibonacci number is divisible by 5.
- The numbers appear in the pattern odd, odd, even, odd, odd, even.
- All the digits from 0 to 9 occur.
- Every even number occurs four times in the list, and every odd number eight times.

What other patterns can you find among the Fibonacci units?

Literary Quiz (page 11):
The book is *Down with Skool*, written by Geoffrey Willans with cartoons by Ronald Searll, and its hero is Nigel Molesworth. It was Archimedes (not Pythagoras) who, on discovering his law of hydrostatics (nothing to do with triangles) jumped out of his bath and ran down the streets of Syracuse (not Athens) shouting 'Eureka!' (not Q.E.D.). Never mind the grammatical errors.

Four Cubes (page 17):
The four cubes that are 24 less than perfect squares are 1, -8, 1000, and $542\ 939\ 080\ 312 = 8158^3$. Add 24 to each and you get $25 = 5^2$, $16 = 4^2$, $1024 = 32^2$ and $542\ 939\ 080\ 336 = 736\ 844^2$.

Tumbling Tumblers (page 17):
It is not possible to turn the tumblers over two at a time and get them all the same way up.

We use the notation (m, n) to describe the state of the tumblers at any stage: m tumblers UP and n tumblers DOWN. The initial state is $(3, 3)$: m and n are both odd.

If two UP glasses are turned over, the new state is $(1, 5)$.
If two DOWN tumblers are turned over, the state changes to $(5, 1)$.
If one UP tumbler and one DOWN tumbler are turned over, the state remains $(3, 3)$.
In all cases, the new state (m, n) has both m and n odd.

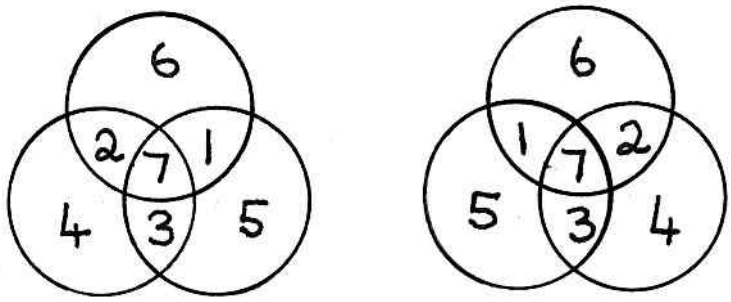
Now consider what happens when two tumblers in the state $(1, 5)$ are turned over. If one UP and one DOWN tumbler are turned over, the new state will be the same. If two UP tumblers, or two DOWN tumblers are turned over, the state reverts to $(3, 3)$.

After every turn, the state (m, n) has both numbers odd.

So no sequence of turns can produce the state $(6, 0)$, with all six tumblers up.

Geometry Test (page 18):
EBCBC DBEAE EEADD EADEB ACDBB

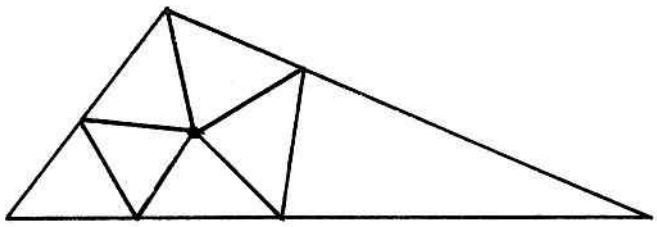
Sevenn Problem (page 24):
Two possible solutions are shown.



Obtuse Dissection (page 25):
At least seven acute-angled triangles are needed in the dissection.

Start with the obtuse angle. This must be divided into two acute angles by a line. However, that line cannot be drawn to the opposite side of the

triangle, for then it would cut the original triangle into two triangles, at least one of which would have an angle of 90° or more. Since we are trying to find a dissection into the the smallest number of acute-angled triangles, the line must terminate in the interior of the triangle.



From there, at least five lines must be drawn, otherwise at least one of the angles would be 90° or more. Drawing these five lines to the sides of the triangle produces a dissection of the original triangle into seven acute-angled triangles.

DID YOU KNOW?

If you toss a coin $2n$ times, where n is large, then the probability of getting exactly n heads and n tails is approximately $\dfrac{1}{\sqrt{n\pi}}$.

The movie *Slumdog Millionaire* was released in India under the name *Slumdog Crorepati*. *Crore* is the Indian-English word for ten million. Indians use *ten lakh* for a million, where *lakh* means one hundred thousand. □

GETTING THE POINT

A system of measuring the size of type font was devised in 18th-Century France by François-Ambrose Didot, and the basic unit was named the *point Didot*, in his honour. It was defined as $\frac{1}{72}$ of a royal inch (which amounts to 0.3759 mm). A measure of 12 points equals one *Cicero*.

This unit was subsequently used as the standard for the Apple Macintosh computer (72 pixels per inch). Sometimes referred to as the *Postscript point*, it has become standard in digital typography. □

THE GOOD OF COUNTING

JOHNSON: Were I a county gentleman, I should not be very hospitable. I should not have crowds in my house.

BOSWELL: Sir Alexander Dick tells me that he remembers having a thousand people in a year to dine at his house: that is, reckoning each person as one, each time that he dined there.

JOHNSON: That, Sir, is about three a day.

BOSWELL: How your statement lessens the idea.

JOHNSON: That, Sir, is the good of counting. It brings every thing to a certainty, which before floated in the mind indefinitely.

(From: *The Life of Johnson*, by James Boswell) □

MATHEMATICAL PIE

Fidget pie is a traditional English dish coming from the Midlands. It was a working man's meal, equivalent to the Cornish pasty. There are many regional variations, but the main ingredients are gammon or ham, apples, onions and potatoes. Recipe books say that the name of the pie comes from the fact that it was originally 'fitched' or five-sided in shape. □

EUCLID

Vachel Lindsay

Old Euclid drew a circle
On a sand-beach long ago.
He bounded and enclosed it
With angles thus and so.
His set of solemn graybeards
Nodded and argued much
Of arc and of circumference,
Diameter and such.
A silent child stood by them
From morning until noon
Because they drew such charming
Round pictures of the moon. □

mathematical digest

No. 171 April 2013

Mersenne Prime Stamp

mathematical digest

Number 171 **April 2013**

Master: Wherefore in all great works are Clerks so much desired? Wherefore are Auditors so well fed? What causeth Geometricians so highly to be enhaunced? Why are Astronomers so greatly advanced? Because that by number such things they finde, which else would farre excell mans mind.

Scholar: Verily, sir, if it bee so, that these men by numbering, their cunning do attain, at whose great works most men do wonder, then I see well I was much deceived, and numbering is a more cunning thing than I took it to be.

(From *The Declaration of the Profit of Arithmeticke* (1570) by Robert Recorde)

Department of Mathematics and Applied Mathematics
University of Cape Town

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Cover: The Liechtenstein postage stamp features a Mersenne prime. Read more about Mersenne primes on pages 2 and 16.

AN IMO FOR AFRICA

The International Mathematical Olympiad (IMO) will be held in South Africa in July 2014 — the first time in Africa.

Since its beginning in 1959, only a handful of African countries have taken part in the IMO. The first to join the IMO was Algeria (1977), followed by Tunisia (1981) and Morocco (1983). South Africa took part for the first time in 1992, followed by Mozambique (2004) and Nigeria (2006), Mauritania (2009), Zimbabwe (2009), Ivory Coast (2010) and Uganda (2012). Not all of these countries are regular participants. At the IMO in Argentina in 2012 there were only six: Ivory Coast, Morocco, Nigeria, South Africa, Tunisia and Uganda.

While the 100 or more countries taking part in the IMO represent about 90% of the world's population, African participants represent less than 40% of Africa's peoples. There is obvious room for improvement, and one of the targets for IMO2014 will be to increase African participation.

It is not simply a matter of sending an invitation to the Ministry of Education of an African country to send a team to Cape Town next year. The IMO is a formidable problem-solving contest, and unprepared students will get humiliatingly low scores. An aspirant IMO country should first of all have its own national Mathematics Olympiad — an essential tool in the process of selecting its IMO team. For African countries, participation in the Pan African Mathematics Olympiad is excellent preparation for the tougher IMO.

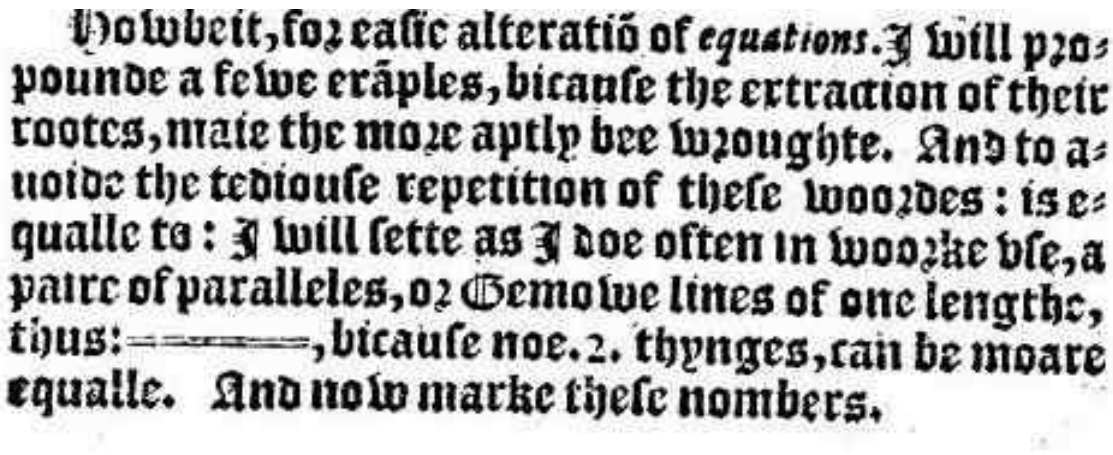
The normal procedure for a new country to join the IMO is to send an Observer first, with the task of finding out how the IMO works, and to learn how other countries select and train their teams. For the IMO in South Africa next year, this condition will be waived: all countries that take part in the 2013 Pan African Mathematics Olympiad will be invited to take part in the first African IMO in 2014.

There are also plans to hold a pre-IMO training programme for all African teams. This project, however, will depend on funding being available. □

ROBERT RECORDE'S EQUALITY SYMBOL

Five hundred years ago, when algebraic notation was being developed in Europe, the word for "equals" was often abbreviated to *ae*, from the Latin *aequalis*.

Today we use the familiar equality sign: two short horizontal parallel lines. It was invented by the Welsh mathematician Robert Recorde, and introduced in his textbook *The Whetstone of Witte*, published in 1557.


Howbeit, for easie alteratiõ of *equations*. I will propounde a fewe eraples, bicause the extraction of their rootes, maie the more aptly bee wroughte. And to auoide the tediouse repetition of these woordes: is equalle to: I will sette as I doe often in woorke vse, a paire of paralleles, or Gemowe lines of one lengthe, thus: =======, bicause noe. 2. thynges, can be moare equalle. And now marke these nombers.

The script is not too easy to read, so here it is again in modern letters:

> Howbeit, for easie alteration of equations, I will propounde a few examples, bicuase the extraction of their roots, maie the more aptly be wroughte. And to avoid the tediouse repetition of these woordes : is equalle to : I will sette as I doe often in woorke use, a pair of paralleles, or Gemowe lines of one length, thus : =======, bicause noe.2.thynges, can be more equalle. And now marke these nombers.

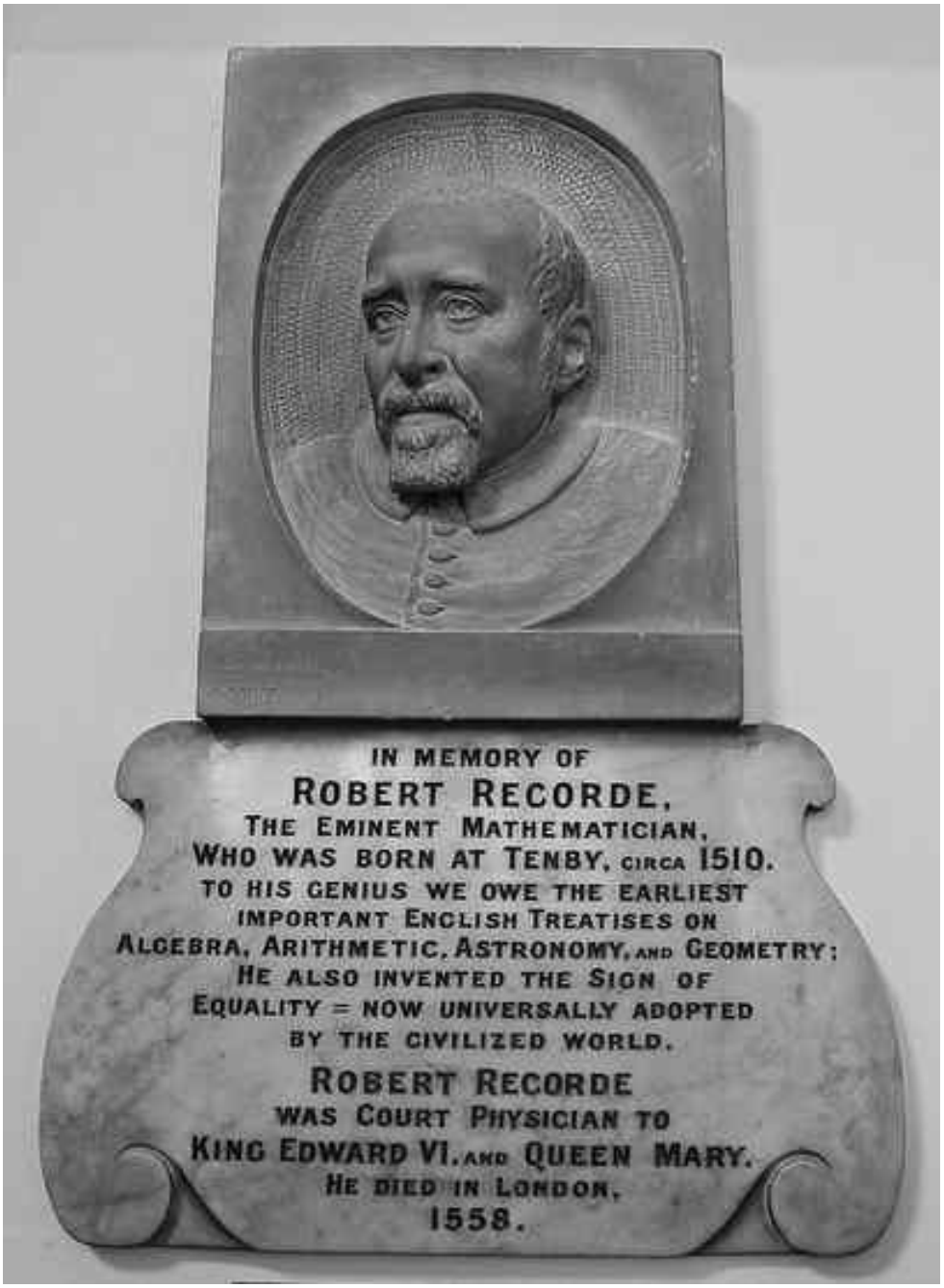
That is pretty understandable, except for the word "Gemowe", which you won't find in modern dictionaries. It means "twin lines", from the Latin *gemellus*.

When Recorde's proposed new symbol for equality became popular, it was quickly realised that such long lines were not essential, and they were shortened to the present form.

Robert Recorde was born in Tenby, Wales in 1510. He wrote a number of important mathematics textbooks and is credited with the introduction of

algebra into England. His *Grounde of Artes*, published in 1543, was a very successful textbook of commercial arithmetic.

Recorde was also a qualified doctor. As the memorial plaque below shows, he rose to the position of physician to King Edward VI, the short-lived son of Henry VIII, and the next monarch, the bloodthirsty Queen Mary. The plaque can be seen in St Mary's church, Tenby.



The plaque does not mentions Recorde's sad end. He unwisely got involved in a legal dispute with the powerful Earl of Pembroke, who successfully sued him for libel. Recorde was fined £1000, a huge sum for those days, which he could not pay. He was thrown into prison, where he died in 1558. □

THE BIGGEST PRIME YET

Which numbers of the form $2^n - 1$ are prime? This problem has fascinated mathematicians for hundreds of years. There are some big prizes for finding large prime numbers of this form.

It is easy to see that if n is composite, then $2^n - 1$ is not prime. This is a consequence of the fact that $x^n - 1$ is divisible by $x - 1$:

$$x^n - 1 = (x - 1)(x^{n-1} + x^{n-2} + \cdots x^3 + x^2 + x + 1).$$

So if $n = pq$, then $2^n - 1 = (2^p)^q - 1$ is divisible by $2^p - 1$ and by $2^q - 1$.

Is the converse true? If p is prime, does it follow that $2^p - 1$ is prime? It is certainly true that for the first four primes 2, 3, 5 and 7, the numbers $2^2 - 1 = 3$, $2^3 - 1 = 7$, $2^5 - 1 = 31$ and $2^7 - 1 = 127$ are all prime, but the pattern breaks down at the next prime: $2^{11} - 1 = 2047 = 23 \times 89$.

Marin Mersenne (1588-1648) was a French monk who spent much time investigating primes of the form $2^n - 1$, and today such primes are called Mersenne primes in his honour. Finding larger and larger Mersenne primes has become a competitive game played by mathematicians.

In 1814 there appeared an article in *A New Mathematical and Philosophical Dictionary* by Peter Barlow, who said "Euler ascertained that $2^{31} - 1 =$ 2 147 483 647 is a prime number ... it is not likely that any person will attempt to find one beyond it."

Barlow was, inevitably, wrong. In 1876 Edouard Lucas proved that $2^{127} - 1$, a 39-digit number, is prime. That was the last time that a Mersenne prime was discovered by hand. No further Mersenne primes were found until computers became available. In 1952 Raphael Robinson used a SWAC computer to find five new Mersenne primes in quick succession, the largest being $2^{2281} - 1$ (687 digits). Over the next 40 years computers found 22 more Mersenne primes.

The next step was using huge networks of computers. The Great Internet Mersenne Prime Search, or GIMPS for short, was launched in 1996 by George Woltman. It links thousands of computers all over the world to test numbers of the form $2^p - 1$, where p is prime. When a computer is not doing anything, GIMPS sets it to work looking for prime numbers.

Anybody can join the search by logging their computer into GIMPS. You may be lucky: your computer might find the next record prime.

In February this year it was announced that Curtis Cooper at the University of Central Missouri in Warrensburg had identified a new record large Mersenne prime:

$$2^{57\ 885\ 161} - 1.$$

This huge number has 17 425 170 million digits. If you were to print it out at 75 digits per line and 50 lines per page it would need 4647 pages.

This was Professor Cooper's third record Mersenne prime: his previous record primes were notched up in 2005 and 2006. They were eclipsed by $2^{43\ 112\ 609} - 1$, found in 2008 by a team at the University of California at Los Angeles.

Since it has taken over four years for the next Mersenne prime to be discovered, it is clear that the task is getting harder. Are there still larger Mersenne primes? There is, in fact, no guarantee that there are infinitely many Mersenne primes. Perhaps they have all been discovered!

Cooper runs GIMPS on nearly a thousand of his university's computers. One of them got lucky: assigned what turned out to be the record Mersenne prime, it took 39 days to check that the number was indeed prime. When the number had been verified by other GIMPS computers, using different hardware and programs, it was recognized as the new record prime.

What's the point? There is little mathematical value in finding prime numbers with millions of digits. However, finding hundred-digit primes is important. Large prime numbers underpin the cryptographic techniques used to make online communications, especially internet banking, secure.

Still, finding a new Mersenne prime can be profitable. Curtis Cooper will receive a $3000 prize from GIMPS for his discovery. And much bigger prizes can be won. The Electronic Frontier Foundation, an internet civil liberties group, is offering prizes of $150 000 and $250 000 to the discoverer of the first prime with at least 100 million and a billion digits, respectively. Previous prizes for primes of one million and ten million digits have already been awarded. The prize money comes from a special donation from an individual EFF supporter, earmarked specifically for the project.

For more on Mersenne primes and GIMPS, see `www.mersenne.org`. □

BEWARE OF MATHEMATICIANS

> The good Christian should beware of mathematicians. The danger already exists that mathematicians have made a covenant with the devil to darken the spirit and confine man in the bonds of Hell.

This notorious quotation about mathematics is often posted on office doors in university mathematics departments. It is taken from a 5th Century Christian treatise *De Genesi ad Litteram* (The Literal Interpretation of Genesis) by Augustine of Hippo (354–430 A.D.).

Augustine was an important Christian theologian and philosopher and Bishop of the town of Hippo Regius, in what is now Algeria. He was declared a Saint in 1298.

Was Augustine really opposed to the practice of mathematics? Not at all. The quotation is, unfortunately, a sloppy translation of Augustine's original Latin, in which the word *mathematici* referred mainly to astrologers. To Augustine, astrologers were no more than "common swindlers".

A more accurate translation of the passage is given by J.H. Taylor in *Ancient Christian Writers* (1982).

> Hence, a devout Christian must avoid astrologers and all impious soothsayers, especially when they tell the truth, for fear of leading his soul into error by consorting with demons and entangling himself with the bonds of such association.

The flawed translation is due to the mathematician (in the modern sense) Morris Kline, and opens the first chapter of his *Mathematics in Western Culture*. Despite that bad start, the book is nevertheless a very good read. First published in 1953, it is still in print after 60 years. □

THE SURPRISE TEST

On a Friday afternoon the maths teacher says: "Next week I will be giving you a test. I'm telling you now, so that you can prepare for it over the weekend. It will take place one day next week, in the usual maths period, but I am not going to tell you on what day the test will take place. It will be a complete surprise."

Over the weekend five classmates get together to study for the test. But first they try to guess on what day the test will be held.

Mona says: "I think the test will be on Monday, because she warned us to work over this weekend and that's a big hint."
Tubby says: "I think it will be on Friday, because she wants us to do well, and that would give us the whole week to prepare."
Wendy says: "It can't be on Friday, because if we have not had the test by Thursday, we will know it must the next day, on Friday, so it won't be a complete surprise. My guess is that it will be on Thursday."
Thora says: "But if we have not had the test by Wednesday, and we know it can't happen on Friday, then it must be on Thursday, so again the test will not be a complete surprise. So I guess it will be on Wednesday"
Frikkie says: "For the same reason, if the test has not taken place by Tuesday, we can expect it for sure on Wednesday and it won't be a surprise. So I think it will be on Tuesday."
Mona says: "For the same reason it can't wait till on Tuesday. I told you it had to to be on Monday."

All five classmates are convinced that they know on what day the test will take place. One of them has to be right, and they suddenly realise that a test during the next week that would be a complete surprise to all of them is not possible.

On Monday morning the class can't wait to confront the teacher and tell her that she can't set a surprise test during the week, because it can't wait till Friday, but then can't be on Thursday, and for the same reason can't be on Wednesday, or Tuesday or Monday. QED!!!

The teacher listens to their arguments, and says: "Well done! Now I can tell you that the test was not going to be on the maths we did last week, but on logic. You have reasoned with perfect logic that what I said was not logically consistent. You have all passed the test. Congratulations, and ... SURPRISE!" □

THE ABC CONJECTURE

Back in 1985 two mathematicians, David Masser and Joseph Oesterlé, proposed a conjecture about the most basic equation in the arithmetic of whole numbers:

$$a + b = c.$$

In broad terms, the conjecture says that if $a+b=c$ (where a, b and c have no common factor) and there are prime numbers that divide either a or b many times, then their frequency is 'balanced out' by larger primes that divide c only a few times.

For example, in the equation $64 + 81 = 145$, 2 divides 64 six times, 3 divides 81 four times, but $145 = 5 \times 29$, with the larger primes 5 and 29 each dividing 145 just once.

The conjecture became known as the ABC conjecture. It was realised that if it could be proved true, then many other results, in particular Fermat's Last Theorem, would follow quite easily.

(Fermat's Last Theorem states that it is not possible to find positive whole numbers a, b, c and n, with $n > 2$, such that $a^n+b^n = c^n$. It was conjectured by the 17th Century French mathematician Pierre de Fermat, but was proved only in 1994 by the British mathematician Andrew Wiles.)

Late last year Shinichi Mochizuki, a mathematician at the University of Kyoto, announced that he had proved the ABC conjecture. Mochizuki is a well-known mathematician with a number of major mathematical advances already to his credit, so his claim was taken seriously, and mathematicians immediately starting analyzing his proof. They found that his approach involved "elliptic curves" (curves with equations of the form $y^2 = x^3+ax+b$), which were the key to cracking Fermat's Last Theorem. But there the similarity stopped. Mathematicians were dismayed to find that not only was the proof hundreds of pages long, but Mochizuki's methods were novel and difficult to comprehend, and relied on understanding a long series of earlier publications that had not received much attention before. It may take some time before the world jury of mathematicians will be able to pronounce on the correctness of Mochizuki's arguments.

The ABC conjecture can be stated more precisely, using the concept of a "square-free" number: one that cannot be divided by the square of any number other than 1. The numbers 42 (= 2.3.7), 165 (= 3.5.11) and 1001 (= 7.11.13) are square-free numbers, for example, but 54 and 75 are not,

being divisible by 9 and 25, respectively. In the factorization of a square-free number, the prime factors appear only once.

Given this idea, the "square-free part" of a number n, written $\mathrm{sqp}(n)$, is defined to be the product of all the prime factors in the factorization of n, taking each prime factor only once. For example:

$$\mathrm{sqp}(120) = \mathrm{sqp}(2^3.3.5) = 2.3.5 = 30, \text{ and}$$
$$\mathrm{sqp}(18\ 513) = \mathrm{sqp}(3^2.11^2.17) = 3.11.17 = 561.$$

In these terms, the ABC conjecture states that, for integers $a + b = c$, the ratio $\mathrm{sqp}(abc)^r/c$ always has some minimum value greater than zero for any value of r greater than 1.

The ABC conjecture can be expressed as follows: For every $r > 1$, there are only finitely many triples of coprime positive integers $a + b = c$ such that $c > \mathrm{sqp}(abc)^r$.

For example, if $a = 3$ and $b = 125$, so that $c = 128$, then $\mathrm{sqp}(abc) = 30$ and $\mathrm{sqp}(abc)^2/c = 900/128$. In this case, in which $r = 2$, $\mathrm{sqp}(abc)^r/c$ is nearly always greater than 1, and always greater than zero.

Prime numbers are the basic atoms of our number system. The ABC conjecture is intriguing because it establishes a connection between the prime factors of two numbers and the prime factors of their sum. It says something new about the relation between addition and multiplication, and suggests there are some deep properties of those simple building blocks of our number system which we still do not really understand. □

LOTTERY LIMERICK

A lottery is a taxation
Upon all the fools in creation.
  And heaven be praised
  It is easily raised.
Credulity's always in fashion.

The verse above comes from *The Lottery*, a "ballad opera" written by Henry Fielding and performed in Drury Lane, London in 1732. It makes fun of the false excitement of the lottery and those who buy lottery tickets in the hope of making a fortune. □

THE ELEGANCE OF MATHS

Angela Kirykowicz
B.Sc. student, University of Cape Town

The terms 'beauty' and 'elegance' are widely used in the mathematics circle, which tends to leave those uninitiated a bit perplexed. As a general rule, most people are used to applying these terms to a wide range of disciplines including Art, Music and Dance, but not to Mathematics.

This speaks of the inadequate way Maths is taught in school; formulas are prescribed without discussing where they come from, or what they actually describe. The history of mathematics, like the history of any subject, is full of quirks and surprise; who would have thought that the mother of computing, Ada Lovelace, was the daughter of the famous poet Lord Byron?

Mathematics is the language of the universe; everything that we are familiar with, or will be familiar with, can be written using its code. The typical saying that nothing in biology makes sense except in the light of evolution, can also be applied to maths; namely, nothing in the universe can be understood except in the light of mathematics.

It is true that concepts can be explained without using mathematics, but the understanding is only superficial; in order to truly know how the universe works, you have to learn and use its language.

As a language, Maths is difficult to grasp. However, this is true of practically everything. Some people never learn to cook or write well, which doesn't deter others from mastering those subjects. It's the same for Maths; all you need is time, perseverance, and an inquiring mind. The tools you will acquire can be applied to any facet of life, from spotting faulty arguments to appreciating the implied beauty it conveys.

It is very difficult to describe how Maths is beautiful; it is just something implied in its structure, similarly to how people appreciate good art — such as Van Gogh's *Midnight in Paris*, you implicitly know that it is beautiful without having to explain why.

Elegance is a bit different to beauty; it usually applies to proofs or formulas that are short or unexpected. A physical example would be Einstein's formula $E = mc^2$, where E is the amount of energy produced by an object with a mass of m and c is the speed of light in a vacuum, which is constant.

Although seemingly innocuous, it is a profound result that shows that matter and energy are interchangeable.

Karl Marx once remarked that when people are learning a new language, they automatically translate it back into the one they already know. This is true for those trying to learn Maths; eventually, your translation malfunctions.

Part of the problem is the way Mathematics is currently taught in school; students are force-fed a myriad of formulas and concepts, often out of context. This creates a vicious cycle of students who struggle with Maths and become parents themselves who cannot help if their children also struggle.

A remedy is to teach Mathematics as if you are teaching a foreign language; you communicate it through its symbols and equations, explain where they come from in the language you know and, ultimately, encourage students to master it themselves. □

(This article was first published in the UCT student newspaper *Varsity*, 12 March 2013, and is reproduced with the Editor's and the author's permission.)

The author writes:
I was born in Durban, South Africa, before moving to Canada at the age of four. I lived in Canada for three and a half years before moving to England, where I lived for ten years. I have been living in Cape Town for over a year; I came to study a Bachelor of Science, and I am in my second year. I am currently studying Genetics, Biochemistry, and Mathematics (although I am not quite sure what I will be doing next year); also, no it is not an odd combination.

I would stress that the important bit is (especially for those high school students), that if I went into a time machine and told my younger self that I would be studying Mathematics at University, I would not have believed it. The fact is that I was not that good at Math; I had to have extra help. My Year 9 maths teacher even said to my parents that I would likely get a D on a bad day, or a C on a good day for GCSE Mathematics. Thankfully, my parents didn't believe him, nor did I; I actually went on to study Mathematics and Further Mathematics at A-Level, and the rest is history. The important thing with Math (and any subject in general) is to persevere, and to believe in your own abilities. □

JEWISH PROBLEMS

Russian society has a sad history of anti-Semitism, from the 19th Century pogroms of the Russian Tsars to Stalin's paranoid delusions in 1952 of a plot by Jewish doctors to kill the Soviet leadership.

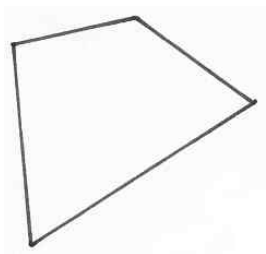
Anti-Semitism in the Soviet Union extended even into the world of mathematics. Being Jewish could count against selection for the USSR International Mathematical Olympiad team. In 1978 the USSR did not enter a team into the IMO, because (it was said) Jewish students had taken all the top positions in the team selection process.

Entrance into top Russian universities is by competitive examination, and the maths examinations were deliberately biased against Jewish applicants. Oral examinations are part of the Russian tradition, and Jewish students applying for admission into the Department of Mechanics and Mathematics of Moscow State University were put into separate rooms and given much more difficult problems than ordinary Russian students.

The devilish part of this plan to exclude Jewish students was the creation of problems with ingeniously short solutions that were nevertheless very difficult to find. If the Moscow professors were criticised for giving Jewish students "hard" problems, these ingenious quickies could be trotted out with the comment " But, comrade, we gave them these problems, and look how easy the solutions are!"

These "Jewish problems" have been collected and published in books, magazines and on the internet. Here are just five of them. Can you solve them? (You don't have to be Jewish!)

1. Given just the lengths of the four sides of a quadrilateral, many different quadrilaterals can be constructed. Which of the different possible quadrilaterals has the largest area?

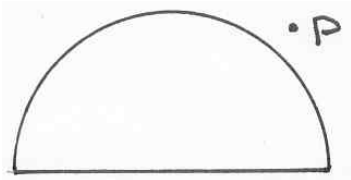


2. Find all functions $f(x) : \mathbb{R} \to \mathbb{R}$ such that for any a and b the following inequality holds:
$$f(a) - f(b) \leq (a - b)^2.$$

3. Prove that $\sin 10^\circ$ is irrational.

4. Which is larger, $\log_2 3$ or $\log_3 5$?

5. A semicircle is drawn in the plane, with its diameter. Show how to drop a perpendicular from a point P outside the semicirle onto the diameter, using only a ruler (straightedge).



Solutions can be found on page 31.

Go to `http://www.tanyakhovanova.com/coffins.html` for more "Jewish problems". □

NUMBER ODDITY

In the equation

$$567^2 = 321489$$

the digits from 1 to 9 are used exactly one.

There is another 3-digit number with this unusual property. What is it?

The answer is on page 31. □

FACTORIAL FACT

Here's an interesting little observation about factorials.

$$10! = 3! \times 5! \times 7! = 6! \times 7!$$

Can you find another factorial which can be expressed as the product of factorials in two different ways? It's easier than you think.

The answer is on page 31. □

PERCENTAGE ERROR

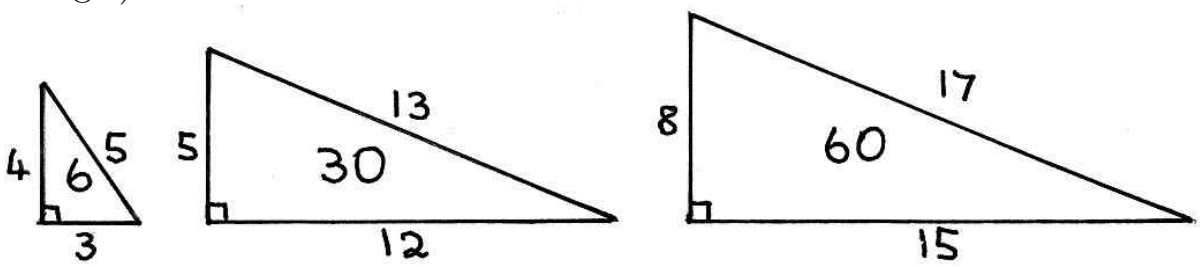
[Transport MEC Robin Carlisle] said 62 people had been killed on Western Cape roads compared with 43 over the same period in 2011, an increase of 31%.

(*The Times*, 14 December 2012) □

CONGRUENT NUMBERS

A *congruent number* is a positive integer that is the area of a right-angled triangle whose three sides are all rational numbers.

It is clear that 6 is a congruent number, since it is the area of a (3, 4, 5) Pythagorean triangle. So is 30 (it is the area of a (5, 12, 13) Pythagorean triangle).



The (8, 15, 17) Pythagorean triangle shows that 60 ($= \frac{1}{2} \times 8 \times 15$) is a congruent number. If the sides of the triangle are divided by 2, the area is divided by 4. So $60/4 = 15$ is a congruent number.

The problem of determining which numbers are congruent is over a thousand years old, first studied (in a slightly different form) by the Persian mathematician al-Karaji. He was inspired by the work of the Greek mathematician Diophantus over 800 years earlier.

In 1225, Fibonacci (famous for his problem about multiplying rabbits) showed that 5 and 7 are congruent numbers. To show that 5 is congruent, start with the (9, 40, 41) Pythagorean triangle (of area 180) and scale its sides down by a factor of 6 to a triangle of area $180/36 = 5$.

Showing that 7 is congruent requires discovering the Pythagorean triangle (175, 288, 337) ($175^2 + 288^2 = 337^2$). Then scale the sides down by a factor of 60 to get (35/12, 24/5, 337/60), a Pythagorean triangle with area $\frac{1}{2} \times \frac{35}{12} \times \frac{24}{5} \times \frac{337}{60} = 7$.

As these examples show, proving that a number is congruent may require a whole lot of numerical ingenuity.

It is easy to see that if n is a congruent number, so in k^2n for any natural number k (simply multiply the sides of the corresponding Pythagorean triangle by k). Similarly, if n is a congruent number that contains a square factor k^2, then n/k is a congruent number.

The sequence of congruent numbers starts with 5, 6, 7, 13, 14, 15, 20, 21, 22, 23, 24, 28, 29, 30, 31, 34, 37, 38, 39, 41, 45, 46, 47, ...

Is 1 a congruent number? That's not quite obvious. In 1659 Fermat (famous for his Last Theorem) proved that it is not.

After a thousand years, the general problem of determining which numbers are congruent is not completely solved, in the sense that there is no general way of determining whether a given integer is congruent or not. In 1982 the American mathematician Jerrold Tunnell found a formula for determining whether a number is congruent, but the correctness of his test depends on another famous unsolved problem in mathematics, the Birch and Swinnerton-Dyer Conjecture. This conjecture was formulated in the early 1960s by two Cambridge mathematicians, Bryan Birch and Peter Swinnerton-Dyer, and is still unresolved after more than 50 years. It is one of the seven Millennium Problems identified by the Clay Mathematics Institute, which has offered a prize of one million dollars for its solution. It is unfortunately not possible to explain this conjecture in simple terms.

The congruent number problem is, however, easy to state but very difficult to crack. It is regarded as one of the major unsolved problems in number theory. □

MATHS FROM MACBETH

In Shakespeare's *Macbeth* the three witches' magic incantation "Double double toil and trouble" can be turned into a numerical puzzle.

In the addition below

$$DOUBLE + DOUBLE + TOIL = TROUBLE$$

each letter represents a digit from 0 to 9, in a one-to-one fashion.

Can you work out the numbers? The answer is on page 31. □

QUOTE

Statistically, six out of seven dwarves are not happy.

(Snow White) □

MERSENNE SNIPPETS

The first four Mersenne primes

$$2^2 - 1 = 3, \quad 2^3 - 1 = 7, \quad 2^5 - 1 = 31, \quad 2^7 - 1 = 127$$

were known to the Greek mathematicians of antiquity. The fifth Mersenne prime was discovered in 1496, by an anonymous mathematician. It is $2^{13} - 1 = 8191$. The minor planet number 8191 is named Mersenne.

In 1963 the Mersenne prime $2^{11213} - 1$ was discovered at the University of Illinois, U.S.A. They were so proud of their discovery that they used it as a meter mark on their mail.

The Mersenne prime $2^{13466917} - 1$, discovered in 2001, was celebrated on a Liechtenstein postage stamp.

□

RUGBY RANKINGS

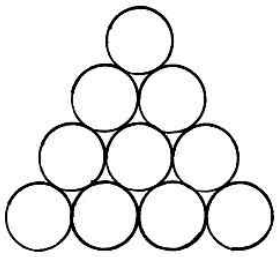
The International Rugby Board regularly publishes the latest rankings of world rugby teams on their website `www.rugbyworldcup.com`. On 3 December 2012 the IRB announced:

> Argentina and Samoa both had a rating of 78.71 but when expanded to 15 decimal places the difference between the two was minimal (78.709236088306938 to 78.708853582562098), enabling the Samoans to become the first Pacific Islanders to occupy a top eight spot in the rankings.

Fifteen-decimal place accuracy seems excessive, especially when the numbers actually differ from the third place on. □

GOLDEN OLDIES

1. Ten coins are arranged in a triangle as shown. Turn the triangle upside down by moving just three coins.



2. The matchsticks show an incorrect equation. Make the equation correct by moving just one matchstick.

3. Coke and a cup together cost R11. The Coke costs R10 more than the cup. How much does the cup cost?
4. In a certain university there are 7 students for every professor. If the number of students is S and the number of professors is P, write down an equation relating S and P.
5. There were ten sheep in a field. During the night all but three were eaten by jackals. How many sheep were left?
6. There were fifty birds in a tree. With one shotgun blast the farmer kills 20% of the birds. How many birds are left in the tree?

Answers on page 31. □

NUMBER PATTERN

$$1 \times 1 = 1^2$$
$$121 \times (1+2+1) = 22^2$$
$$12321 \times (1+2+3+2+1) = 333^2$$
$$1234321 \times (1+2+3+4+3+2+1) = 4444^2$$
$$123454321 \times (1+2+3+4+5+4+3+2+1) = 55555^2$$
$$12345654321 \times (1+2+3+4+5+6+5+4+3+2+1) = 666666^2$$
$$1234567654321 \times (1+2+3+4+5+6+7+6+5+4+3+2+1) = 7777777^2$$

And so on. □

THE DIGITS OF PI

Here is π to a thousand decimal places.

3.
1415926535 8979323846 2643383279 5028841971 6939937510 5820974944
5923078164 0628620899 8628034825 3421170679 8214808651 3282306647
0938446095 5058223172 5359408128 4811174502 8410270193 8521105559
6446229489 5493038196 4428810975 6659334461 2847564823 3786783165
2712019091 4564856692 3460348610 4543266482 1339360726 0249141273
7245870066 0631558817 4881520920 9628292540 9171536436 7892590360
0113305305 4882046652 1384146951 9415116094 3305727036 5759591953
0921861173 8193261179 3105118548 0744623799 6274956735 1885752724
8912279381 8301194912 9833673362 4406566430 8602139494 6395224737
1907021798 6094370277 0539217176 2931767523 8467481846 7669405132
0005681271 4526356082 7785771342 7577896091 7363717872 1468440901
2249534301 4654958537 1050792279 6892589235 4201995611 2129021960
8640344181 5981362977 4771309960 5187072113 4999999837 2978049951
0597317328 1609631859 5024459455 3469083026 4252230825 3344685035
2619311881 7101000313 7838752886 5875332083 8142061717 7669147303
5982534904 2875546873 1159562863 8823537875 9375195778 1857780532
1712268066 1300192787 6611195909 2164201989

John Donne (1572-1631) wrote a poem *Upon the Translations of the Psalms by Sir Philip Sidney, and the Countess of Pembroke, His Sister*, which condemns attempts to find an exact value of pi, or to "square a circle". Donne viewed the evaluation of π as a heretical attempt to rationalize God:

> Eternal God–for whom who ever dare
> Seek new expressions, do the circle square,
> And thrust into straight corners of poor wit
> Thee, who art cornerless and infinite.

Perhaps Donne was right. The first 144 digits of π add up to 666, which many scholars say is the *Number of the Beast*, because of the verse in the Book of Revelation chapter 13, verses 17 and 18:

> Here is wisdom. Let him that hath understanding count the number of the beast: for it is the number of a man; and his number is Six hundred threescore and six.

Why 144? Well, $144 = (6 + 6) \times (6 + 6)$.

The Feynman point is a sequence of six 9s that begins at the 762nd decimal place of the decimal expansion of π. It is named after physicist Richard Feynman, who once stated during a lecture he would like to memorize the digits of π until that point, so he could recite them and then say "nine nine nine nine nine nine and so on", suggesting that π is rational.

Pop star Kate Bush did not get that far in her album *Aerial*, in which she sang the first 100 digits of π.

There is no end to the digits of π.

The allusion in the cartoon is to the best-selling novel by Yann Martel,

The Life of Pi, which is all about an Indian teenager called Pi, and has nothing to do with the number.

Wislawa Szymborska, the winner of the 1996 Nobel Prize for Literature, wrote a poem about π.

The admirable number pi:
three point one four one.
All the following digits are also initial,
five nine two because it never ends.
It can't be comprehended six five three five at a glance,
eight nine by calculation,
seven nine or imagination,
not even three two three eight by wit, that is, by comparison
four six to anything else
two six four three in the world.
The longest snake on earth calls it quits at about forty feet.
Likewise, snakes of myth and legend, though they may hold out a bit longer.
The pageant of digits comprising the number pi
doesn't stop at the page's edge.
It goes on across the table, through the air,
over a wall, a leaf, a bird's nest, clouds, straight into the sky,
through all the bottomless, bloated heavens.
Oh how brief – a mouse tail, a pigtail – is the tail of a comet!
How feeble the star's ray, bent by bumping up against space!
While here we have two three fifteen three hundred nineteen
my phone number your shirt size the year
nineteen hundred and seventy-three the sixth floor
the number of inhabitants sixty-five cents
hip measurement two fingers a charade, a code,
in which we find hail to thee, blithe spirit, bird thou never wert
alongside ladies and gentlemen, no cause for alarm,
as well as heaven and earth shall pass away,
but not the number pi, oh no, nothing doing,
it keeps right on with its rather remarkable five,
its uncommonly fine eight,
its far from final seven,
nudging, always nudging a sluggish eternity
to continue. □

LOTTERY QUOTE

> In order to have a better chance for some of the great prizes, some people purchase several tickets, and others, small shares in a greater number. There is not, however, a more certain proposition in mathematics than the more tickets you adventure upon, the more likely you are to be a loser. Adventure upon all the tickets in the lottery, and you lose for certain; and the greater the number of your tickets the nearer you approach this certainty.

(Adam Smith, in *The Wealth of Nations*, published in 1776) □

PARTY PROBLEMS

At his birthday celebration in April this year John said "My age is the sum of the digits of the year in which I was born."

When was John born?

At the same party Mary said "I will be x years old in the year x^2."

When was Mary born?

Answers may be found on page 31. □

POETRY AND MATHEMATICS

David Hilbert was one of the greatest mathematicians of the 20th Century. One day he noticed that one of his students was no longer attending his lectures. On making enquiries, he was told that the student had given up mathematics and taken up poetry instead.

Hilbert's response was "Good - he did not have enough imagination to become a mathematician." □

CLASSICAL QUOTE

Odd numbers are the gods' delight.

(Virgil) □

DIVIDING THE HERD

An old induna was dying, and summoned his three sons to tell them how he wished to divide his herd of cows between them. He said that his eldest son should get half the cows, the second son one-third and the youngest son one-eighth of the cows.

Shortly afterwards the old man died. His sons counted the herd and found that they numbered 23. They began to argue about how to divide the cows, since 23 could not be divided by 2 or by 3 or by 8.

Unable to agree on how the herd should be divided, they went off to the chief to ask him how they should carry out their father's wishes.

The chief listened to them carefully and was silent, deep in thought, for some time. "Bring me the cows," he said.

The three sons rounded up the 23 cows and drove them into the chief's kraal. The chief summoned his herdboy, his youngest son. "Bring me my favourite cow, the fat cow with the twisted horn," he ordered. The herdboy drove the fat cow with the twisted horn into the kraal with the young men's cows.

The chief looked at the herd, and said. "Now there are 24 cows. I shall now divide them. As your father said, you, the oldest son, will get half, namely 12 cows. The middle son must get one-third of the herd, which is 8 cows, and the youngest son must get one-eighth of the herd, which is 3 cows. The three of you have now received twelve cows and eight cows and three cows, which is twenty-three cows. The twenty-fourth cow, my own cow, the fat one with the twisted horn, will remain mine. I have spoken. Go."

The three sons rounded up their cows and drove them home, marvelling at the wise way in which the chief had solved the problem.

Back in their village they told everybody the story of how the wise chief had told them how to divide the cows.

Some time later, another old man, who also had three sons, lay dying in the same village. He called his three sons together, and told them his wishes: "You, my oldest son, are to have half my herd. My middle son will get one-third of the herd, and the youngest will get one-sixth of my herd."

When he died, the three sons gathered their father's cows together. There were 31 cows in the herd. Once again they could not divide the herd

exactly, but they knew that the chief would be able to solve the problem. So they drove their cows to the chief's kraal and asked for his judgement on how to divide the cows between them.

The chief sighed. "I am tired of having to solve problems about dividing cows. This time you must pay me one cow for my judgement."

The sons agreed, and chief chose the fattest cow from their herd as his payment and told his youngest son, his herdboy, to drive it away to join the chief's herd.

The chief then gave his judgement to the three young men. "You have paid my fee. Now you have 30 cows. Your father has ordered that the eldest son will get half the herd, which is 15 cows. The second son will get one-third, which is 10 cows. The third son will get one-sixth, which is 5 cows. Since fifteen cows and ten cows and five cows is thirty cows, the herd is divided as your father wished. I have spoken. Go. And when you get back to your village, tell all the old men that they must learn about fractions before they die." □

FIVE NINES

In the world of financial software development "five 9's" refers to a system which is available 99.999% of the time.

Here's what the nines mean in "downtime per year":

90% ("one nine"): 36.5 days

99% ("two nines"): 3.65 days

99.9% ("three nines"): 8.76 hours

99.99% ("four nines"): 52.56 minutes

99.999% ("five nines"): 5.26 minutes

99.9999% ("six nines"): 31.5 seconds

99.99999% ("seven nines"): 3.15 seconds

A reader, who is a software developer, recently overheard a colleague boasting that his system's availability was "nine 5's". He was embarrassed when it was pointed out that this meant that his system was available on average for 55.5555555% of a year, which meant that it had an expected downtime of 162.2222223 days per year. □

WHAT'S THE USE OF LOGARITHMS?

The Richter scale

California is famous for its earthquakes, especially the devastating San Francisco quake of 1906. So it is not surprising that the first scientific measurement of the energy of an earthquake was devised in California.

In 1935 Charles Richter devised what is now known as the Richter scale for measuring earthquakes. The scale is logarithmic. The magnitude of an earthquake is defined as the logarithm (base 10) of the ratio of the amplitude of waves measured by a seismograph to an arbitrary small amplitude. Measurements are in the range from 0 to 10, and are given accurate to one decimal place. Because the scale is logarithmic, a magnitude 6 earthquake has an amplitude 10 times that of a magnitude 5 earthquake. In terms of energy release, an increase of one unit on the Richter scale corresponds to a $10^{1.5} \approx 32$ times increase in the amount of energy released, and an increase of two units corresponds to a $10^3 = 1000$ times increase in energy.

The Richter scale is used to set building codes for earthquake resistance.

The 1906 San Francisco earthquake was probably of magnitude 8.3 on the Richter scale. By comparison, the most destructive earthquake in South African history (Tulbagh, 1969) measured 6.3 on the Richter scale.

Although the Richter scale is often cited in media reports of earthquakes, it has largely been supplanted in the scientific world. Geophysicists found that the Richter scale was not quite satisfactory for measuring large earthquakes, and devised the Moment Magnitude Scale, which is also logarithmic:

$$M_w = \frac{2}{3}\log_{10} M_0 - 6.0$$

where M_0 is the seismic moment in Newton metres (10^7 dyne.cm) and the subscript w means mechanical work accomplished. The constants in the equation were set so that the numbers are consistent with the measurements of the old Richter scale: an earthquake measure of 5.0 on one scale is also 5.0 on the other.

The pH scale

The pH measurement is a measure of acidity/alkalinity of a substance. The formula is logarithmic:

$$pH = -\log_{10}(a_{H^+})$$

where a_{H^+} is the measure of hydrogen ion activity. The pH scale runs from 0 (acid) to 14 (alkaline), with pure water measuring 7. The measure is very widely used in agriculture, chemistry, engineering, environmental science, forestry, medicine, nutrition, oceanography and water treatment.

The decibel scale

The *decibel* (dB) is a unit for measuring the relative power of sounds. It is one-tenth of a *bel* (B), named in honour of the inventor of the telephone, Graham Alexander Bell. The decibel measure is ten times the logarithm of the power ratio of the sounds:

$$L_{dB} = 10 \log_{10} \left(\frac{P_1}{P_0} \right)$$

On the decibel scale, the smallest audible sound (near total silence) is 0 dB. A sound 10 times more powerful is 10 dB. A sound 100 times more powerful than near total silence is 20 dB. A sound 1,000 times more powerful than near total silence is 30 dB. Here are some common sounds and their decibel ratings:

Silence - 0 dB
A whisper - 15 dB
Normal conversation - 60 dB
Snoring - 85 decibels
A rock concert or a jet engine - 120 dB
A gunshot - 140 dB

Three minutes is the safe maximum time to listen to 120 decibels. A noise of 90 decibels for 8 hours will cause loss of hearing. Any noise over 140 decibels will cause immediate hearing damage.

The Weber-Fechner Law

When we estimate the weights of two objects, the just-noticeable difference between two weights is proportional to the weight: one can tell the difference between 10 grams and 11 grams much more readily than the difference between 1000 grams and 1001 grams. Similar observations apply to our sensitivity to sound and light. It turns out that human sensitivity to change is proportional to the logarithm of the intensity of the stimulus, rather than the intensity itself. This is the Weber-Fechner Law, named after two 19th Century German psychologists. □

VARSITY ARITHMETIC TEST

Arithmetic, after Reading and Writing, is a pillar of basic education. Being able to add, subtract, multiply and divide is a core subject in the early years of primary school, followed by fractions, percentages, decimals, powers and roots. Careers in science, engineering and commerce all require good arithmetic skills.

Here is a set of number problems to test your ability in numerical calculations (for university study). Give yourself 90 minutes to tackle the problems, then check your answers. There is no pass/fail mark in this test, but any errors you make will highlight a weakness in your ability to handle numbers.

Of course, no calculators are allowed.

1. $52 \times 61 + 49 \times 47 + 47 \times 52 + 61 \times 49$ is equal to

   (A) 11808 (B) 10908 (C) 12728 (D) 13658 (E) 14628

2. What is the 2013th digit in the decimal expansion of $\frac{5}{13}$?

   (A) 2 (B) 3 (C) 6 (D) 9 (E) 4

3. If $10^{2013} - 2013$ is written as an integer, what is the sum of its digits?

   (A) 17845 (B) 18112 (C) 18440 (D) 18669 (E) 18923

4. How many numbers on the list

$$\sqrt{1025}, \sqrt{1026}, \sqrt{1027}, \cdots, \sqrt{1088}$$

   are closer to 32 than to 33?

   (A) 31 (B) 32 (C) 33 (D) 34 (E) 35

5. $\dfrac{(\sqrt{2}-1)(2+\sqrt{2})}{(\sqrt{2}+1)(2-\sqrt{2})}$ is equal to

   (A) 1 (B) $\sqrt{2}$ (C) 2 (D) $2\sqrt{2}$ (E) $1/\sqrt{2}$

6. $\left((4\sqrt{2})^{\frac{1}{5}} + (3\sqrt{3})^{\frac{1}{3}}\right)^2$ is equal to

   (A) 5 (B) $\sqrt{24} + \sqrt{25}$ (C) $2 + 5\sqrt{3}$ (D) $5 + 3\sqrt{2}$
   (E) $3\sqrt{2} + 2\sqrt{3}$

7. $(3\sqrt{2} - 2)^3$ is equal to

   (A) $90\sqrt{2} - 116$ (B) $80\sqrt{2} - 96$ (C) $64\sqrt{2} - 72$ (D) $60\sqrt{2} - 84$
   (E) $48\sqrt{2} - 56$

8. 12.5×0.02 is equal to

(A) 25 (B) $\frac{5}{2}$ (C) $\frac{5}{8}$ (D) $\frac{1}{4}$ (E) $\frac{1}{40}$

9. $\frac{1}{2-\sqrt{3}} + (\sqrt{3}-1)^2$ is equal to

(A) $2+\sqrt{3}$ (B) 3 (C) $2-\sqrt{3}$ (D) 2 (E) $6-\sqrt{3}$

10. The price of an item is increased by 25% and the new price is later reduced by 16%. The overall effect is an increase of

(A) 9% (B) 8% (C) 7% (D) 6% (E) 5%

11. Fresh apricots have a moisture content of 80%. When left in the sun to dry, they lose 75% of their moisture content. What is the moisture content of dried apricots?

(A) 5% (B) 10% (C) 25% (D) 40% (E) 50%

12. The five-digit number 2013A is divisible by 9. What is the remainder when it is divided by 7?

(A) 3 (B) 4 (C) 1 (D) 5 (E) 6

13. $1 + \cfrac{2}{3 + \cfrac{4}{5 + \frac{6}{7}}}$ is equal to

(A) $\frac{233}{151}$ (B) $\frac{232}{153}$ (C) $\frac{231}{155}$ (D) $\frac{234}{157}$ (5) $\frac{235}{159}$

14. $(3^2)^3$ is equal to

(A) 18 (B) 243 (C) 729 (D) 6561 (E) 27

15. What is the sum of the first hundred digits of the decimal expansion of $\frac{6}{7}$?

(A) 398 (B) 453 (C) 560 (D) 583 (E6) 601

16. Which is the largest of the following numbers?

(A) 2^{175} (B) 3^{125} (C) 4^{100} (D) 5^{75} (E) 6^{50}

17. Fifty numbers have an average of 76. Forty of these numbers have an average of 80. The average of the other ten numbers is

(A) 60 (B) 4 (C) 72 (D) 40 (E) 78

18. The sum of the largest and the smallest of the numbers 0.5129, 0.9, 0.89 and 0.289 is

(A) 1.189 (B) 0.8019 (C) 1.428 (E) 1.179 (E) 1.4129

19. Which of the following is true?
(A) $\pi = \frac{22}{7}$ (B) $\pi = 3.14$ (C) $\pi = 3.142$ (D) $\pi < \frac{22}{7}$ (E) $\pi > \frac{22}{7}$

20. 244×909 is equal to
(A) 221 792 (B) 221 793 (C) 221 794 (D) 221 796 (E) 221 798

21. Which of the following is a prime number?
(A) 451 (B) 452 (3) 459 (D) 465 (E) 467

22. Which of the following statements is true?
(A) $10^8 < 5^{12} < 2^{24}$ (B) $2^{24} < 10^8 < 5^{12}$ (C) $10^8 < 2^{24} < 5^{12}$
(D) $5^{12} < 10^8 < 2^{24}$ (E) $2^{24} < 5^{12} < 10^8$

23. $\dfrac{1}{7} + \dfrac{1}{30} + \dfrac{1}{42}$ is equal to
(A) $\dfrac{1}{5}$ (B) $\dfrac{4}{21}$ (C) $\dfrac{1}{6}$ (D) $\dfrac{7}{30}$ (E) $\dfrac{8}{35}$

24. If $a = 0.027027027\cdots$ and $b = 0.909090\cdots$ are recurring decimals, then $a + b$ is equal to
(A) $\dfrac{381}{407}$ (B) $\dfrac{835}{892}$ (C) $\dfrac{402}{397}$ (D) $\dfrac{329}{352}$ (E) $\dfrac{572}{611}$

25. What is the remainder when 123456789 is divided by 8?
(A) 1 (B) 3 (C) 5 (D) 7 (E) 0

26. What is the remainder when 1234×56789 is divided by 9?
(A) 0 (B) 2 (C) 3 (D) 5 (E) 8

27. $\dfrac{\sqrt{12} + \sqrt{48}}{\sqrt{27} + \sqrt{75}}$ is equal to
(A) $\dfrac{\sqrt{10}}{\sqrt{17}}$ (B) $\dfrac{3}{4}$ (C) $\dfrac{\sqrt{60}}{\sqrt{102}}$ (D) $\dfrac{2}{3}$ (E) $\dfrac{4}{7}$

28. $\dfrac{(\sqrt{2} - 1)^2}{2 - \sqrt{2}}$ is equal to
(A) $\sqrt{2} - 1$ (B) $2 - \sqrt{2}$ (C) $1 - \dfrac{1}{\sqrt{2}}$ (D) $3 - 2\sqrt{2}$ (E) $2 - \dfrac{3}{\sqrt{2}}$

29. Which of the following numbers is not a multiple of 7?
(A) 78 428 (B) 91 770 (C) 14 784 (D) 21 063 (E) 21 503

30. Which of the following integers is a perfect square?
(A) 31 415 (B) 92 653 (C) 58 978 (D) 31 684 (E) 27 182

Answers are on page 31. □

PI MILLION DOLLARS FOR HACKERS

At the third annual Pwnium hacking contest, Google offered prize money totalling pi million dollars (that's $3 141 592 to the nearest dollar) for any hacker who could punch a hole in its Chrome OS and browser. The contest was held in Vancouver, Canada, 6–8 March.

At the previous Pwnium contest, Chrome was successfully breached, but this year it emerged victorious, even after the contest had been extended by several hours. Finally the Google team announced "We just closed out the competition. We did not receive any winning entries but we are evaluating some work that may qualify as partial exploits." □

MATHEMATICAL WORDS

Many mathematical words have different meanings when they are used in real life.

What are the mathematical words defined by the following terms?

Scottish land agent	factory output	essential
percussion instrument	marriage	liquid mixture
financial instrument	forecast	road junction

The answers are on page 31. □

NUMBER MYSTERIES

"The average call to a call centre is answered after 38 seconds. In a lifetime you will spend 1,2 years waiting for call centres to answer."

(Radio advert for Wesbank on 567 Cape Talk)

"A bus leaves every 20 minutes from 05:30 to 22:00."

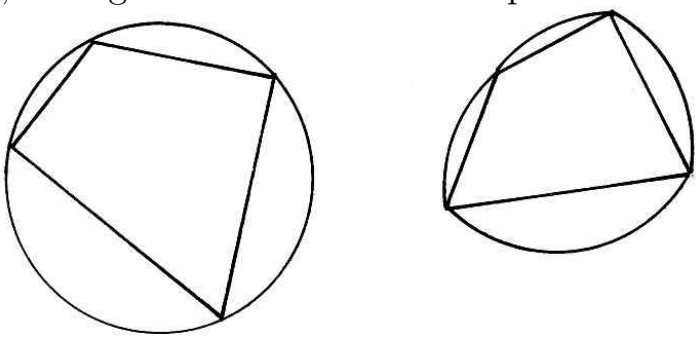
(Billboard at Cape Town International Airport)

(Submitted by Peter Waker) □

ANSWERS

Jewish Problems (page 12):

1. Consider the cyclic quadrilateral with the four given sides. If the quadrilateral is flexed, with the four segments of the circle attached to the sides, the figure will have the same perimeter as the circle.



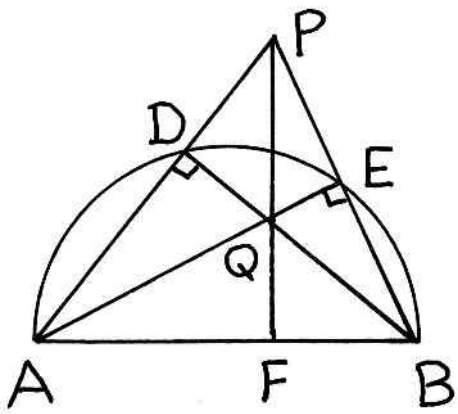
But a circle is the figure that encloses the maximum area for a given perimeter. Therefore the cyclic quadrilateral has the maximum area.

2. Set $a = x + h$ and $b = x$. Then the inequality becomes
$$-h \leqslant \frac{f(x+h) - f(x)}{h} \leq h$$
for all x and h. Letting $h \to 0$ gives $f'(x) = 0$ for all x, and therefore f is constant.

3. Note that $\frac{1}{2} = \sin 30^\circ = 3\sin 10^\circ - 4\sin^3 10^\circ$.
Setting $x = 2\sin 10^\circ$, this reduces to the equation $x^3 - 3x + 1 = 0$. Any rational root of this equation must be an integer, and the only possibilities (± 1) do not work. It follows that x is not rational.

4. Since $2^3 < 3^2$, $3 < 2\log_2 3$.
Since $5^2 < 3^3$, $2\log_3 5 < 3$.
So $\log_3 5 < \frac{3}{2} < \log_2 3$.

5. Let the diameter of the circle be AB. Join PA and PB, meeting the semicircle at D and E respectively.
Join DB and EA, intersecting at Q.
Draw PQ, cutting the diameter in F.
Since $BD \perp AP$ and $AE \perp PB$ (angles in a semicircle), they are altitudes of triangle PAB. Since the altitudes of a triangle are concurrent, $PF \perp AB$.



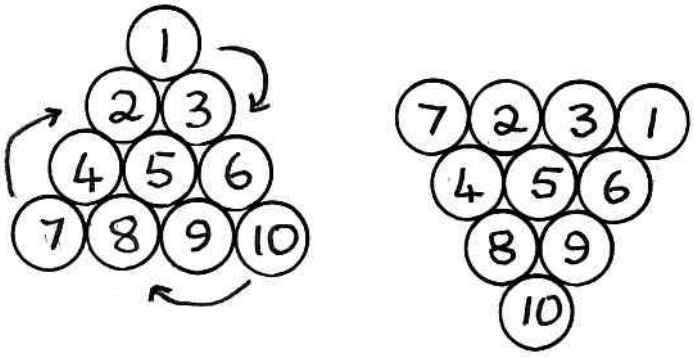
Number Oddity (page 13):
$854^2 = 729316$

Factorial Facts (page 13):
$2! = 0! \times 2! = 1! \times 2!$

Maths from Macbeth (page 15):
$789064 + 789064 + 1936 = 1598063$

Golden Oldies (page 17):

1.



2.

3. 50c (Did you fall into the trap and guess R1?)
4. No, it's not $7S = P$, but $7P = S$.
5. 3
6. None. The dead birds fell to the ground and the rest flew away.

Party Problems (page 21):

John was born in 1992, and in April 2013 was $1+9+9+2 = 21$ years old.

Mary was born in 1980 and will be 45 years old in $2025 = 45^2$.

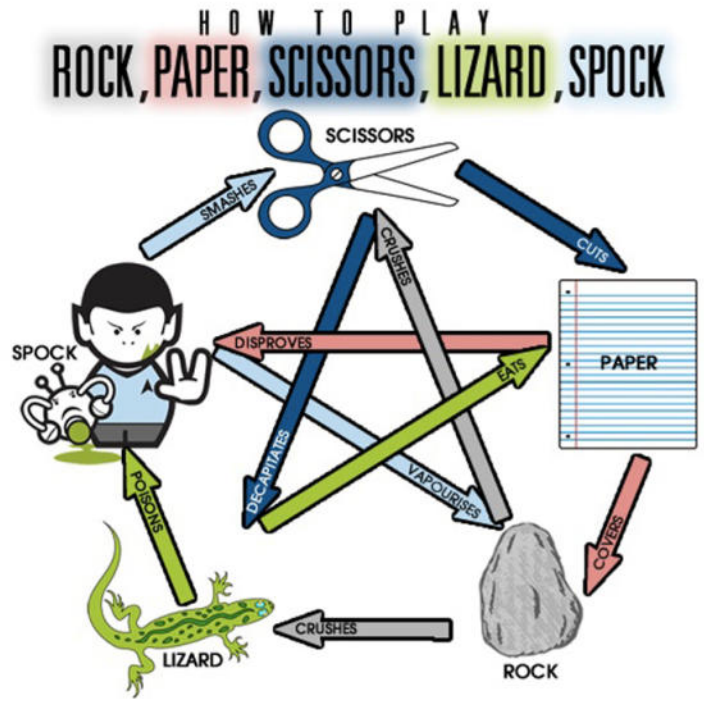
Varsity Arithmetic Test (page 26):

BEBBA BADEE ECACB CAAED EBAAA EBCED

Mathematical Words (page 29):
Factor, triangle, derivative, product, integral, projection, intersection, union, solution. □

mathematical digest

No. 172 July 2013



The Lizard-Spock Expansion

mathematical digest

Number 172 **July 2013**

If nature leads us to mathematical forms of great simplicity and beauty — by forms I am referring to coherent systems of hypothesis, axioms, etc. — to forms that no-one has previously encountered, We cannot help thinking that they are 'true', that they reveal a genuine feature of nature... You must have felt this too: the almost frightening simplicity and wholeness of the relationships which nature suddenly spreads out before us and for which none of us was in the least prepared.

(Werner Heisenberg, in a discussion with Albert Einstein)

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Cover: The cover design is by Nathanthenerd. The "Lizard-Spock Expansion" is described on page 6.

IMO 2014 IN AFRICA

In just one year from now, the International Mathematical Olympiad (IMO) will take place in South Africa.

This will be the culmination of several years of planning. The first step was to put together a bid to host the event, and in 2011 the IMO Jury (the Team Leaders of all the participating countries) approved South Africa as the host country for 2014.

Then the real work began. Universities and cities were invited to bid to host the event, and the University of Cape Town was chosen over stiff competition from Durban and Johannesburg. With more than 100 countries and over 1000 participants, the IMO is an expensive operation, since the host country undertakes to pay all the accommodation expenses of the visiting teams. With the endorsement of the Department of Basic Education and the Department of Science and Technology, and the support of all the top scientific societies in the country, a major fund-raising drive was mounted. The campaign was successful: by June 2013 the core funding for IMO2014 had been secured from a range of government and corporate sources.

IMO 2014
Cape Town - South Africa

While the countries taking part in the IMO currently represent over 90% of the world's population, participation by African countries is poor. Of the 54 countries in Africa, only Morocco, Tunisia, Nigeria and South Africa are regular IMO participants. These four, and Uganda, are the only African countries sending teams to the 2013 IMO in Colombia. South Africa is mounting a campaign to increase African participation, and it is hoped that the number of African participants will be at least doubled next year.

The IMO is a famously tough competition. Without thorough preparation, new teams can end with dismal scores. The South African IMO team always gets together for an intensive training camp before an IMO, and there are plans to expand next year's pre-IMO camp to include all African teams.

IMO2014, the first IMO to be held on the African continent, will be an "IMO for Africa". □

STEPHEN FRY ON MATHS

When he [my father] grasped the completeness of my ignorance and my incompetence he did not gulp or gasp, I'll give him that. He stuck by his own beliefs and went right back to the beginning. He taught me something that I did not understand: the equals sign. I knew what $2 + 2 = 4$ meant.

I did not understand, however, even the rudimentary possibilities that flowed from that. The very thought of an equals sign approximating a pair of scales had never penetrated my skull. That you could do anything to an equation, so long as you did the same to each side, was a revelation to me. My father, never once flinching at such staggering ignorance, moved on. There came the second revelation, even more beautiful than the first.

Algebra.

Algebra, I suddenly saw, is what Shakespeare did. It is metonym and metaphor, substitution, transferal, analogy, allegory: it is poetry. I had thought its a's and b's were nothing more fruitless (if you'll forgive me) than apples and bananas.

Suddenly I could do simultaneous equations.

Quadratic equations I pounced on because there was a formula you could remember for solving them. My father was not interested in my remembering formulas. Any fool can remember a formula. He wanted me to see why.

So we went back to the Greeks, to Pythagoras and Euclid.

Oh shit. Geometry. I hated geometry.

He decided that we would set out together, as if we knew nothing, to prove the suggestion that, so far as right-angles triangles were concerned, the square on the hypotenuse might well turn out to be equal to the sum of the squares on the other two sides.

Proof?

How could you prove such a thing ? The whole idea was completely alien to me. I suggested spending an entire day drawing right-angled triangles

of different sizes and checking. If they all conformed to this theorem then that would suit me fine.

Ho-no.

I don't remember the proof, I remember it took in circles and segments and sectors and angles which were temporarily designated the value Theta. But I remember that we got there and that I followed it all the way and I remember too that when the final, triumphant QED! was written on the bottom of the page I felt a thrill of genuine joy.

I can't claim my father made a mathematician out of me. I still speak only stumbling, schoolboy Maths with an atrocious English accent. I never quite got the hang of vectors. I have damned Descartes eternally for the foul things. My father can vectorise anything from Dutch Elm Disease, to a sunrise, to the act of opening a tin of beans. Plotting things against things, writing $4x = (x^2 - y^2)$ on a graph line, that sort of runic weirdness — absolute mystery to me.

(From *Moab is My Washpot*, the autobiography of Stephen Fry) □

NON-COMMUTATIVE SYSTEMS ANALYSIS

In the world of financial software development "five 9's" refers to a system which is available 99.999% of the time.

Here's what the nines mean in real terms:

90% ("one nine"): 36.5 days downtime per year

99% ("two nines"): 3.65 days

99.9% ("three nines"): 8.76 hours

99.99% ("four nines"): 52.56 minutes

99.999% ("five nines"): 5.26 minutes

99.9999% ("six nines"): 31.5 seconds

99.99999% ("seven nines"): 3.15 seconds

But just recently a software developer was overheard referring to his system's availability as "nine 5's". He was embarrassed when it was pointed out that his system was available on average for 55.5555555% of a year, namely 202.7777776 days. □

SUMS OF CONSECUTIVE SQUARES

David Allison

We all know that $3^2 + 4^2 = 5^2$. It's natural to ask whether there are any other solutions of the equation $n^2 + (n+1)^2 = (n+2)^2$. Well, that's a quadratic equation and $n = 3$ is not the only solution. You can easily find the other solution. It's unexciting.

We can get a bit more fun from the equation

$$n^2 + (n+1)^2 = (n+r)^2 \tag{1}$$

where n and r are positive while numbers. We've already dealt with the case $n = 2$; if we try $n = 3$ or 4 our equation only has irrational roots. Let's try another line of attack.

We already know a solution of (1) with $r > 2$, even if we haven't known that we knew it. For $(-4)^2 + (-3)^2 = 5^2$, and so $n = -4$, $r = 9$ is a solution of (1).

If we put $r = 9$ and solve the quadratic we find that its other solution is $n = 20$. That is,

$$20^2 + 21^2 = 29^2,$$

as you can easily verify. But then also

$$(-21)^2 + (-20)^2 = 29^2,$$

and this is the solution of (1) with $n = -21,\ r = 50$.

If we put $r = 50$ in (1) and solve the quadratic we find the other solution, $n = 119$. Thus

$$119^2 + 120^2 = 169^2,$$

One can go on like this indefinitely.

For, suppose that we know that (1) holds for some positive integers n and r. Then also

$$\big(-(n+1)\big)^2 + (-n)^2 = (n+r)^2. \tag{2}$$

If we put $x = -(n+1)$, then $n + r = -x - 1 + r = x + (r - 1 - 2x)$, so that $x = -(n+1)$ satisfies the equation

$$x^2 + (x+1)^2 = (x+R)^2, \tag{3}$$

where $R = r + n + (n+1) = 2n + r + 1$.

Since $x = -(n+1)$ is one root, and the sum of the roots is $2R - 2$, the other root is

$$x = 2R - 2 + (n+1) = 4n + 2r + 2 - 2 + (n+1) = 5n + 2r + 1.$$

Thus we have shown that if the numbers n and r satisfy (1), so also do the numbers N and R, where $N = 5n + 2r + 1$ and $R = 2n + r + 1$.

So the process we used to obtain our expressions for 29^2 and 169^2 as sums of two consecutive squares does, as we expected, carry on indefinitely. With more work it can be shown that all the solutions of (1) can be found this way, but that's another story. □

TREBLE

To *treble* something means to multiply by three. But in music, *treble* means something different. The treble clef in piano music is what is played by the right hand on the higher notes of the piano. In choral music a boy's voice, before it breaks, is referred to as treble. In a peal of bells, the smallest bell (giving out the highest note) is referred to as the treble bell.

What is the connection between this musical use of *treble* to indicate high notes, and the numerical meaning of the word?

In European choral music the different voices were classified in about 1500 into three levels. The lowest voice were called *Tenor* (what we would call baritone today) and the next level was called *Motetus* (high tenor). The highest register was the third, called *Triplex*. The choirboys who sang at this level came to be called *trebles*. □

WORD PLAY

The ten words

ALE INLAY GRAB GRAIL MORE MOTH QUIET SOIL TRYING UNTO

can be divided into five pairs so that each pair is an anagram of a mathematical word.

Find the five mathematical words. Answers are on page 31. □

THE LIZARD-SPOCK EXPANSION

Many generations of children have played rock-paper-scissors to decide who starts a game — or just for the fun of the game itself.

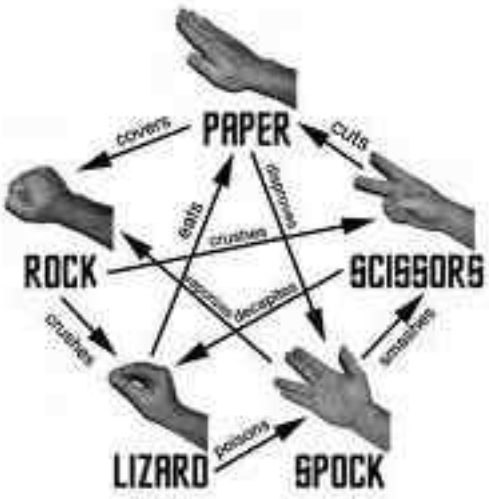
In the original rock-paper-scissors game, two players face each other with their right hands behind their back. They simultaneously shoot out their hands, showing either rock (a fist), paper (a flat palm) and scissors (a V-for-victory sign). The winner is decided on the basis that rock crushes scissors, scissors cuts paper and paper covers rock.

In the popular TV show *The Big Bang Theory*, the super-geek Sheldon Cooper expands the game by adding two more elements: Lizard and Spock.

In the Lizard-Spock expansion, Spock is an open hand with a gap between the third and fourth fingers and Lizard is formed by the hand with a sock-puppet type mount.

The rules, Sheldon says, are:

"Scissors cuts paper, paper covers rock, rock crushes lizard, lizard poisons Spock, Spock smashes scissors, scissors decapitates lizard, lizard eats paper, paper disproves Spock, Spock vaporizes rock, and as it always has, rock crushes scissors."



After the Big Bang episode the new game surged in popularity, and soon featured on mugs and T shirts. Big Bang Theory fans were surprised to learn that the game was not new — it was in fact invented by Sam Kass and Karen Bryla, who were greatly amused at its popularity.

In extending the original three-gesture game, two gestures had to be added. Why is it not possible to construct a four-gesture game? □

GOLDBACH VARIATIONS

The famous Goldbach Conjecture asserts that every even number greater than 2 can be expressed as the sum of two prime numbers. Thus

$$4 = 2 + 2,\ 6 = 3 + 3,\ 8 = 3 + 5, \cdots .$$

Although mathematicians believe that the Goldbach conjecture is true, nobody has been able to prove its truth since it was first formulated over 200 years ago by Christian Goldbach.

Goldbach himself gave a weaker form of the conjecture, that every *odd* number greater than 5 can be expressed as the sum of *three* primes.

It is clear that if the original Goldbach conjecture is true, then so is the weaker form. For, given an odd number greater than five, just subtract 3 and you to get an even number greater than 2, which is the sum of two primes. Adding the prime number 3, the original number is now the sum of three primes.

However, there is no way of going the other way.

Last year a major step in cracking the weak Goldbach conjecture was made: Terence Tao proved that all odd numbers are the sum of at most five primes. See MATHEMATICAL DIGEST **168** (July 2012) for the full story.

In May this year the final step was taken. Harald Helfgott, a Peruvian-born mathematician now working at the École Normale Supérieure in Paris, published a paper online that claimed to reduce Terence Tao's *five* to *three*.

Helfgott's proof is built on pioneering research dating back to the 1930s, when the Russian mathematician Ivan Vinogradov proved that all odd numbers below a certain number C, were the sum of three primes. The problem was that C turned out to be of the order of $10^{6846168}$, quite beyond the bounds of computation then, or now, or any time in the future.

Vinogradov's bound was gradually reduced over the next 70 years, and in 2002 it stood at 10^{1346}, still out of computational range. Helfgott's achievement was to cut the bound down to a manageable 10^{30}. With the help of David Platt of the University of Bristol, all odd numbers below that level were checked by computer, and shown to be the sum of three primes. The weak Goldbach conjecture has been laid to rest. □

FAKE PRIME RECORD

Two American mathematicians have discovered a 300-billion-digit "fake" prime number. What is fake about it, and is there any practical importance in their discovery?

Once upon a time some pure (and purist) mathematicians liked to boast that the study of prime numbers was completely useless, and that made number theory the most beautiful topic in mathematics. The boast is no longer true. While the study of prime numbers is still a beautiful topic, today they are useful — even indispensable — in our daily life. Very large prime numbers are the key to "unbreakable" codes, which lie at the heart of security in the transmission of data over the internet.

Proving that a very large number n is prime is a difficult problem. The only way is to divide n by the successive prime numbers 2, 3, 5, 7, 11, $\cdots$ less than $\sqrt{n}$ and see if there is a remainder. If there are no remainders, n is prime. This is a hugely time-consuming process, even with the fastest computers.

Back in the 17th Century the French mathematician Pierre de Fermat proved that, if p is a prime number, then for every positive integer a the number $a^p - a$ is divisible by p.

So if you want to show that a number n is composite, you just have to produce a value of a such that $a^n - a$ is not divisible by n. Moreover, you only have to check the values of a from 2 to n. This is the "Fermat primality test".

Unfortunately, the test is not watertight. It can happen that $a^n - a$ is divisible by n for all a, and yet n is composite. The smallest such number is 561: for all natural numbers a, $a^{561} - a$ is divisible by 561, yet 561 is not prime ($561 = 3 \times 11 \times 13$). So the "Fermat primality test" can fail.

Numbers that pass the Fermat primality test, but are not prime, are called *Fermat pseudoprimes*, or colloquially "fake primes". After 561, the next fake prime is 1105. They are thinly scattered. While there are nearly 80 000 primes in the first million numbers, there are only 43 fake primes in the same range.

In February this year Andrew Shallue and Steven Hayman of Illinois Wesleyan University in Bloomington found a record new fake prime, a 300-billion-digit monster. It dwarfed the record huge genuine prime number

of 17 million digits, found just the week before: $2^{57\,885\,161} - 1$. (See *The biggest prime yet*, MATHEMATICAL DIGEST **171** (April 2013), pp 4-5.)

Even though they are much rarer than primes, it is easier to find large fake primes than large genuine primes. Though there is a small chance of failure, using a fake prime rather than a true prime in a cryptographic system may be acceptable to the user. A study of fake primes therefore has practical importance in data security.

Reference

Largest fake prime number holds 300 billion digits, by Jacob Aron (*New Scientist*, 14 February 2013). □

PERCENTAGE ERRORS

Premier Helen Zille says only about 10 percent of the province's economically active population between the ages of 15 and 64 potentially qualify for a senior job in the Western Cape government.

This was because the minimum requirement for a senior job in a provincial government is a three-year degree.

Zille said this means that only 1.7 percent (or 43 738) of black people, 2.1 percent (or 51 658) of coloured people, 0.3 percent of Indian people and 6 percent of white people would qualify.

This works out to be just 10.1 percent of the economically active.

(Report in the *Cape Argus*, 18 April 2013)

Unsecured lending credit in the South African consumer credit market has shown a dramatic 389% increase, from a total of R40.9-billion in December 2007 to a total of R159.3-billion in December 2012.

(Moody's Investor Service, reported in both *The Times* and the *Cape Times*, 28 June 2013) □

BODMAS AND CALCULATORS

In May this year it was reported in the media that, in a national evaluation of teachers, it was found that 46% were unable to work out the correct answer in the calculation

$$10 \times 2 + (6 - 4) \div 2.$$

It appears that the teachers who got it wrong did the calculation from left to right:

> Start with 10, multiply by 2 to get 20, add 6 to get 26, subtract 4 to get 22 and then finally divide by 2 to get 11.

This procedure ignores the well-known BODMAS convention. BODMAS is an acronym for the order of precedence of operations:

> **B**rackets, **O**f, **D**ivision, **M**ultiplication, **A**ddition, **S**ubtraction

According to this convention, the first step in the calculation of

$$10 \times 2 + (6 - 4) \div 2$$

should be to replace $6 - 4$ by 2. That would give

$$10 \times 2 + 2 \div 2.$$

Multiplication and division then receive preference over addition, so the next operations are:

$$10 \times 2 = 20 \qquad \text{and} \qquad 2 \div 2 = 1.$$

The final step is addition:

$$20 + 1 = 21.$$

All that seems unarguable. But what about the following calculation, which has been getting some internet attention lately?

$$6 \div 2(1 + 2)$$

Following BODMAS, the bracket must be treated first, then division, then multiplication. So

$$6 \div 2(1 + 2) = 6 \div 2 \times 3 = 3 \times 3 = 9.$$

Or should we interpret the bracket calculation as $2(1 + 2) = 6$? In that case we have

$$6 \div 2(1 + 2) = 6 \div 6 = 1.$$

Which is correct?

We decided to reach for a calculator. In fact, we did the calculation on six different calculators (three in the Sharp EL531 series, and four in the Casio fx-82 series).

They all gave the answer 1.

That seemed decisive. They all interpreted the first bracket calculation as $2(1+2)$.

Then we used the same calculators, but keyed in the calculation

$$6 \div 2 \times (1+2),$$

which should, of course, give the same answer, because all we have done is filled in the implied multiplication sign between the 2 and the bracket.

This time they all gave the answer 9.

There are several messages here.

Calculators can be unpredictable if you don't understand their internal logic.

Conventions such as BODMAS, supposed to eliminate ambiguity in calculation, can still be ambiguous. How, for example, should one interpret $2 \div 3 \div 4$? Is it $(2 \div 3) \div 4 = 2/12 = 1/6$ or $2 \div (3 \div 4) = 8/3$?

BODMAS says nothing about exponents.

There is an updated version of BODMAS called PEDMAS, which stands for :

Parentheses, **E**xponents, **D**ivision, **M**ultiplication, **A**ddition, **S**ubtraction

And what about 2^{3^4}? Should it be interpreted as $2^{(3^4)} = 2^{81}$ or $(2^3)^4 = 8^4 = 2^{12}$?

The moral of the story is that our normal arithmetic notation is potentially ambiguous. If there is any doubt about the order in which a sequence of operations should be performed, use brackets.

At the back of a drawer we found a thirty-year-old Sharp EL-531 and set it to work. It gave $6 \div 2 \times (1+2) = 9$, as expected, but when we left out the multiplication sign we obtained the surprising

$$6 \div 2(1+2) = 2.2857143.$$

Can you figure out the logic of this vintage calculator?
The answer is on page 31. □

VARSITY READINESS TEST

How ready are you for university maths? Getting an A in the Senior Certificate Mathematics examination (a predictable sequence of routine exercises) is a necessary achievement, but may not be sufficient.

The problems below cover the standard mathematics curriculum, but are more challenging. Give yourself two or more hours to tackle them, without a calculator or formula sheet. Then turn to page 31 to check your answers.

There is no pass/fail mark. Every wrong answer, however, exposes a weakness in your mathematical preparation that needs to be addressed before your first lecture in Mathematics I at your chosen university.

1. The graph of $y = (x-3)(1-x)$ is tangent to the graph of $y = kx^2$. Determine k.
   (A) $\frac{2}{5}$ (B) $\frac{1}{3}$ (C) -2 (D) $\sqrt{2}$ (E) $-\frac{1}{2}$
2. The straight line $3x + y - 9 = 0$ is tangent to the curve $y = f(x)$ at $x = 4$. What is the value of $f(4) + f'(4)$?
   (A) -6 (B) 7 (C) 4 (D) -3 (E) 6
3. The solution of the inequality $\log_2 x + \log_2(x-3) < 2$ is
   (A) $-1 < x < 4$ (B) $x > 0$ (C) $0 < x < 4$ (D) $3 < x < 4$ (E) $x > 3$
4. The roots of the equation $x^2 - 2x - 7 = 0$ are a and b. Which of the following equations has roots $a+1$ and $b+1$?
   (A) $x^2 - 3x - 8 = 0$ (B) $x^2 - x - 6 = 0$ (C) $x^2 - 4x - 4 = 0$
   (D) $x^2 + 2x - 8 = 0$ (E) $x^2 + 2x - 5 = 0$.
5. In triangle ABC, with $AB = c$, $BC = a$ and $CA = b$,
   $$\frac{4 \times \text{Area } ABC}{b^2 + c^2 - a^2}$$
   is equal to
   (A) $\cos A$ (B) $\tan A$ (C) $\sin 2A$ (D) $\frac{1}{2}\cos A$ (E) $2\sin A$
6. If $f(x) = 3x + 5$ and $g(x) = 4x + 7$, then $g(f(x)) - f(g(x))$ is equal to
   (A) x (B) 0 (C) $x + 1$ (D) 1 (E) $x - 1$
7. In triangle ABC, D, E and F lie on sides AB, BC and CA, respectively, so that $BD = BE$ and $CE = CF$. If $\angle A = x$, what is the size of $\angle DEF$?
   (A) $90° + x$ (B) $180° - 2x$ (C) $90° + 2x$ (D) $90° - \frac{1}{2}x$ (E) $180° - x$

8. Let f be a function defined by the equation $f(x) = 3^x$, where x is a real number. Which of the following is true for all real numbers a and b?

(A) $f(ab) = f(a) + f(b)$ (B) $f(ab) = f(a)f(b)$
(C) $f(ab) = 3f(ab-1)$ (D) $f(a+b) = f(a) + f(b)$
(E) $f(3ab) = f(ab+3)$

9. If $f(2x+3) = 4x+5$, then $f(4x+5)$ is equal to

(A) $5x+6$ (B) $6x+7$ (C) $7x+8$ (D) $8x+9$ (E) $9x+10$

10. If $\cos 2x = \frac{23}{49}$ and $0° < x < 90°$, then $\cos x$ is equal to

(A) $\frac{4}{5}$ (B) $\frac{6}{7}$ (C) $\frac{1}{13}$ (D) $\frac{2}{3}$ (E) $\frac{7}{11}$

11. The length of the common chord of the two circles

$$x^2 + y^2 = 5 \text{ and } x^2 - 8x + y^2 + 11 = 0$$

is

(A) $\sqrt{5}$ (B) 2 (C) $\sqrt{3}$ (D) $\frac{1}{2}\sqrt{5}$ (E) $\frac{3}{2}$

12. In triangle ABC, D lies on BC such that AD is perpendicular to BC. If $AB = 19$, $BC = 15$ and $CA = 16$, then $BD - DC$ is equal to

(A) $5\sqrt{2}$ (B) $4\sqrt{3}$ (C) $\sqrt{47}$ (D) $4\sqrt{51}$ (E) 7

13. If $\dfrac{xy}{3x+2y} = \dfrac{1}{5}$ and $\dfrac{xy}{5x+7y} = 1$, then $\dfrac{xy}{x+y}$ is equal to

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

14. If $\dfrac{\sin 3x}{\sin x} = \dfrac{7}{3}$ and $0° < x < 90°$, determine $\dfrac{\cos 3x}{\cos x}$.

(A) $\dfrac{1}{4}$ (B) $\dfrac{3}{4}$ (C) $\dfrac{1}{3}$ (D) $\dfrac{2}{3}$ (E) $\dfrac{1}{2}$

15. In a convex quadrilateral $ABCD$, $\angle BAD = 80°$, $\angle ABC = 75°$, $\angle ADC = 65°$ and $AB = AC$. What is the size of $\angle BDC$?

(A) $10°$ (B) $15°$ (C) $20°$ (D) $25°$ (E) $30°$

16. If A and B are two points in the plane, the set of all points P such that AP is perpendicular to BP lie

(A) on a straight line
(B) on a parabola
(C) on a hyperbola
(D) on a circle
(E) on two straight lines

17. If $a > 1$ is a rational number, which of the following is true of the roots of
$$(a-1)x^2 + 2ax - x + 2 = 0?$$
(A) The roots are equal and rational.
(B) The roots are both irrational.
(C) The roots are not real.
(D) The roots have opposite signs.
(E) Both roots are negative rational numbers.

18. For which positive value of k do the equations $x^2+y^2 = 3$ and $x+y = k$ have exactly one solution?
(A) $\sqrt{5}$ (B) $2\sqrt{2}$ (C) $\sqrt{7}$ (D) $\sqrt{6}$ (E) $\sqrt{10}$

19. $\tan 3x$ is equal to
(A) $\dfrac{\tan x - 2\tan^3 x}{2 + 3\tan^2 x}$ (B) $\dfrac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}$ (C) $\dfrac{2\tan x - 3\tan^3 x}{3 - 2\tan^2 x}$
(D) $\dfrac{3\tan x - 2\tan^3 x}{1 - 2\tan^2 x}$ (E) $3\tan x$

20. The function $y = f(x)$ has a straight line graph. If $f(0) = 3$ and $f(f(0)) = 5$, then $f(f(f(0)))$ is equal to
(A) 7 (B) $\dfrac{19}{3}$ (C) $\dfrac{11}{5}$ (D) 4 (E) $\dfrac{8}{7}$

21. The sum of the roots of the equation $9^{2x+1} - 28 \cdot 3^{2x} + 3 = 0$ is
(A) $-\dfrac{1}{2}$ (B) $\dfrac{5}{2}$ (C) -2 (D) $\dfrac{1}{3}$ (E) 10

22. One of the roots of the equation $2x^2 - 11x + 15 = m$ is 5.
What is the other root?
(A) -5 (B) 3 (C) 6 (D) 10 (E) $\dfrac{1}{2}$

23. If $\log_3 6 = a$ and $\log_6 5 = b$, then $\log_3 10$ is equal to
(A) $a + b + \dfrac{1}{a}$ (B) $ab + a - 1$ (C) $\dfrac{a}{b} + a + b$ (D) $ab - b + 1$ (E) $a + b + 1$

24. What is the value of $\dfrac{2\cos 40^\circ - \cos 20^\circ}{\sin 20^\circ}$?
(A) 2 (B) $\sqrt{3}$ (3) $2\sqrt{2}$ (4) $\sqrt{5}$ (5) 3

25. Two numbers a and b are such that $a^4 < b^3$ and $b^6 < a^5$.
Which of the following is true?
(A) $a > 1$ and $0 < b < 1$ (B) $0 < a < 1$ and $b > 1$ (C) $ab > 1$
(D) $0 < a < 1$ and $0 < b < 1$ (E) $a > 1$ and $b > 1$ □

IMO IN COLOMBIA

The 54th International Mathematical Olympiad will take place in Santa Marta, Colombia, from 17 to 27 July. Over 100 teams will be taking part.

The South African team is
Lauren Denny (grade 12, Rustenburg Girls' High School)
Nashlen Govindasamy (grade 11, Star College)
Andrew McGregor (grade 9, Rondebosch Boys' High School)
Shaylan Lalloo (grade 12, Pearson High School)
Bronson Rudner (grade 9, South African College High School)
Robin Visser (grade 11, St George's Grammar School)

The Team Leader is Dr Jay van Zyl (Rhodes University) and the Deputy Leader is Dirk Basson (University of Stellenbosch).

South Africa's participation in the IMO is a project of the South African Mathematics Foundation, with sponsorship from Harmony Gold.

You can follow the IMO through the website `www.imo-official.org`. □

PAMO IN NIGERIA

This year's Pan African Mathematics Olympiad will take place in Abuja, Nigera, from 23 June until 2 July.

The South African team of four is

Nashlen Govindasamy (grade 11, Star College)
Andrew McGregor (grade 9, Rondebosch Boys' High School)
Yaseen Mowzer (grade 9, Fairbairn College)
Bronson Rudner (grade 9, South African College High School)

The team Leader is Maciek Stankiewicz and the Deputy Leader is Phil Labuschagne.

The team's participation is coordinated by the South African Mathematics Foundation and sponsored by Harmony Gold. □

UCT INVITATIONAL MATHEMATICAL CHALLENGE

The top contestants in the annual UCT Mathematics Competition (which took place this year on 17 April) were invited to write a follow-up paper, the UCT Invitational Mathematical Challenge.

Two papers were set: Junior (grades 8 and 9) and Senior (grades 10, 11 and 12). They were written on Saturday 18 May. The problems are below.

Each paper was allotted two hours. No calculators were allowed, and only answers were required.

Junior Challenge Problems

1. What is the largest possible angle in an isosceles triangle, if the difference between the largest and the smallest angle is $9°$?

2. Two unusual dice each have their faces labelled 1, 2, 3, 5, 7, 9. The two dice are thrown, and the numbers shown on their upper faces added up. How many different totals can be obtained?

3. The faces of a cube are to be painted either red or black. In how many different ways can this be done? (Two cubes are regarded as the same if one can be rotated to look like the other.)

4. Adam cycles at a constant rate and stops for a five-minute break at the end of every kilometre. Eve cycles at a constant rate which is three-quarters the rate that Adam cycles, but Eve takes a five-minute break at the end of every two kilometres. Adam and Eve begin cycling at the same time and arrive at the 50-kilometre mark at exactly the same time. How many minutes has it taken them?

5. Of the students attending a school party, 60% of the students are girls, and 40% of the students like to dance. After these students are joined by 20 more boy students, all of whom like to dance, the party is now 58% girls. How many students now at the party like to dance?

6. A right circular cone has base radius r and height h. The cone lies on its side on a flat table. As the cone rolls on the surface of the table without slipping, the point where the cone's base meets the table traces a circular arc centered at the point where the vertex touches the table. The cone first returns to its original position on the table after

making 17 complete rotations. The value of h/r can be written in the form $\sqrt{n}$, where m and n are positive integers and n is not divisible by the square of any prime. Find $m+n$.

7. A circle of radius 15 intersects another circle of radius 20 at right angles. What is the difference between the areas of the non-overlapping portions?

8. At a school morning market Alice buys 10 toffees, 4 pieces of fudge and a chocolate, costing R7.85. Bob buys 7 toffees, 3 pieces of fudge and a chocolate, costing R7.05.

   What did Charlie pay for one toffee, one piece of fudge and a chocolate?

9. An *ascending* number is one in which each successive digit is greater than the one before. Examples are 15, 238, 3456, etc.

   A *descending number* is one in which each successive digit is less than the one before. Examples are 953, 876, 32, etc.

   Find the three-digit descending number which is the square of an ascending number.

10. Three three-digit numbers a, b, and c are made of the nine digits $1, 2, 3, \cdots, 9$, with each digit used exactly once. If the ratio $a : b : c = 1 : 3 : 5$, determine a, b and c.

Senior Challenge Problems

1. When $(x+2)(2x+3)(3x+4)(4x+5)$ is multiplied out to give a polynomial of degree 4, what is the sum of its coefficients?

2. Find positive integers m and n such that $m^2 = n^2 + 9n + 9$.

3. In the trapezium $ABCD$, AB and DC are parallel. The point O lies inside the trapezium, such that $AB = AO = BO = DO$ and $CD = CO = CB$. Find the size of $\angle ADO$.

4. How many integers n, with $1 \leq n \leq 1000$, have the property that n^n is a square number?

5. In quadrilateral $ABCD$, AB is parallel to DC. If $AD = 11$, $DC = 7$ and $\angle D = 2\angle B$, determine AB.

6. Solve for x, y and z:

$xy + 3x + 2y = 26$
$yz + 5y + 3z = 9$
$zx + 2z + 5x = 2$

7. Ten points on a circle are joined in all possible ways, and no three lines meet in a common point inside the circle. How many intersection points are formed inside the circle?

8. Find a positive real value of x such that

$$(12x - 1)(6x - 1)(4x - 1)(3x - 1) = 5.$$

9. A sequence $\{a_n\}$ is defined by the following rule:
$a_1 = 20$, $a_2 = 13$ and $a_{n+1} = \dfrac{a_n + 1}{a_{n-1}}$ (for $n > 1$).
Determine a_{2013}.

10. In the square $ABCD$, E is the midpoint of BC. The lines AC and DE are joined, intersecting in F. Determine the ratio Area $ABEF$: Area CDF.

Solutions are on page 22. □

POINTS TO PONDER

If there is a 50-50 chance that something can go wrong, then nine times out of ten it will.

Decimals have a point.

Natural numbers are better for your health.

To a mathematician, real life is a special case.

Mathematics is like love, a simple idea, but it can get complicated.

Parallel lines actually do meet, but they are very discrete.

These days, even the most pure and abstract mathematics is in danger of finding an application.

Philosophy is a game with objectives and no rules. Mathematics is a game with rules and no objectives. □

ABEL PRIZE 2013

The Norwegian Academy of Science and Letters has awarded the Abel Prize for 2013 to the Belgian mathematician Pierre Deligne. He received the Abel Prize "for seminal contributions to algebraic geometry and for their transformative impact on number theory, representation theory, and related fields", to quote the Abel committee. Pierre Deligne received the Abel Prize from H.M. King Harald at an award ceremony in Oslo on the 21st of May.

The Abel Prize commemorates Niels Henrik Abel, Norway's most famous mathematician, who was born in 1802. Abel is remembered for resolving an old problem in algebra, that of finding a solution formula for the general quintic (fifth-degree) equation

$$ax^5 + bx^4 + cx^3 + dx + e = 0.$$

A method for solving the quadratic equation $ax^2 + bx + c = 0$ had been known in antiquity, while the cubic equation ($ax^3 + bx^2 + cx + d = 0$) and quartic equation ($ax^4 + bx^3 + cx^2 + dx + e = 0$) had been cracked by Italian mathematicians in the 16th Century. For the next 250 years mathematicians struggled to find a similar solution formula for the quintic. Abel's surprising discovery was a proof that is not possible to find a general formula, based on simple algebraic manipulation of the coefficents, to solve the quintic. His groundbreaking papers became known only after his tragic death from tuberculosis at the age of 26.

Pierre Deligne was born in Belgium in 1944. He was a recipient of a Field Medal in 1978 for his proof of the Weil Conjectures in 1973. Fields Medals have been likened to the Nobel Prizes (there is no Nobel Prize in Mathematics), but they are very different from Nobel Prizes, being awarded only every four years, and only to mathematicians under the age of 40. In many respects the Abel Prizes are the true equivalent of the Nobel Prizes, as they are awarded every year for lifetime contributions to mathematical research. □

THE 2013 UCT MATHEMATICAL OLYMPIAD

The top students in each of the Junior and Senior divisions of the UCT Mathematical Challenge were invited to take part in the UCT Mathematical Olympiad, held at UCT on Saturday 25 May. Two hours were given to provide full solutions of the problems below.

Junior paper

1. Prove that the 5-digit number $3aaa7$ is divisible by 37, for any digit a.
2. Find all positive integers m and n such that $20m + 13n = 2013$.
3. Can the numbers from 1 to 2013 be partitioned into two or more subsets, not necessarily of the same size, such that in each subset the largest number is equal to the sum of the other numbers in the subset?
4. Ten coins are in a row, with five showing HEADS and five TAILS.

   You are told to pick up any two coins and turn them over. Repeating this operation as many times as you like, can you end up with all coins showing HEADS?
5. It is possible to find two consecutive squares whose sum is again a square. An obvious example is $3^2 + 4^2 = 5^2$.

   (a) Find another example.

   (b) Prove that the sum of three consecutive squares can never be a perfect square.
6. A solid wooden cube is painted on four of its sides and then cut up into a number of identical small cubes. If 54 small cubes are found to be painted on just two sides, how many small cubes are painted on just one side?

Senior paper

1. It is possible to find two consecutive squares whose sum is again a square. An obvious example is $3^2 + 4^2 = 5^2$.

   (a) Find another example.

   (b) Prove that the sum of three consecutive squares can never be a perfect square.

2. Find all positive integers m and n such that $20m + 13n = 2013$.
3. Prove that $a\sqrt{\dfrac{2b}{a^2+1}} + b\sqrt{\dfrac{2a}{b^2+1}} \leq a + b$, for all positive numbers a and b.
4. Prove that $\sqrt[3]{\sqrt{52}+5} - \sqrt[3]{\sqrt{52}-5} = 1$.
5. A solid wooden cube is painted on four of its sides and then cut up into a number of identical small cubes. If 54 cubes small cubes are found to be painted on just two sides, how many small cubes are painted on just one side?
6. In triangle ABC, $\angle A = 30°$ and $\angle B = 50°$. A point M lies on the side AC such that $CM = CB$. Prove that $BM = AC$.

Solutions are on page 27. □

PORRIDGE PROBLEM

Here's another type of logical problem. Suppose that Pamela Potter's pease porridge is putrid provided that Pablo Picasso painted potted palms. Also, either Pablo Picasso painted potted palms, or Peter Piper did not pick a peck of pickled peppers. Also, there are two possibilities: either Peter Piper picked a peck of pickled peppers or else it is impossible that both Pablo Picasso did not paint potted palms and that Pamela Potter's porridge is not putrid. The question is: Is Pamela's porridge putrid?

See page 31 for the answer.

(From a Gresham Lecture by Robin Wilson) □

NAMES WITH A SECRET

The names of ten people hiding a mathematical secret are listed below.

Arlon Edwards, Pat Wood, Cath Reeves, Rudolfo Urquart, Jeff Ivesen, Sibisi Xumba, Rose Venables, Leigh Towers, Jenni Needham, Kate Niven

What are they concealing? The answer is on page 31. □

CHALLENGE SOLUTIONS

Junior Challenge

1. If the largest angle is $a°$, then the other two angles are either:

   $a°$ and $(a-9)°$, giving $a+a+a-9=180$, so that $a=63$,

   or $(a-9)°$ and $(a-9)°$, giving $a+a-9+a-9=180$, so that $a=66$.

   So the largest possible angle is $66°$.

2. The following sums can be obtained: 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 16, 18.

3. If the colours are red (R) and black (B) then we have the following possibilities:

   0R, 6B: 1; 1R, 5B: 1; 2R, 4B: 2; 3R, 3B: 2; 4R, 2B: 2; 5R, 1B: 1; 6R, 0B: 1.

   $1+1+2+2+2+1+1=10$

4. Let Adam's speed be u km/h. Then Eve's speed is $3u/4$.
   During the 50 km ride Adam stops 49 times, for a total stoppage time of $5/49 = 245$ minutes, and rides for $50/u$ minutes.
   During the same ride, Eve stops 24 times, for a total stopping time of $24 \times 5 = 120$ minutes, and rides for $50/\frac{3}{4}u = 200/3u$ minutes.

   Since they start and end the ride at the same time, we have the equation

$$\begin{aligned} 245 + 50/u &= 120 + 200/(3u) \\ \Rightarrow 125 + 50/u &= 200/(3u) \\ \Rightarrow 375u + 150 &= 200 \\ \Rightarrow u = 50/375 &= 2/15. \end{aligned}$$

   So Adam cycles for $\dfrac{50}{2/15} = 375$ minutes, taking $375 + 245 = 620$ minutes for the ride.

5. Let the number of boys and girls at the party be b and g, respectively. The ratio of boys to girls is $\frac{40}{60} = \frac{2}{3}$, giving the equation $3b = 2g$.

   After 20 more boys arrive, the ratio of boys to girls is $\frac{b+20}{g} = \frac{42}{58} = \frac{21}{29}$, giving the equation $29b + 580 = 42g$.

   Solving the two equations gives $b = 232$ and $g = 348$.

So the number of boys and girls at the party is $232 + 348 = 580$, and 40% of them like to dance, namely 232. After the extra 20 boys (all of whom like to dance) arrive, the total number of those who like to dance is 252.

6. The slant height of the cone is $\sqrt{h^2 + r^2}$, and this is the radius of the circle round which the cone rotates. The circumference of this circle is $2\pi\sqrt{h^2 + r^2}$, which is 17 times the circumference of the the base of the cone (which is $2\pi r$).

   So we have the equation

   $2\pi\sqrt{h^2 + r^2} = 17 \times 2\pi r$

   $\Rightarrow h^2 + r^2 = 289r^2$

   $\Rightarrow h^2 = 288r^2$

   $\Rightarrow h/r = \sqrt{288} = 12\sqrt{2}$, and $12 + 2 = 14$.

7. If the areas of the non-overlapping portions are a and b, and the common portion has area x, then $a + x = (15)^2\pi$ and $b + x = (20)^2\pi$. So $b - a = 400\pi - 225\pi = 175\pi$.

8. If the prices are t for a toffee, f for a piece of fudge and c for a chocolate, we obtain two equations:

$$10t + 4f + c = 785 \qquad \textit{and} \qquad 7t + 3f + c = 705$$

   Multiply the first equation by 2 and the second equation by 3 and subtract, to get $t + f + c = 545$.

9. The smallest and largest three-digit descending numbers are 210 and 987, respectively. So the number required will be the square of a number between 15 ($15^2 = 225$) and 31 ($31^2 = 961$)

   Since the descending number n^2 cannot end in 9, n cannot end in 3 or 7. So the possible values of n are 15, 16, 18, 19, 24, 25, 26, 28, 29 which have squares 225, 256, 324, 625, 676, 784 and 841. Only 841 is a descending number.

10. Suppose $a = a_1a_2a_3, b = b_1b_2b_3, c = c_1c_2c_3$, and $a < b < c$.
    Then $a_1 = 1$. (If $a_1 > 1$, then $c > 1000$.)
    Next, $c_3 = 5$ (since c is a multiple of 5 and the digit 0 is not allowed).
    Also a_3 is odd (otherwise $c_3 = 0$) and therefore b_3 is odd (since $b = 3a$).

    Now $b_1 > 2$ (since otherwise $a < 100$) and b_3 is odd (since a_3 is odd).
    Further, $b_1 < 6$ (since $b_1 \geq 6$ implies $a \geq 200$).
    Since $c_3 = 5$, we have only two possibilities for b_1: 3 or 4.

Noting that b is a multiple of 3, we list the possible values of b, with the corresponding values of a and c below:

b	327	369	387	423	429	483	489
a	109	123	129	141	143	161	163
c	545	615	645	705	715	805	815

and the only triplet using all nine digits is $a = 129, b = 387, c = 645$.

Senior Challenge

1. If $p(x)$ is a polynomial $a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$, the sum of its coefficients $a_0 + a_1 + a_2 + \cdots + a_n$ is equal to $p(1)$. So the sum of the coefficients of $(x + 2)(2x + 3)(3x + 4)(4x + 5)$ is $3 \times 5 \times 7 \times 9 = 945$.

2. The discriminant of the quadratic in n must be a perfect square:

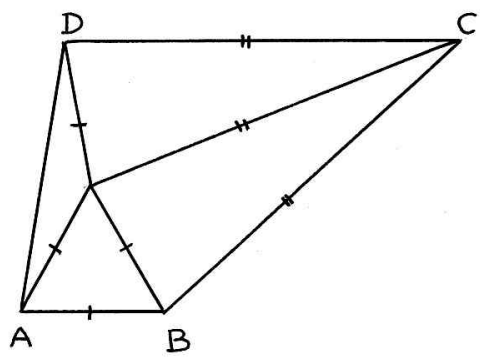
$$81 - 4(9 - m^2) = k^2.$$

This simplifies to $(k + 2m)(k - 2m) = 45$. Since k and m are integers, we consider the factorisations of 45:

$k + 2m = 45, k - 2m = 1$ gives $m = 11, k = 23$ and $n = 7$.

No other factorisations of 45 give positive values for m and n.

3. Since triangle ABO is equilateral, $\angle OAB = \angle OBA = \angle AOB = 60°$. Let $\angle ADO = x$. Since $OA = OD$, $\angle DAO = x$ and hence $\angle AOD = 180° - 2x$.
Since $AB||DC$, $\angle BAD + \angle ADC = 180°$, giving $\angle ODC = 120° - 2x$.
Since $CD = CO$, $\angle DOC = 120° - 2x$.



Since triangles COD and COB are congruent, $\angle COB = 120° - 2x$.

Considering the angles around O, we have

$$(180° - 2x) + 60° + (120° - 2x) + (120° - 2x) = 360°$$

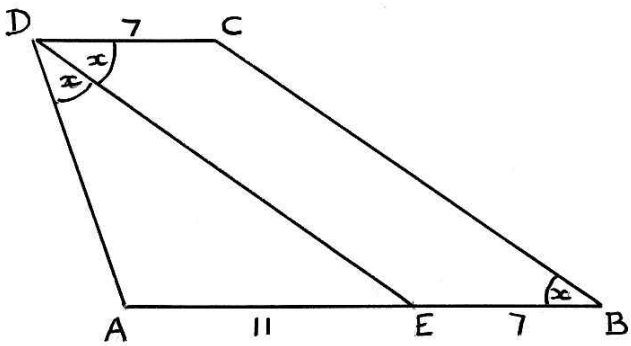
and it follows that $x = 20°$.

4. If n is even, then n^n is square. There are 500 even numbers between 1 and 1000.

If n is odd, then n^n can only be square if n itself is a square. So we must count the odd squares less than 1000. Since $31^2 = 961 < 1000 < 1024 = 32^2$, the odd squares in the range from 1 to 1000 are $1^2, 3^2, 5^2, \cdots, 31^2$: 16 altogether.

So there are $500+16 = 516$ values of n for which n^n is a perfect square.

5. Let $\angle D = 2x$. Let ED be the bisector of $\angle D$, with E on AB. Then $\angle ADE = \angle EDC = \angle EBC$. $EBCD$ is a parallelogram, with $EB = 7$. So $AB = AE + EB = 11 + 7 = 18$.



6. The equations can be written in the form

$(x+2)(y+3) = 32$
$(y+3)(z+5) = 24$
$(z+5)(x+2) = 12$

Multiply them together and taking square roots gives two cases:

(1) $(x+2)(y+3)(z+5) = \sqrt{32 \times 24 \times 12} = 96$.

Dividing this equation by each of the above equations give $z+5 = 3$, $x+2 = 4$ and $y+3 = 8$. So $x = 2, y = 5, z = -2$.

(2) $(x+2)(y+3)(z+5) = \sqrt{32 \times 24 \times 12} = -96$.

Dividing this equation by each of the above equations give $z+5 = -3$, $x+2 = -4$ and $y+3 = -8$. So $x = -5, y = -11, z = -8$.

7. An intersection point is formed by two lines, and each line is defined by two points. Conversely, when four points on the circle are joined in all possible ways, just one intersection point is formed inside the circle. So the number of intersection points is equal to the number of ways of choosing four of the ten points, namely

$$\binom{10}{4} = \frac{10.9.8.7}{1.2.3.4} = 210.$$

8. Putting $x = \dfrac{y}{12}$, the equation becomes

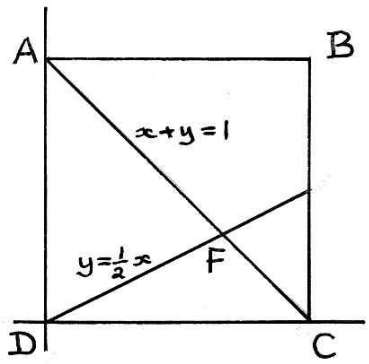
$$(y-1)(y-2)(y-3)(y-4) = 120 = 2.3.4.5.$$

So $y = 6$ and hence $x = \frac{1}{2}$.

9. The terms of the sequence are 20, 13, $\dfrac{7}{10}$, $\dfrac{17}{130}$, $\dfrac{21}{13}$, 20, 13, $\cdots$ and the sequence recurs in cycles of five. So $a_{2013} = a_3 = \dfrac{7}{10}$

10. Set the square in a coordinate system with vertices $A(0,1)$, $B(1,1)$, $C(1,0)$ and $D(0,0)$. Then the equation of line DE is $y = \frac{1}{2}x$ and the equation of AC is $x+y = 1$. Solving the equations gives the coordinates of F: $(\frac{2}{3}, \frac{1}{3})$.

From the figure it is seen that
Area triangle $ADF = \frac{1}{2} \cdot 1 \cdot \frac{2}{3} = \frac{1}{3}$
Area triangle $ECF = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{12}$
Area triangle $CDF = \frac{1}{2} \cdot 1 \cdot \frac{1}{3} = 16$.
It follows that the area of quadrilateral $ABEF$ is $1 - \frac{1}{3} - \frac{1}{12} - \frac{1}{6} = \frac{5}{12}$. So
Area $ABEF$: Area $CDF = 5 : 2$.



GOOD MATHS EQUALS SUCCESS

Children with top maths skills at the end of primary school will go on to earn considerably more than middle-ability classmates later in life, new UK research suggests. A pupil who scores highly in the subject can expect to receive about £2100 (about R28 500) extra each year by the time they reach 30, it was claimed.

The report by the London-based Institute for Fiscal Studies found that a decent grasp of maths at the age of 10 could add more that 7% to a child's eventual earning power. Researchers also found evidence of a wage premium for good reading skills, although the effect was considerably less marked than for maths.

The disclosure underlines the importance of ensuring that pupils have a good understanding of adding, subtracting and dividing by the time they leave primary school.

Last year, figures showed that one in six pupils failed to reach the basic standard expected for their age by the time they started secondary education.

(*Sunday Times*, 10 March 2013) □

OLYMPIAD SOLUTIONS

Junior paper

1. $3aaa7 = 30007 + 10 \times a \times 111 = 37 \times 811 + 30 \times a \times 37$.

2. We begin by noting that in any solution pair (m, n), $m \leq 100$, and that $m = 100$ and $n = 1$ satisfies the equation.

   Now suppose that we have two different solutions of the equation:

   $$20m + 13n = 2013 \text{ and } 20p + 13q = 2013.$$

   Then $20(m - p) = 13(q - n)$.

   Since 20 and 13 have no common factors, it follows that $m - p$ is a multiple of 13 and $q - n$ is a multiple of 20. Starting with our first solution for (m, n), namely $m = 100$ and $n = 1$, we obtain all the other solutions by subtracting successive multiples of 13 from 100, obtaining (87, 21), (74, 41), (61, 61), (48, 81), (35, 101), (22, 121) and (9, 141).

   Alternatively:

   Since $20m$ ends in 0, $13n$ must end in 3, and so the last digit of n must be 1.

   On the other hand, $13n < 2013$, so $n \leq 154$. So the possible values of n are 1, 11, 21, 31, ... 151.

   Since 2013 is one more than a multiple of 4, and $20m$ is a multiple of 4, $13n$ must be be one more than a multiple of 4.
   So n is one more than a multiple of 4, and the possible values of n are: 1, 21, 41, 61, 81, 101, 121, 141.
   The corresponding values of m are 100, 87, 74, 61, 48, 35, 22, 9.

   Putting them together: $(m, n) =$ (100, 1), (87, 21), (74, 41), (61, 61), (48, 81), (35, 101), (22, 121) or (9, 141).

3. Such a partition is not possible. If in each subset of the partition the largest number is the sum of the other numbers, then the sum of the numbers in that subset is even, and therefore the sum of all the numbers is even. But the sum of the numbers from 1 to 2013 is $\frac{1}{2} \times 2013 \times 2014 = 2013 \times 1007$: an odd number.

4. The answer is No.

   We can describe the state of the coins by the notation (m, n): m coins HEADS and n coins TAILS. The initial state is $(5, 5)$: both m and n are odd.

We shall prove that if the state is (m, n), with both numbers odd, before two coins are turned over, the new state will again have both numbers odd.

If one coin showing HEADS and one coin showing TAILS are turned over, the state is unchanged at (m, n): both numbers are odd.

If two HEADS are turned over, the state changes from (m, n) to $(m-2,\, n+2)$. Both numbers are still odd.

If two TAILS are turned over, the state changes from (m, n) to $(m+2,\, n-2)$. Both numbers are still odd.

So there is no way that we can turn coins over to achieve the state $(10, 0)$ (in which both numbers are even).

5. (a) $20^2 + 21^2 = 29^2$

   (b) Note first that $n^2 + (n+1)^2 + (n+2)^2 = 3n^2 + 6n + 5$, which leaves a remainder of 2 on division by 3.

   Note next that a square number is of the form $(3n)^2, (3n+1)^2 = 9n^2 + 6n + 1$ or $(3n+2)^2 = 9n^2 + 12n + 4$, so is either divisible by 3 or leaves a remainder of 1 on division by 3.

   So the sum of three consecutive squares can never be a square.

6. Suppose the cube is cut into n^3 small cubes.

   If the two unpainted faces are opposite each other, the cubes along the four edges connecting these two faces will be painted on just two sides. So $4n$ cubes will be painted on just two sides. Since 54 is not a multiple of four, we must conclude that the two unpainted faces are adjacent.

   In this case there are $4(n-1) + (n-2) = 5n - 6$ cubes with just two faces painted. Setting $5n - 6 = 54$ gives $n = 12$.

   Suppose that the front and top face of the $12 \times 12 \times 12$ cube are unpainted. Then on each of the back and bottom faces there are 110 small cubes painted on one side only, and 121 on each of the two side faces: 462 altogether.

Senior paper

1. See Junior paper question 5.

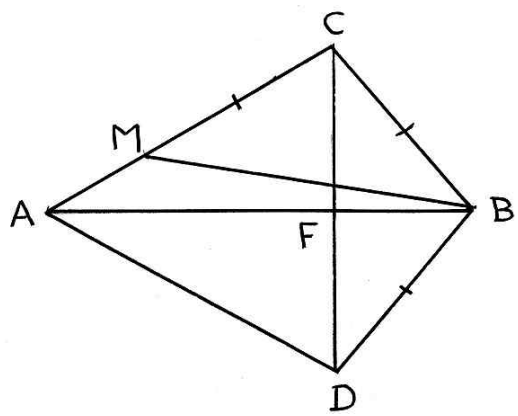
2. See Junior paper question 2.

3. $a\sqrt{\dfrac{2b}{a^2+1}} + b\sqrt{\dfrac{2a}{b^2+1}} = \sqrt{ab}\Big(\sqrt{\dfrac{2a}{a^2+1}} + \sqrt{\dfrac{2b}{a^2+1}}\Big) \leq 2\sqrt{ab}$, since $x^2+1 \geq 2x$ for all values of x.

4. Let $X = \sqrt[3]{\sqrt{52}+5} - \sqrt[3]{\sqrt{52}-5}$.
Using the identity $(a-b)^3 = a^3 - b^3 - 3ab(a-b)$, we deduce that
$$\begin{aligned} X^3 &= (\sqrt{52}+5) - (\sqrt{52}-5) \\ &\qquad - 3\big(\sqrt[3]{\sqrt{52}+5}\cdot\sqrt[3]{\sqrt{52}-5}\big)\big(\sqrt[3]{\sqrt{52}+5} - \sqrt[3]{\sqrt{52}-5}\big) \\ &= 10 - 3\sqrt[3]{52-25}\cdot X \\ &= 10 - 9X. \end{aligned}$$
So $X^3 + 9X - 10 = 0$.
Since $X^3+9X-10 = (X-1)(X^2+X+10)$, and the quadratic equation $X^2+X+10 = 0$ has no real solutions, the only real solution of the equation $X^3+9X-10=0$ is $X=1$.

5. See Junior paper question 6

6. Construct $BD = BC$ with $\angle CBD = 100°$, and join CD and DA.
Since triangle BCD is isosceles, with $\angle CBD = 100°$, it follows that $\angle BCD = \angle BDC = 40°$. Since $CM = CB$ (given), triangles BCD and CMB are congruent, and hence $BM = DC$.



Since $\angle CBA = 50°$, and $\angle CFB = 90°$, the line BA is the perpendicular bisector of BC, and the figure is a kite, with BA bisecting $\angle CAD$. It follows that $\angle CAD = 60°$ and triangle ADC is equilateral. So $BM = DC = AC$. □

DAM RIGHT

"They built the dams too big, so they take a long time to fill. They should have built them smaller, they would fill quicker and we would have plenty of water and never run out."

(Spokesman on water affairs from an anonymous town council) □

UCT MATHEMATICS OLYMPIAD RESULTS

Junior division

1. Bronson Rudner (South African College High School)
1. Yaseen Mowzer (Fairbairn College)
3. Andrew McGregor (Rondebosch Boys' High School)

Senior division

1. Lauren Denny (Rustenburg Girls' High School)
2. Robin Visser (St. George's Grammar School)
3. Musa Omar (Livingstone High School)

HOW OLD?

On his birthday in April this year John said "My age is the sum of the digits of the year in which I was born."

When was John born?

Mary said "I will be x years old in the year x^2."

When was Mary born?

Answers may be found on page 31. □

CHEWING GUM AIDS MEMORY

Previous research has already found that chewing gum can boost concentration in visual memory tasks. But, in a new study, Cardiff University researchers have found that it also enhances concentration in audio memory tasks.

The study examined 38 participants who were split into two groups and were scored on how accurately and quickly they could detect a sequence of odd numbers. Those who chewed gum had quicker reaction times and more accurate results than those who did not.

(*Sunday Times*, 10 March 2013) □

ANSWERS

Word Play (page 3):
GRAB + ALE → ALGEBRA
UNTO + SOIL → SOLUTION
QUIET + INLAY → INEQUALITY
TRYING + MORE → TRIGONOMETRY
LOATH + GRIM → LOGARITHM or ALGORITHM

BODMAS and Calculators (page 10):
In doing the calculation $6 \div 2(1 + 2)$, the 1980s vintage Sharp EL 531 ignored the brackets and worked out $6 \div 21 + 2 = 2.2857143$.

Varsity Readiness Test (page 12):
BADCB DDCDB BEACB DEDBB AEBBD

Porridge Problem (page 21):
If we let P denote the statement "Pamela's porridge is putrid", Q the statement "Picasso painted potted palms" and R the statement "Peter Piper picked a peck of pickled peppers", then:

(1) P implies Q ; (2) either Q or not R is true; (3) either R or not (not$P$ and not Q) is true. Suppose now that P is false. Then, by (1) Q is false, and so by (2), R is false. But then (3) is contradictory. So it follows that P is true: Pamela's porridge is putrid.

Names with a Secret (page 21):
ArlONEdwards, PaTWOod, CaTHREEves, RudolFOURquart, JefFIVEsen, SibiSIXumba, RoSEVENables, LEIGHTowers, JenNINEedham, KaTENiven

How Old (page 30):
Answers: John was born in 1992, and will be $1 + 9 + 9 + 2 = 21$ years old in 2013.

Mary was born in 1980 and will be 45 years old in $2025 = 45^2$.

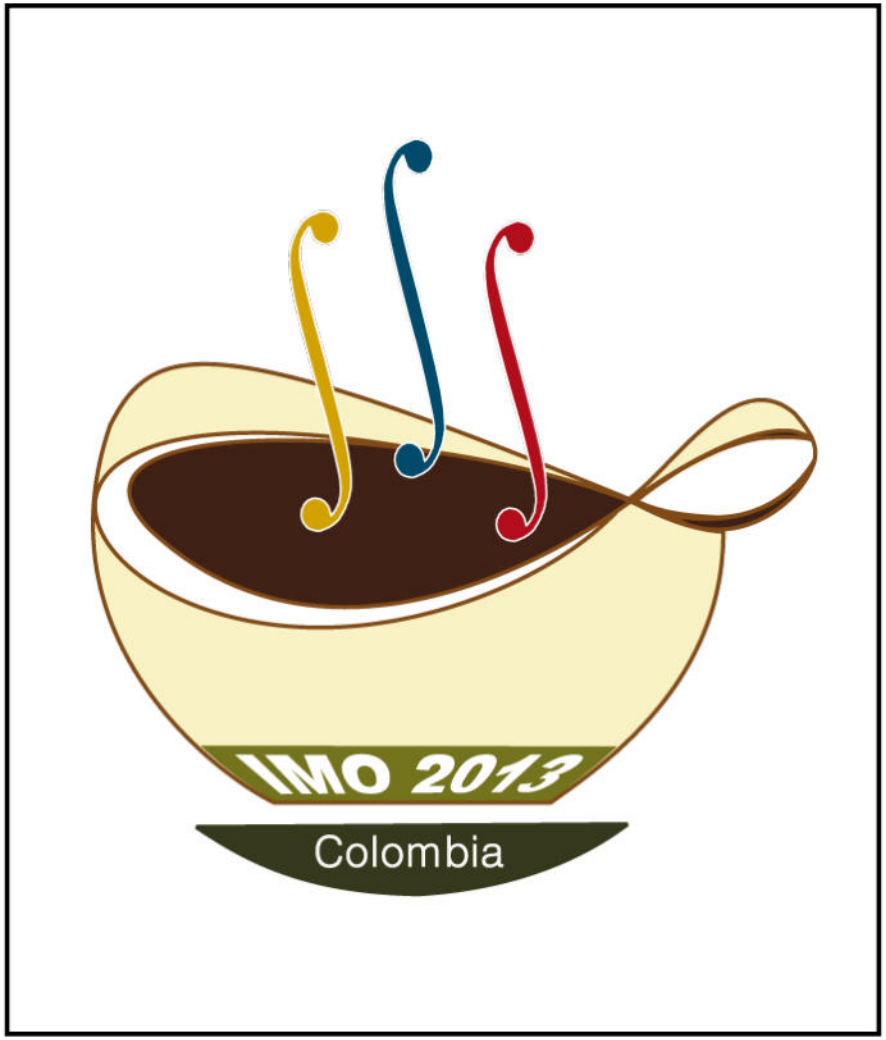
QUOTE

I have to keep my mind active. I have a capacious memory for stuff. I can remember masses of dialogue, poetry, number games. I play number games to keep my mind alive.

(Antony Hopkins, in the *Sunday Times*, 7 April 2013) □

mathematical digest

No. 173 October 2013



mathematical digest

Number 173 **October 2013**

In the mathematics I can report no deficience, except that it be that men do not sufficiently understand the excellent use of the pure mathematics, in that they do remedy and cure many defects in the wit and faculties intellectual. For if the wit be too dull, they sharpen it; if too wandering, they fix it; if too inherent in the sense, they abstract it. So that as tennis is a game of no use in itself, but of great use in respect it maketh a quick eye and a body ready to put itself into all postures; so in the mathematics, that use which is collateral and intervenient is no less worthy than that which is principal and intended.

(Roger Bacon, 13th Century philosopher and Franciscan friar)

Department of Mathematics and Applied Mathematics
University of Cape Town

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Cover: Articles about the International Mathematical Olympiad are on pages 1 - 3.

IMO 2014

Cape Town - South Africa

THE ELEMENTS OF OLYMPIADS

Mary Falk de Losada
Jury Chairman, IMO 2013

There are so many reasons for taking part in problem-solving competitions in mathematics, and so many different benefits to be expected that I will be able to name only a few.

There is the element of fun. Challenges, among them those found in the IMO problems, almost always are taken up because they have an element of fun in them. Fun can lead to engagement, and develop into a passionate calling. And the IMO combines the elements of the highest of goals with the best of fun.

There is the element of surprise. Surprises make life interesting, joggling our routines; persons, places, facts, problems that we already know can be precious to us, but surprises make us adjust our attitudes and actions in new ways that enrich our lives.

There is the element of beauty: a beautiful problem created by a beautiful mind, is more than satisfying, it is joyful and uplifting. And so is a beautiful solution such as those many of you will create and that will fill your teachers and families with pride.

There is the element of learning, learning about mathematics, about what our mathematical mentors expect of us, about what we are capable of doing, about adjusting our own expectations.

There is the element of belonging, belonging to a tradition of many millennia, a human endeavor, a community of thinkers, creators of knowledge and understanding. And with the community comes friendship: many of the friendships you have formed with your mentors, fellow team members and contestants from other countries will turn into lifelong companionships and lasting collaborations.

There is the element of truth. When you are facing a problem there is no way to bluff, to fool yourself or others, you are alone, constantly assessing yourself and your own progress, you have a real opportunity to take to heart Shakespeare's wonderful advice about above all being true to your own self.

(Extract from Welcoming Address at the 54th International Mathematical Olympiad in Santa Marta, Colombia) □

IMO IN COLOMBIA

The 54th International Mathematical Olympiad took place in Santa Marta, a resort city on the Caribbean coast of Colombia, from 17 to 27 July. Although over 100 teams were expected, there were some last-minute cancellations, and the total number of countries ended up at 97.

The South African team was: Lauren Denny (grade 12, Rustenburg Girls' High School), Nashlen Govindasamy (grade 11, Star College), Andrew McGregor (grade 9, Rondebosch Boys' High School), Shaylan Lalloo (grade 12, Pearson High School), Bronson Rudner (grade 9, South African College High School) and Robin Visser (grade 11, St George's Grammar School).

The team was led by Dr Jay van Zyl (Rhodes University), with Deputy Leader Dirk Basson (University of Stellenbosch).

The IMO is a complex organization. The Leaders arrive a few days before the teams to set the question papers from a list of problems put together by an international committee. The problems have to cover the four general fields of algebra, geometry, combinatorics and number theory, and must be completely new. Two papers consisting of just three problems each are selected, to be tackled on two successive days. The papers are then translated into over 50 languages, from Afrikaans to Vietnamese. For this year's papers, see `www.imo-official.org/problems`.

The teams arrived as the papers were being finalised, and the IMO began officially the next day with a lavish Opening Ceremony. It featured many speeches, some thoughtful, some braggingly political, interspersed with much exuberant Latin American music and ending with the traditional parade of flag-waving teams led by weird and wonderful Mardi Gras characters and flamboyant dancers.

The next two days were hard work for the teams: four-and-a-half hours of problem solving each day. Meanwhile the Leaders relaxed, visiting the picturesque old town of Cartagena (with echoes of the Spanish colonial times and Pirates of the Caribbean).

After the papers had been written it was back to work for the Leaders, who had to mark their own team's papers, then present their assessments

to an international team of Coordinators, whose job was to ensure that all the scores were fairly decided.

But now the teams could take it easy, which is what one does in the tropics in summer. The Santa Marta resort is stretched across a long beachfront, so there were the warm waters of the Caribbean to enjoy, with beach volleyball for the energetic. An enormous swimming pool was popular in the steamingly hot climate, especially when the beach sand was too hot to walk on.

But the maths wasn't over. For the enthusiasts there was a programme of lectures, the most popular of which proved to be one by Harald Helfgott on his recent proof of the "weak Goldbach conjecture": every odd number greater than 5 can be expressed as the sum of three prime numbers. (See *Mathematical Digest* **172** (July 2013) for more details.)

After two days of intense wrangling behind closed doors, the Leaders were agreed on the scores and the award of medals to the top half of the contestants: Gold, Silver and Bronze in the ratio $1 : 2 : 3$. Team scores were also totted up, and in a close contest the winning country was China, with 208 points out of a possible 242, just ahead of South Korea (204). The USA was third on 190 points, and the rest of the top ten were Russia (187), North Korea (184), Singapore (182), Vietnam (180), Taiwan (176), United Kingdom (171) and Iran (168).

The South African team was ranked 58th. Robin Visser and Shaylan Lalloo won Bronze Medals, while Lauren Denny, Nashlen Govindasamy and Andrew McGregor were awarded Honourable Mentions for solving at least one problem completely.

The Closing Ceremony was held at the memorial to the South American liberator Simon Bolivar (who died in 1830 in Santa Marta). A military wreath-laying was followed by speeches and music and by the presentation of Gold, Silver and Bronze medals. The evening ceremony ended with a ceremonial handing-over of the IMO flag by the Colombians to the South African team, and a video inviting everybody to come to Cape Town in 2014 to take part in the first IMO ever to be held on the African continent.

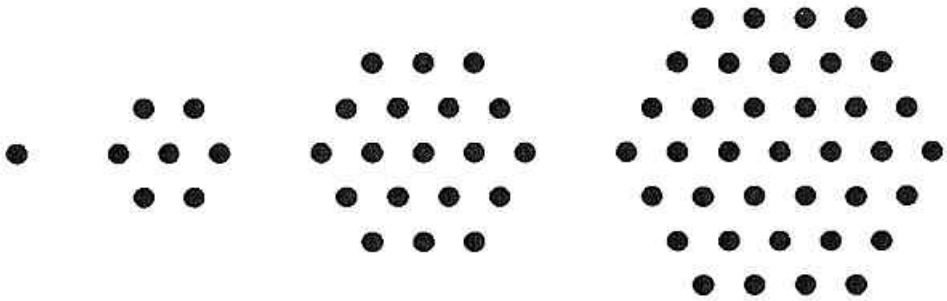
South Africa's participation in the IMO is a project of the South African Mathematics Foundation, with sponsorship from Harmony Gold.

Websites: `www.imo2014.org.za` (view the IMO welcome video here) and `www.imo-official.org`.
Post your comments on Facebook and Google+. □

HEXAGONAL NUMBERS AGAIN

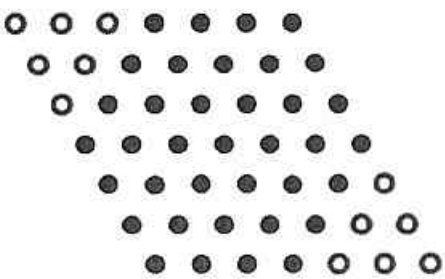
Duncan Samson
Rhodes University

Polygonal numbers such as triangular, square, pentagonal and hexagonal numbers can be represented diagrammatically by regular arrays of dots. The nth hexagonal number can be determined from the formula $H_n = 3n^2 - 3n + 1$.



In *Mathematical Digest* **169** (October 2012) three ways of deriving the formula for hexagonal numbers were presented. This was followed up in *Mathematical Digest* 170 (January 2013) with an additional method of deriving the formula. Two further visual approaches are described here. Each method uses the approach of visualising the hexagonal array in terms of other polygonal numbers, specifically triangular and square numbers.

The first method involves seeing the nth hexagonal number as being embedded in the $(2n-1)$th square number:

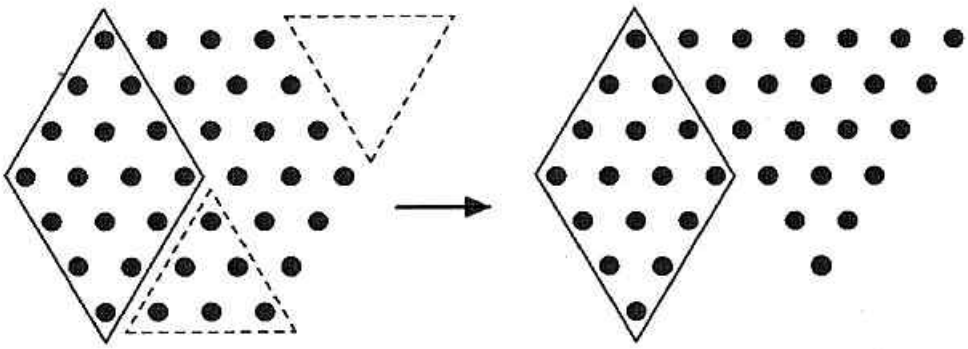


If one looks at the hexagonal arrays, then the longest row of dots (i.e. the central row) forms the sequence of odd numbers, 1, 3, 5, 7, , the nth term of which can be expressed as $2n-1$. Thus, in general, the nth hexagonal number can be seen as the $(2n-1)$th square number less two multiples of

the $(n-1)$th triangular number. We thus have:

$$\begin{aligned} H_n &= (2n-1)^2 - 2\left(\frac{(n-1)(n)}{2}\right) \\ &= 4n^2 - 4n + 1 - (n^2 - n) \\ &= 3n^2 - 3n + 1 \end{aligned}$$

The second method involves a bit of dynamic visualization whereby the nth hexagonal number can be seen as the sum of the nth square number and the $(2n-2)$th triangular number.



We thus have:

$$\begin{aligned} H_n &= n^2 + \left(\frac{(2n-2)(2n-1)}{2}\right) \\ &= n^2 + \left(\frac{4n^2 - 6n + 2}{2}\right) \\ &= 3n^2 - 3n + 1 \end{aligned}$$

It is also worth noting that since the nth square number can be expressed as the sum of the nth and $(n-1)$th triangular numbers, the nth hexagonal number can consequently be expressed as the sum of the nth, $(n-1)$th and $(2n-2)$th triangular numbers. It is left to the reader to verify this. □

FLAW OF AVERAGES

Plans based on average assumptions are wrong on average.

Far better an approximate answer to the right question, which is often vague, than the exact answer to the wrong question, which can always be made precise.

I have found that teaching probability and statistics is easy. The hard part is getting people to learn the stuff.

(From: *The Flaw of Averages*, by Sam L. Savage, Wiley, 2009) □

HANUKKAH AND THANKSGIVING

This year features an unusual event for American Jews: Thursday 28 November is both Thanksgiving and the first day of Hanukkah. This will never happen again.

Thanksgiving has its origin in harvest festivals in Europe, and was first celebrated on the American continent by the Mayflower Pilgrims in 1621. Over the years it was held on different days in the different US states, until President Abraham Lincoln decreed in 1863 that it should take place on the last Thursday of November. That rule was changed by President Franklin D. Roosevelt in 1942 to the fourth Thursday of November.

Hannukah (meaning "Dedication") commemorates the rededication of the Second Temple in Jerusalem after the Maccabean Revolt of the 2nd century BC. Judas Maccabeus had led a successful uprising against the Greek-Syrian leader Antiochus, who had outlawed the Jewish religion and defiled the Temple.

In purifying the Temple the wicks of the menorah miraculously burned for eight days, even though there was only enough sacred oil for one day's lighting. So Hanukkah is also known as the Festival of Lights, and is celebrated for eight days.

On the Jewish calendar the first day of Hanukkah is the 25th day of Kislev. In terms of the Gregorian calendar, Hanukkah varies from late November to early December, with the earliest possible date being November 28 – the latest possible date of Thanksgiving. The two festivals last fell on the same date in 1888, but that was when Thanksgiving was held on the last, not the fourth, Thursday of November (the 29th that year).

Thanksgiving can fall on any of the seven dates 22–28 November. The Jewish lunar-solar calendar runs on a 19-year cycle, so one might expect the dates of Thanksgiving and Hanukkah to coincide about once every $7 \times 19 = 133$ years. However, Hanukkah and Thanksgiving will never again coincide. The Jewish calendar is slowly getting out of step with the solar calendar (it is too long by about 6 minutes and 25 seconds, which amounts to an error of one day in 231 years). So in the future Hanukkah will have to be held later than 28 November, and the joint Thanksgiving-Hanukkah celebration will never happen again.

There is, however, a little flexibility possible in thinking of Hanukkah and Thanksgiving coinciding. In 2070 and again in 2165 Thanksgiving will be on 28 November, with Hanukkah on 29 November. In Jewish tradition a day ends, and the new day starts, at sunset, while the Gregorian calendar starts a new day at midnight. So the first candle on the Hanukkah menorah will be lit on the night of Thanksgiving in those years. □

NEGATIVE THOUGHTS

> Mary said "I will be x years old in the year x^2." When was Mary born?

This little puzzle featured in *Mathematical Digest* **172** (July 2013), and the answer was given in the same edition: Mary was born in 1980, and will be 45 years old in $2025 = 45^2$.

In a letter to the Editor, Professor John Barrow (Cambridge University) pointed out that there is a negative way of getting the same answer. Born in 1980, Mary could claim to have been -44 years old in $1936 = (-44)^2$.

The puzzle was invented by the 19th Century English mathematician August De Morgan, who used to tell people that he was x years old in the year x^2. Since he was born in 1806, he was 43 years old in $1849 = 43^2$. De Morgan is not, however, on record as noting that he was also -42 years old in $(-42)^2 = 1764$.

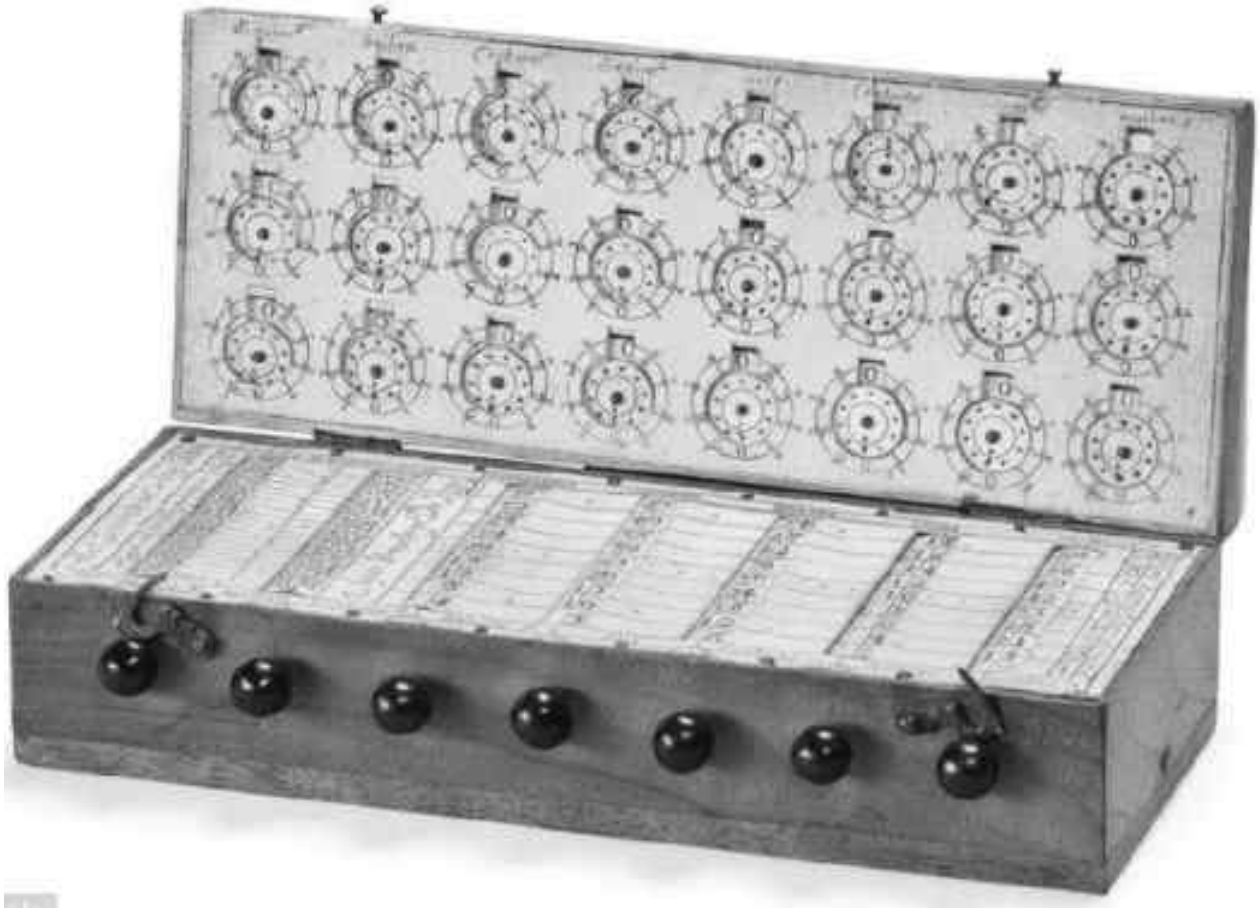
What is the next year that will entitle a person born in that year to make De Morgan's claim "I was x years old in the year x^2"?

More generally, how often do these special "De Morgan years" occur?

Answers may be found on page 31. □

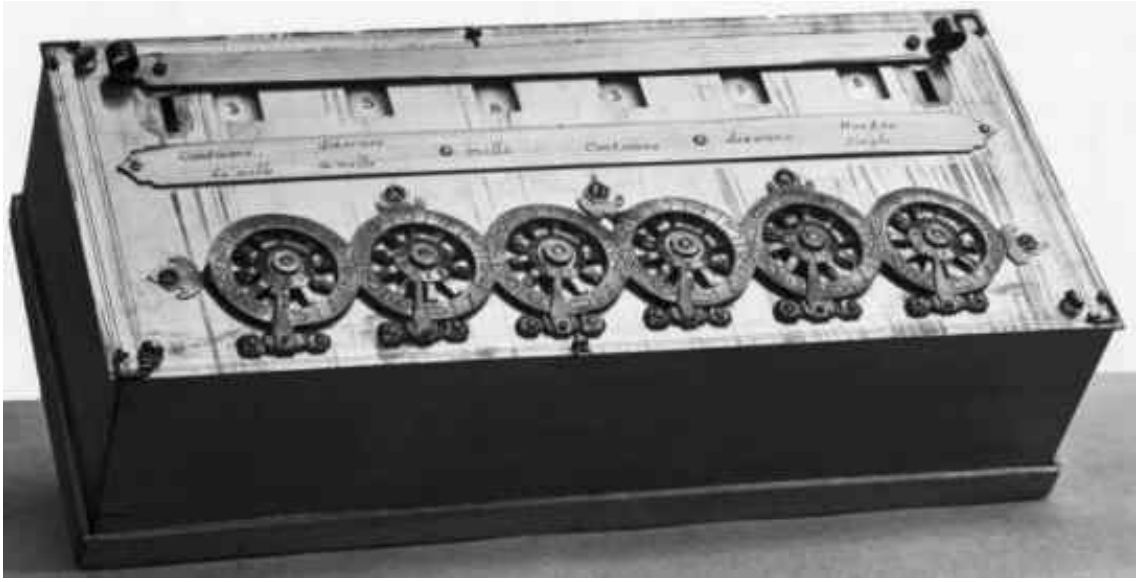
THE GRILLET CALCULATOR

One of the first portable mechanical calculators, dating from the 17th Century, is to be auctioned, and expected to raise £100 000.



The calculator was designed and built by a Parisian watchmaker, René Grillet. Only three of his devices have survived, and one of these is to go under the hammer at Christie's, the famous auctioneer, this month (October).

Grillet's calculator was made of wood. It was contained in a walnut box measuring 14.5 cm by 32.5 cm. In the lid are 24 rotating dials arranged in three rows of eight and used for addition and subtraction. The body of the box contains a set of rotating cylinders to perform multiplication. The dials and cylinders were not connected, so performing a calculation involving both addition and multiplication could become complicated. The dials were based on a calculator designed earlier by Blaise Pascal, while the multiplying rods were an adaptation of Napier's Bones, an early device for multiplying numbers invented by the Scottish mathematician John Napier in 1617.



Blaise Pascal (1623-1662), the French philosopher, mathematician, scientist and inventor, designed and built the first mechanical calculator at the age of 19 to help his father, the King's Commissioner for Taxes in the city of Rouen, with the endless arithmetic that the job required. The Pascaline, as it was called, was a heavy and very expensive device, with complicated brass cogs and wheels, and was not a commercial success. Brillet's calculator was much lighter, but it too was not a commercial success. □

QUICK QUIZ

Answer these quick quiz questions as quickly as you can.

1. Do this without pencil and paper or a calculator. Start with one thousand and add ten. Then add a thousand and then add twenty. Then add a thousand and then add thirty. Then add a thousand and then add forty. What total do you get?
2. Mary is the youngest daughter in a family of five girls. The older four daughters are called Nana, Nene, Nini and Nono. What is the name of the fifth daughter?
3. Write in figures: eleven thousand, eleven hundred and eleven.

Now turn to page 31 to find out how many you got right. □

PAMO IN NIGERIA

The 22nd Pan African Mathematics Olympiad took place in Abuja, Nigeria, from 23rd June to 2nd July, 2013.

Teams of up to four from 11 countries took part: Benin, Burkina Faso, Côte D'Ivoire, Gambia, Ghana, Malawi, Nigeria, Senegal, South Africa, Tanzania and Tunisia.

The South African team was Nashlen Govindasamy (Star College, Durban), Andrew McGregor (grade 9, Rondebosch Boys' High School), Yaseen Mowser (grade 9, Fairbairn College) and Bronson Rudner (grade 9, South African College High School). The Team Leader was Dr Maciek Stankiewicz (UCT), with Deputy Leader Phil Labuschagne.

Two $4\frac{1}{2}$-hour papers, each consisting of three problems, were written on two successive days (see page 11). Each problem was worth 7 points.

Henry Anioba (Nigeria) topped the ranking with a perfect sore of 42, winning a Gold Medal. Gold Medals were also won by Ferchiou Zouhaier and Khayeche Mohamed Mehdi of Tunisia. For South Africa, Andrew McGregor took 4th place, Yaseen Mowser and Bronson Rudner tied on 6th place and Nashlen Govindasamy took 10th place. All four South Africans won Silver Medals.

In the overall ranking, Tunisia came first with 123 points, followed by South Africa (99) and Nigeria (90). □

SHAW ON STATS

It is the mark of a truly intelligent person to be moved by statistics.

(George Bernard Shaw) □

ERRATUM

Mathematical Digest 172 (July 2013):
On page 7 (*Goldbach Variations*), the sentence in the last but one paragraph should read "... Ivan Vinogradov proved that all odd numbers **above** a certain number C were the sum of three odd primes." □

PAMO PROBLEMS

First Day

1. A positive integer n is such that $n(n+2013)$ is a perfect square.

(a) Show that n cannot be prime.
(b) Find a value of n such that $n(n+2013)$ is a perfect square.

2. Find all functions $f : \mathbf{R} \to \mathbf{R}$ such that $f(x)f(y) + f(x+y) = xy$ for all real numbers x and y.

3. Let $ABCDEF$ be a convex hexagon with $\angle A = \angle D$ and $\angle B = \angle E$. Let K and L be the midpoints of the sides AB and DE respectively.

Prove that the sum of the areas of triangles FAK, KCB and CFL is equal to half of the area of the hexagon if and only if

$$\frac{BC}{CD} = \frac{EF}{FA}.$$

Second Day

4. Let $ABCD$ be a convex quadrilateral with AB parallel to CD. Let P and Q be the midpoints of AC and BD, respectively.

Prove that if $\angle ABD = \angle CBD$, then $\angle BCQ = \angle ACD$.

5. The cells of an $n \times n$ board with $n \geq 5$ are coloured black or white so that no three adjacent squares in a row, column or diagonal are the same colour. Show that for any 3×3 square within the board, two of its corner squares are coloured black and two are coloured white.

6. Let x and y be real numbers such that $x < y < z < 6$. Solve the system of inequalities

$$\frac{1}{y-x} + \frac{1}{z-y} \leq 2$$

$$\frac{1}{6-z} + 2 \leq x.$$

$\square$

TWO ROUTES TO ROOT TWO

Without access to a calculator, how can you determine root two? Here are two routes.

The continued fraction method

Since $(\sqrt{2}-1)(\sqrt{2}+1)=1$, it follows that $\sqrt{2}-1=\dfrac{1}{2+(\sqrt{2}-1)}$.

By repeatedly using this equation, we have

$$\begin{aligned}
\sqrt{2} &= 1+(\sqrt{2}-1)\\
&= 1+\frac{1}{2+(\sqrt{2}-1)}\\
&= 1+\cfrac{1}{2+\cfrac{1}{2+(\sqrt{2}-1)}}\\
&= 1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+(\sqrt{2}-1)}}}\\
&= 1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+(\sqrt{2}-1)}}}}
\end{aligned}$$

and so on.

This is called a "continued fraction".

By dropping the term $(\sqrt{2}-1)$ from each of the above fractions, we obtain successive approximations to $\sqrt{2}$:

$$1,\ 1+\frac{1}{2}=\frac{3}{2},\ 1+\cfrac{1}{2+\cfrac{1}{2}}=\frac{7}{5},\ 1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2}}}=\frac{17}{12},\ 1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2}}}}=\frac{41}{29}.$$

Squaring each of these successive approximations gives the fractions

$$1,\ \frac{9}{4},\ \frac{49}{25},\ \frac{289}{288},\ \frac{1681}{841}$$

which are alternately less than and greater than 2. So the successive approximations we have obtained are, alternately, less than and greater than $\sqrt{2}$.

The sequence of approximations continues:

$$1,\ \frac{3}{2},\ \frac{7}{5},\ \frac{17}{12},\ \frac{41}{29},\ \frac{99}{70},\ \frac{239}{169},\ \frac{577}{408}\cdots,$$

and there is a simple recurrence relation giving the successive terms: the next term after $\dfrac{p}{q}$ is $\dfrac{p+2q}{p+q}$.

In decimals, the successive approximations are

$$1.5,\ 1.4,\ 1.416...,\ 1.4137...,\ 1.41428...$$

showing that the values converge slowly to $\sqrt{2} = 1.414213562...$.

A second approach turns out to be more efficient.

The square root algorithm

The key idea is that if $a > \sqrt{2}$ then $2/a < \sqrt{2}$.

On the other hand, if $a < \sqrt{2}$ then $2/a > \sqrt{2}$.

So $\sqrt{2}$ lies between a and $2/a$, and therefore the average of these two approximations, namely

$$\frac{1}{2}\Big(a + \frac{2}{a}\Big)$$

should be a better approximation to $\sqrt{2}$ than a.

Starting with 1 as our first approximation, the second approximation is

$$\frac{1}{2}\Big(1 + \frac{2}{1}\Big) = \frac{3}{2},$$

the third approximation is

$$\frac{1}{2}\Big(\frac{3}{2} + \frac{2}{3/2}\Big) = \frac{17}{12},$$

and the fourth approximation is

$$\frac{1}{2}\Big(\frac{17}{12} + \frac{2}{17/12}\Big) = \frac{577}{408}.$$

These approximations are the first, second, fourth and eighth terms of the continued fraction approximations we obtained above.

The fifth approximation is

$$\frac{1}{2}\Big(\frac{577}{408}+\frac{2}{577/408}\Big)=\frac{665857}{470832},$$

which turns out to be the sixteenth term of the sequence of continued fraction approximations.

The square root algorithm is clearly a more efficient way of calculating $\sqrt{2}$ than via continued fractions.

It turns out that the square root algorithm is stunningly efficient.

Suppose that a is an approximation to $\sqrt{2}$, so that the next approximation a' is $\frac{1}{2}\Big(a+\frac{2}{a}\Big)$.
Then

$$0<a'-\sqrt{2}=\frac{1}{2}\Big(a+\frac{2}{a}\Big)-\sqrt{2}=\frac{a^2+2-2\sqrt{a}}{2a}=\frac{(a-\sqrt{2})^2}{2a}<(a-\sqrt{2})^2.$$

So if $0<a-\sqrt{2}<10^{-n}$, then $0<a'-\sqrt{2}<10^{-2n}$.

In other words, the number of accurate decimal places doubles with each repetition of the algorithm.

If $\dfrac{665857}{470832}-\sqrt{2}$ is keyed into a calculator, it gives the answer 0.

Does that means that $\sqrt{2}=\dfrac{665857}{470832}$? The calculator suggests it is, but only to the limit of its accuracy. The equality is only approximate. Indeed, if $\sqrt{2}$ and $\dfrac{665857}{470832}$ were exactly equal, then squaring both sides and multiplying up would give

$$665857^2=2\times 470832^2$$

which is just not possible, for the very obvious reason that the left hand side is odd and the right hand side is even.

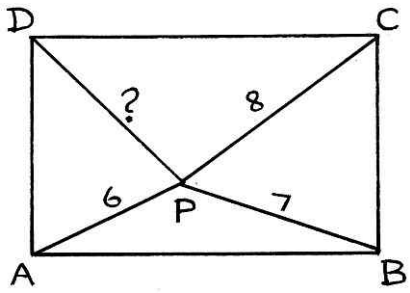
Using a beefed-up version of this odd-versus-even argument (basically due to Euclid), it can be shown that $\sqrt{2}$ is not a rational number — something that was discovered by Pythagoras two-and-a-half millennia ago.

The square root algorithm for finding $\sqrt{2}$ can be adjusted to find $\sqrt{A}$, for any positive number A. If a is an approximation to $\sqrt{A}$, then $\frac{1}{2}(a+A/a)$ is a better approximation. Try $A=3$. With $a=2$, the next three approximations are $\dfrac{7}{4}$, $\dfrac{97}{56}$ and $\dfrac{18817}{10864}$. The last term differs from $\sqrt{3}$ by 2.4×10^{-9}. $\square$

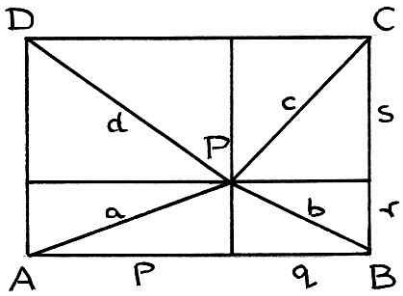
NOT ENOUGH INFORMATION

The following five problems have a common feature: it seems that in each case not enough information has been given.

Nevertheless, the problems can all be solved, because in each case no extra information is needed.



Problem 1
A point P lies inside a rectangle $ABCD$. If $AP = 6$, $BP = 7$ and $CP = 8$, determine DP.



Solution
The dimensions of the rectangle are not given, but it turns out that the distance AD is the same for all rectangles, given the distances AP, BP and CP.

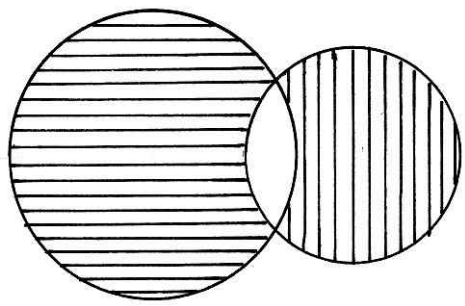
Suppose that the distances from P to the four vertices are a, b, c and d. Then, referring to the figure,

$$a^2 = p^2 + r^2, \quad b^2 = r^2 + q^2, \quad c^2 = q^2 + s^2 \text{ and } d^2 = s^2 + p^2.$$

Adding the first and third equations, and then the second and fourth, shows that $a^2 + c^2 = b^2 + d^2$.

So, if a, b and c are given, then $d^2 = a^2 + c^2 - b^2$, whatever the shape of the rectangle.

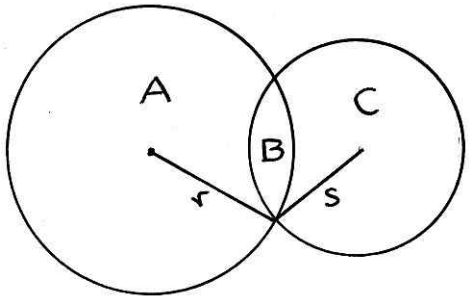
Problem 2
Two circles of radii 3 and 4 intersect as shown.
What is the difference between the areas of the two shaded regions?



Solution

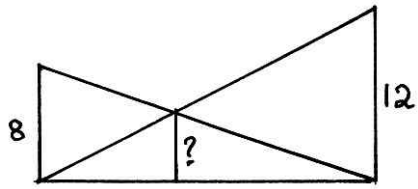
Here it seems that we need to know the distance between the centres of the circles in order to do the calculation.

Suppose that the radii of the circles are r and s, and the areas of the three regions are A, B and C, as shown. then $C - A = (C + B) - (A + B) = \pi r^2 - \pi s^2 = \pi(r^2 - s^2)$.



Problem 3

Two vertical poles of heights 8 and 12 stand on level ground. Two ropes join the top of each pole to the bottom of the opposite pole. Determine the height above the ground at which the two ropes intersect.



It seems that we need to know the distance between the poles in order to determine the height h.

Solution

Using the two pairs of similar triangles in the figure, we have

$$\frac{p+q}{a} = \frac{q}{h} \text{ and } \frac{p+q}{b} = \frac{p}{h}.$$

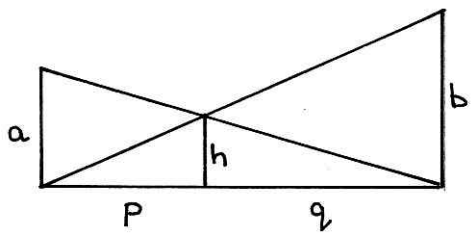
Adding the two equations gives

$$\frac{p+q}{a} + \frac{p+q}{b} = \frac{p+q}{h},$$

from which it follows that

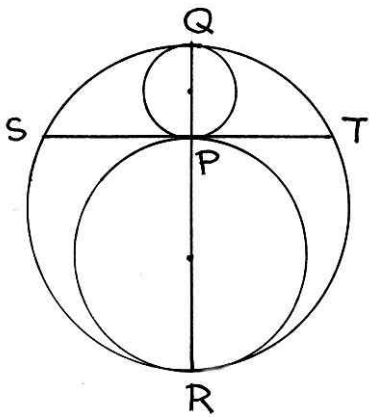
$$h = \frac{ab}{a+b},$$

and the answer is independent of the distance between the poles.



Problem 4

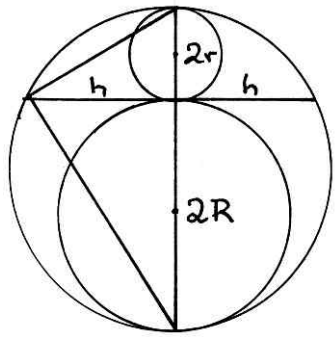
In the figure, the centres of the two inner circles lie on the diameter of the large circle. The two small circles are tangent to each other at P and to the large circle at Q and R. The tangent to the small circles at P cuts the large circle at S and T.



If $ST = 6$, find the area of the region inside the large circle but outside the two small circles.

It appears that we need to know the radii of the two circles.

Solution



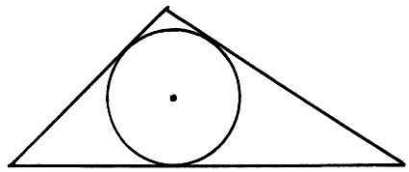
Suppose that the two small circles have radii r and R, and that $ST = 2h$.
Since the two triangles are similar, $h/2r = 2R/h$, giving $h^2 = 4rR$.

Now the area required is

$$\pi(r+R)^2 - \pi r^2 - \pi R^2 = \pi(4rR) = h^2$$

which is independent of R and r.

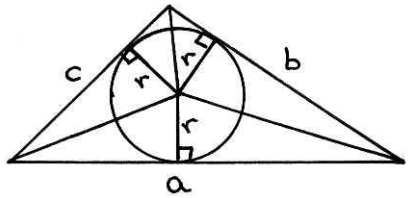
Problem 5



The area of a triangle is 6 square cm^2 and its perimeter is 6 cm. What is the radius of the inscribed circle?

It seems that many different triangles could have the same area and the same perimeter specified. So there seems to be not enough information.

Solution



Suppose that the area of the triangle is A, its perimeter is P and the radius of the inscribed circle is r.

The centre of the inscribed circle divides the triangle into three small triangles, with bases of lengths a, b and c, and the same altitude r. Then

$$A = \frac{1}{2}ar + \frac{1}{2}br + \frac{1}{2}cr = \frac{1}{2}(a+b+c)r = Pr,$$

and it follows that $r = A/P$, whatever the shape of the triangle. □

UCT MATHS COMPETITION BOOK

The report on the 2013 UCT Mathematics Competition has been published. The book contains the question papers, full solutions, detailed statistics and names of prizewinners and top schools. Copies cost R50 and may be obtained using the form at the back of the magazine. □

VARSITY TEST

Are you planning to take a mathematics course at university? This test focuses on some of the basic skills your lecturers will expect you to have. And that's not just for your maths courses. Physics, computer science, biochemistry, engineering, accounting etc etc all require a solid mathematical background, a high level of logical thinking, algebraic expertise, geometric insight and numerical skills.

Merely studying for the predictable Senior Certificate Mathematics examination may not have prepared you adequately for varsity maths, and nowadays a matric maths A is no guarantee that you will be able to cope. So try the problems below to find out how ready you are for varsity maths. The problems don't just check your content knowledge, they require flexible thinking. The bad news is that you should tackle them without a calculator and without a formula sheet. The good news is that there is no time limit. The bad news is that you should work through all the problems in one sitting. The good news is that there is no pass/fail mark. Every wrong answer, however, will highlight a weakness in your mathematical background that needs to be fixed before you arrive on your chosen university campus.

1. $\sin 15^\circ \cos 75^\circ$ is equal to

   (A) $\tan 15^\circ$ (B) $\tan 30^\circ$ (C) $\dfrac{\sqrt{3}}{4}$ (D) $\dfrac{1}{\sqrt{2}}$ (E) $\dfrac{1}{4}$

2. A computer hard drive spins at 7200 rpm. What is its angular velocity in degrees per second?

   (A) 43 200 (B) 54 400 (C) 36 600 (D) 48 480 (E) 24 360

3. If $6^a = 3^b = 2$, then b is equal to

   (A) $\dfrac{a}{2}$ (B) $\dfrac{a}{2} - 1$ (C) $\dfrac{a}{1-a}$ (D) $\dfrac{a+1}{2}$ (E) $\dfrac{2}{a}$

4. The remainder when $2x^4 + 2x^3 - x^2 - 15x + 8$ is divided by $2x^2 - 4x + 1$ is

   (A) $4x + 5$ (B) $3x - 1$ (C) $2x + 3$ (D) $-x + 7$ (E) $x + 4$

5. In triangle ABC, $\tan A = \frac{3}{4}$ and $\sin B = \frac{5}{13}$. The value of $\cos C$ is

   (A) $-\dfrac{33}{65}$ (B) $\dfrac{56}{65}$ (C) $\dfrac{17}{65}$ (D) $\dfrac{41}{65}$ (E) $\dfrac{8}{65}$

6. If $\tan A + \dfrac{1}{\cos A} = 2$, then $\cos A$ is equal to

(A) $\dfrac{6}{11}$ (B) $\dfrac{5}{9}$ (C) $\dfrac{3}{7}$ (D) $\dfrac{4}{5}$ (E) $\dfrac{7}{8}$

7. Triangle ABC is right-angled at C and D lies on BC. If $AB = 14$, $AD = 10$ and $CD = 5$, the perimeter of triangle ABD is

(A) $24 + 5\sqrt{2}$ (B) $24 + 3\sqrt{3}$ (C) 29 (D) 30 (E) 31

8. If $x^2 - 4x + 3 < 0$ and $y = x^2 + 4x + 3$, then

(A) $y < 0$ (B) $0 < y < 3$ (C) $3 < y < 8$ (D) $8 < y < 24$
(E) $24 < y < 35$

9. The diagonals AC and BD of quadrilateral $ABCD$ intersect at the point E inside the quadrilateral. If $AD = AE$, $BE = CE$ and $\angle ADE = 20°$, then $\angle CBE$ is equal to

(A) 50° (B) 65° (C) 70° (D) 75° (E) 80°

10. If $f(x)$ is a quadratic function such that $f(-1) = -3$, $f(1) = 1$ and $f(2) = 12$, find $f(3)$.

(A) 8 (B) 11 (C) 17 (D) 29 (E) 35

11. Fifty numbers have an average of 76. Forty of these numbers have an average of 80. The average of the other ten numbers is

(A) 60 (B) 4 (C) 72 (D) 40 (E) 78

12. If $a - b = 3$ and $ab = 7$, then $\dfrac{1}{a} - \dfrac{1}{b}$ is equal to

(A) $\dfrac{1}{3}$ (B) $-\dfrac{3}{7}$ (C) $\dfrac{7}{3}$ (D) $-\dfrac{1}{3}$ (E) $\dfrac{1}{7}$

13. $87 \times 24 + 43 \times 56 + 56 \times 87 + 24 \times 43$ is equal to

(A) 7200 (B) 8100 (C) 8400 (D) 9000 (E) 9100

14. If $\sin x = \dfrac{7}{11}$, then $\cos 2x$ is equal to

(A) $\dfrac{\sqrt{73}}{121}$ (B) $\dfrac{8}{119}$ (C) $\dfrac{4}{11}$ (D) $\dfrac{\sqrt{77}}{11}$ (E) $\dfrac{23}{121}$

15. Town A is 13 km west of Town B. Town B is 4 km south of Town C. Town D is 3 km west of Town C.

The sum of the distances AD and BC is

(A) between 13 and 14 (B) between 14 and 15
(C) between 15 and 16 (D) between 16 and 17
(E) between 17 and 18

16. Which of the following is a prime number?

(A) 451 (B) 452 (3) 459 (D) 465 (E) 467

17. The graph of $5x - 3y - 7 = 0$ is translated 3 units up and 2 units to the right. The equation of the new graph is

(A) $5x - 3y + 7 = 0$ (B) $3x - 5y + 4 = 0$ (C) $5x - 3y - 8 = 0$
(D) $3x - 5y - 4 = 0$ (E) $5x - 3y + 4 = 0$

18. If $\dfrac{1}{n} + \dfrac{1}{n(n-1)} = \dfrac{1}{4}$, then n is equal to

(A) 3 (B) 4 (C) 5 (D) 6 (E) 7

19. An equilateral triangle is cut into four equilateral triangles, each with a perimeter of 12 cm. What is the perimeter (in cm) of the original equilateral triangle?

(A) 18 (B) 24 (C) 30 (D) 36 (E) 48

20. Water from Tap A will fill a bath in 4 minutes, and water from Tap B will fill the same bath in 6 minutes. How long (in seconds) will it take to fill the bath if both taps are running together?

(A) 120 (B) 125 (C) 128 (D) 132 (E) 144

21. If $a - b + c = 2$, $b - c + a = -3$ and $c - a + b = 5$, then $a + b + c$ is equal to

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

22. Solve for x: $sx^2 - t(1 + s^2)x + st^2 = 0$.

(A) $x = s$ or t (B) $x = s + t$ or $s - t$ (C) $x = st$ or $\dfrac{t}{s}$
(D) $x = st$ or $s + t$ (E) $x = st$ or $\dfrac{1}{st}$

23. If $f(g(x)) = x^2 + 3x + 2$ and $g(x) = x + 1$, then $f(x)$ is equal to

(A) $x^2 + x$ (B) $2x + 2$ (C) $x + 2$ (D) $x^2 + 4x + 3$ (E) $x^2 + 5x + 6$

24. Which of the following statements is true?

(A) $11^6 < 5^9 < 2^{21}$ (B) $2^{21} < 11^6 < 5^9$ (C) $11^6 < 2^{21} < 5^9$
(D) $5^9 < 11^6 < 2^{21}$ (E) $2^{21} < 5^9 < 11^6$

25. $\log_2 500$ is equal to

(A) $4 + 2\log_2 5$ (B) $1 + 3\log_2 5$ (C) $3 + 2\log_2 10$
(D) $3 + 2\log_2 5$ (E) $2 + 3\log_2 5$

THE 2013 INTERPROVINCIAL MATHEMATICS OLYMPIAD

Western Province took top honours in this year's Interprovincial Mathematics Olympiad (IPMO), winning both divisions, and seeing their Senior B team come second.

The (IPMO) was launched in 1990 with just two teams: Western Province and Natal. After a couple of years it split into Junior and Senior divisions while the number of participating provinces grew steadily. This year saw 56 teams from 14 provinces getting together on Saturday 7 September for an afternoon of friendly maths rivalry. The term "province" is only loosely related to the official provincial division of South Africa, and provinces are allowed to enter as many teams in each division as they like.

The format of the contest has remained pretty much the same since it began. Teams of ten meet in different provincial centres around the country and write two papers. The first is an individual multiple-choice paper, lasting one hour. Then there is a break for refreshments and discussion of team strategy before the teams tackle a ten-problem paper together, submitting just one set of answers. With the individual paper worth a maximum of 100 points (making 1000 points for the whole team) and the second round paper counting another 1000 points (100 per question), a team's maximum score is 2000. Papers are marked immediately and scores submitted by email, fax or phone to a central organiser. Results are made known by the end of the afternoon.

The top Junior team was Western Province on 1830 points, followed by Eastern Cape (1676) UKZN (1640), Free State/Northern Cape (1566) and Gauteng South (1561).

Western Province dominated the Senior division. Their A team headed the rankings with 1568 points, a good way ahead of their nearest rivals, the Western Province B team on 1326. Close behind was Eastern Cape (Port Elizabeth) on 1292 points, followed by Gauteng South (1265) and Gauteng North C (1211).

The 2013 IPMO was organized by Dr Sudan Hansraj, of the School of Mathematical Sciences, University of KwaZulu-Natal on behalf of the South African Mathematics Foundation, with sponsorship provided by the Actuarial Society of South Africa and Casio. □

IPMO 2013 PAPERS

Junior Individual paper

1. What is the value of $\dfrac{300+10}{20-10}$?

   (A) 400 (B) 40 (C) 4 (D) 31 (D) 310

2. When $x = 3$, the expression $(x-1)(x-3)(x-5)(x-7)$ has the value ...

   (A) 16 (B) 105 (C) -16 (D) 0 (E) -105

3. The expression $8p - 5p + 2p - p$ simplifies to

   (A) $4p$ (B) $4p^4$ (C) 4 (D) $80p^4$ (E) $-80p^4$

4. What is the value of $\sqrt{36 \times \sqrt{16}}$?

   (A) 12 (B) 144 (C) 24 (D) 26 (E) 96

5. A full can of paint has a mass of 1000g. If half the paint is used up, the mass of the can and the remaining paint is 700g. What is the mass of the empty paint can?

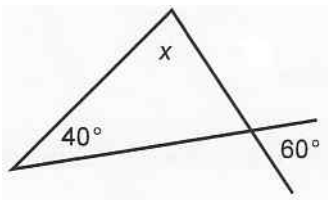
   (A) 600 g (B) 500 g (C) 350 g (D) 400 g (E) 300 g

6. In the addition sum shown, a digit, either the same or different, can be placed in each of the two blank boxes. What is the sum of the two missing digits?

$$\begin{array}{r} 863 \\ \square 91 \\ 7\square 8 \\ \hline 2182 \end{array}$$

   (A) 9 (B) 11 (C) 13 (D) 3 (E) 7

7. What is the measure of angle x in the diagram?



   (A) 20° (B) 80° (C) 100° (D) 120° (E) 60°

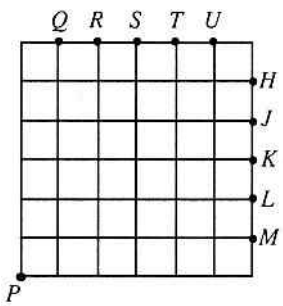
8. In a certain month, three of the Sundays have dates that are even numbers. The 10th day of this month is a

   (A) Saturday (B) Sunday (C) Monday (D) Tuesday (E) Wednesday

9. Piet drives 60 km south, 40 km west, 20 km north and 10 km east. What is the distance from his starting point to his finishing point?

   (A) 30 km (B) 50 km (C) 40 km (D) 70 km (E) 35 km

10. In the 6 by 6 square grid shown, two lines are drawn from point P to two other points in such a way that these two lines divide the grid into three regions of equal area. The two other points are



(A) M and Q (B) L and R (C) K and S (D) H and U (E) J and T

11. A square of side 8 cm is divided into two rectangles by a single straight line. The larger rectangle has a perimeter $\frac{3}{2}$ times the smaller one. Find the ratio of the areas of the larger to the smaller rectangle.

(A) $4:1$ (B) $2:1$ (C) $\frac{9}{4}:1$ (D) $\frac{3}{2}:1$ (E) $16:1$

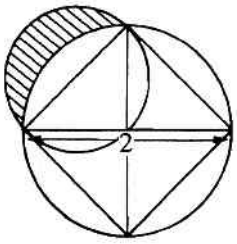
12. Pages m and n are on the same side of the same sheet of a double sheet of newspaper. What page will be on the same sheet as page $2m$?

(A) $2n$ (B) $n+m$ (C) $n+m+1$ (D) $n-m-1$ (E) $n-m$

13. How many 5-digit numbers of the form 33□□6 are perfect squares?

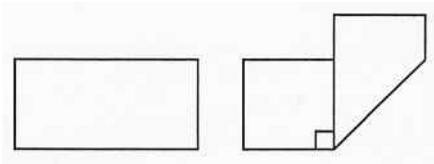
(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

14. The figure shows a square inscribed in a circle of diameter 2 cm. Find the area shaded, in cm^2.



(A) 1 (B) $\pi - 2$ (C) $\dfrac{1}{2}$ (D) 2 (E) $\dfrac{\pi}{2}$

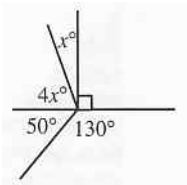
15. A rectagular piece of paper measures 17 cm by 8 cm. It is folded so that a right angle is formed between the two segments of the original bottom edge, as shown. What is the area of the new figure in cm^2?



(A) 104 (B) 81 (c) 72 (D) 168 (E) 64

Senior Individual paper

1. Find the value of x in the diagram shown.



(A) 22.5 (B) 25 (C) 20 (D) 36 (E) 18

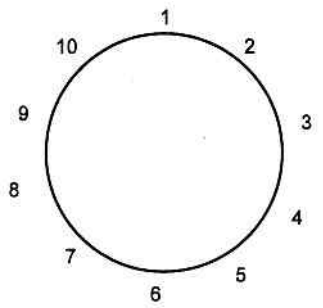
2. During hibernation, a bear loses 20% of its original mass. After hibernation, its mass is 220 kg. What was its mass, in kilograms, just before hibernation?

   (A) 176 (B) 264 (C) 240 (D) 275 (E) 1100

3. $20132013 \times 20132013 + 20131013 + 20132014$ is the square of

   (A) 20142013 (B) 20142014 (C) 20132014 (D) 20142031 (E) 2013

4. In the diagram, the numbers 1 to 10 are placed around a circle. Zama crosses out 1, then 4 and then 7. Continuing in a clockwise direction, she crosses out every third number of those remaining, until only two numbers are left. The sum of these two number is



   (A) 13 (B) 10 (C) 8 (D) 14 (E) 17

5. A triangle with perimeter 7 has integer side lengths. What is the maximum possible area of such a triangle?

   (A) $\frac{\sqrt{35}}{4}$ (B) $\frac{\sqrt{7}}{4}$ (C) $\frac{49}{4}$ (D) $\frac{1}{7}$ (E) $\frac{22}{7}$

6. Two integers, each lying between 60 and 70, that divide $2^{48} - 1$ are

   (A) 61 & 64 (B) 63 & 67 (C) 65 & 69 (D) 63 & 65 (E) 62 & 68

7. A positive integer X, leaves the same remainder when divided into 5814, 5430 and 5958. What is the largest value of X?

   (A) 24 (B) 48 (C) 72 (D) 96 (E) 144

8. Suppose $4^{X_1} = 5$, $5^{X_2} = 6$, $6^{X_3} = 7$, $\ldots$, $126^{X_{123}} = 127$, $127^{X_{124}} = 128$. What is the value of the product $X_1 \times X_2 \times \cdots \times X_{124}$?

   (A) $\frac{7}{2}$ (B) $2\sqrt{5}$ (C) $4\sqrt{2}$ (D) $\frac{9\sqrt{2}}{2}$ (E) $5\sqrt{2}$

9. What is the largest integer that is a divisor of $P(n) = n^5 - 5n^3 + 4n$, for all positive integers n?

   (A) 8 (B) 15 (C) 120 (D) 240 (E) 260

10. In $\triangle ABC$, $AB = AC = \sqrt{2}$ and $BC = 2$. Suppose D is a point on line BC such that C lies between B and D and $AD = 2$. What is the length of the segment BD?

    (A) $\sqrt{2}+1$ (B) $\sqrt{2}-1$ (C) $2+\sqrt{3}$ (D) $\sqrt{3}+1$ (E) $\sqrt{3}-1$

11. The largest prime factor of $3^{14} + 3^{13} - 12$ is

(A) 13 (B) 37 (C) 73 (D) 87 (E) 3

12. The number of perfect cubes between 1 and 1000001, that are exactly divisible by 7, is:

(A) 10 (B) 7 (C) 14 (D) 49 (E) 21

13. PS is a line segment of length l and O is the midpoint. A semicircular arc is drawn with PS as the diameter. Let X be the midpoint of the arc. Q and R are two points on the arc PXS, such that QR is parallel to PS and PS is the tangent to the semicircular arc drawn with QR as a diameter. What is the are of $QXROQ$ (bounded by two semicircular arcs)?

(A) $\frac{l^2}{8}(\pi - 1)$ (B) $\frac{l^2}{4}(\pi^2 + 1)$ (C) $\frac{\pi l^2}{4}$ (D) $\frac{\pi l^2}{16} - 1$ (E) $\frac{(\pi - 1)l^2}{2}$

14. Fermat's last theorem states that no positive integer triplet (a, b, c) is a solution of $a^3 + b^3 = c^3$. How many positive integer triplets are a solution of $a^3 + b^3 = c^4$?

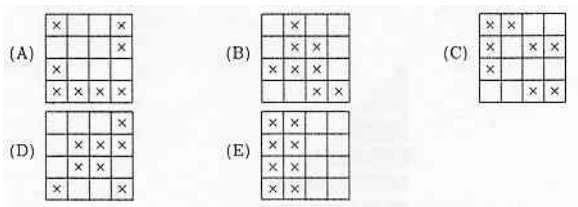
(A) 0 (B) 1 (C) 4 (D) 5 (E) infinitely many

15. The product $1 \times 2 \times 3 \times \cdots \times 2013$ has how many zeros at the end?

(A) 0 (B) 100 (C) 200 (D) 501 (E) 2013

Junior team paper

1. In each of these five diagrams there are 16 little squares, some with crosses. Which of these diagrams can be cut up into four congruent parts each containing the same number of crosses?



2. How many four digit numbers have each digit a perfect square?

3. What is the 2013th digit after the point in the decimal form of $\frac{5}{27}$?

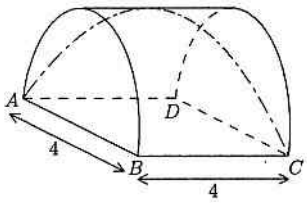
4. One of the greatest mathematicians ever, Carl Gauss, was born in 1777 and died in 1855. Find one solution to the *alphametric* shown. Letters must be replaced by digits in a one-to-one fashion.

$$\begin{array}{r} 1\ 7\ 7\ 7 \\ 1\ 8\ 5\ 5 \\ +\ C\ A\ R\ L \\ \hline G\ A\ U\ S\ S \end{array}$$

5. Find the sum of the digits of the number $100^{2013} - 2013$.

6. Show all the ways the number 365 can be written as the sum of two or more consecutive squares.

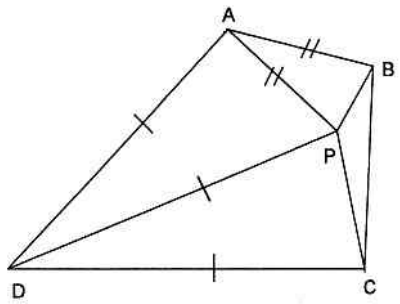
7. The figure shows half a cylinder. An ant walk from A to C on the surface of the cylinder as shown. What is the shortest possible distance the ant travels?



8. Find the sum of all the numbers in the grid shown.

1	2	3	...	49	50
2	3	4	...	50	51
3	4	5	...	51	52
⋮	⋮	⋮	⋱	⋮	⋮
49	50	51	...	97	98
50	51	52	...	98	99

9. In the diagram, point P is inside quadrilateral $ABCD$. Also, $DA = DP = DC$ and $AP = AB$. If $\angle ADP = \angle CDP = 2x$, $\angle BAP = x + 5°$, and $\angle BPC = 10x - 5°$, what is the value of x in degrees?



10. A sequence is constructed as follows:

$$\begin{aligned} a_1 &= 2012 \\ a_2 &= 2^2 + 0^2 + 1^2 + 2^2 = 9 \\ a_3 &= 9^2 = 81 \\ a_4 &= 8^2 + 1^2 = 65 \end{aligned}$$

If this pattern continues, find a_{2013}.

Senior team paper

1. How many non-negative integral values of x satisfy the equation $\left[\frac{x}{5}\right] = \left[\frac{x}{7}\right]$, where $[x]$ denotes the largest integer $\leq x$.

2. Point A lies between two parallel lines ℓ_1 and ℓ_2. The distance from A to ℓ_1 is 2, from A to ℓ_2 is 4. Points B in ℓ_1 and C in ℓ_2 are chosen so that $\angle BAC = 60°$ (fig.1). What is the smallest possible area of $\triangle ABC$?

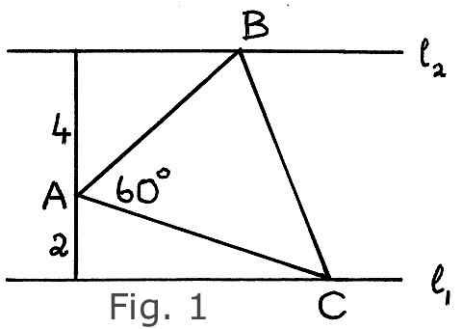


Fig. 1

3. One of the greatest mathematicians ever, Carl Gauss, was born in 1777 and died in 1855. Find one solution to the *alphametric* shown. Letters must be replaced by digits in a one-to-one fashion.

$$\begin{array}{r} 1\ 7\ 7\ 7 \\ 1\ 8\ 5\ 5 \\ +\ C\ A\ R\ L \\ \hline G\ A\ U\ S\ S \\ \hline \end{array}$$

4. Find the sum of all the numbers in the grid shown.

1	2	3	...	49	50
2	3	4	...	50	51
3	4	5	...	51	52
⋮	⋮	⋮	⋱	⋮	⋮
49	50	51	...	97	98
50	51	52	...	98	99

5. It is known that all angles of a 2013-polygon $A_1A_2\ldots A_{2013}$ are equal. In addition

$$|A_1A_2| \leq |A_2A_3| \leq \cdots \leqslant |A_{2012}A_{2013}| \leqslant |A_{2013}A_1|.$$

Determine the ratio $\dfrac{|A_{2013}A_1|}{|A_1A_2|}$.

6. Determine all integers $n > 1$ that can be divided by any integer m, $1 \leq m \leq \sqrt{n}$.

7. Determine all pairs of positive integers such that the square of one of the integers plus three times the other integer is a perfect square.

8. The first $2n$ natural numbers $1,\ 2, \ldots, 2n-1,\ 2n$ are split in two groups, n numbers in each. The numbers from the first group $a_1 < a_2 < \cdots < a_n$ are written in ascending order and the numbers from the second group $b_1 > b_2 > \cdots > b_n$ are written in descending order. Determine the sum

$$|a_1 - b_1| + |a_2 - b_2| + \cdots + |a_n - b_n|.$$

9. Diagonals cut a regular n-polygon into several non-overlapping triangles. (We call this a non-overlapping triangular partition of the polygon: see fig.2 where $n = 6$). Determine the total number of different non-overlapping triangular partitions of a regular octagon.

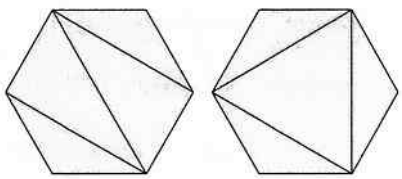


Fig. 2.

10. A convex quadrilateral $ABCD$ is extended as shown in fig.3. Determine the area of $A'B'C'D'$ if it is known that the area of $ABCD$ is equal to 1.

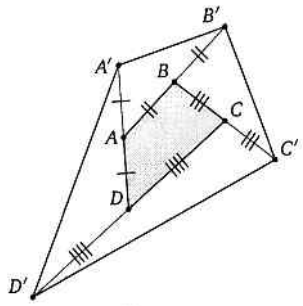


Fig. 3.

NINETY-NINE POINT NINE PERCENT

Advertisements on television and in the press for soap or other anti-bacterial products often claim that they "kill 99.9% of germs". Sometimes advertisers go further: the product is guaranteed to "kill 99.9% of germs *dead*", as if there could be other outcomes of killing. Or perhaps some products only inflict grievous bodily harm on bacteria.

In order to get these 99.9% claims past the advertising standards people, manufacturers have to submit their product to a lab test.

That sounds good, but not when you look at the details.

- In the USA the effectiveness of a bactericide (germ-killer) is determined by a lab test in which 60 slides are inoculated with a specific bacterium. The bactericide is then applied to the slides for ten minutes, after which it must have eliminated the nasty bug on 59 of the slides. (Note that 59 successes out of 60 amounts to a success rate of 98.3%, not 99.9%, but let that pass.)
- If the test fails, the manufacturer can do it again.
- In real life, people do not dot wash their hands or wipe a kitchen work surface with the soap or sanitiser for ten minutes. Ten seconds is more likely.
- The test need only demonstrate that the product is effective on a "representative sample" of bacteria, without saying which bugs the product kills.
- Since dirty hands or surfaces may well contain over a billion bacteria, a 99.9% kill rate translates to a million bacteria still surviving.

Does it matter? Not really. Advertisers know that 99.9% of people will believe anything presented in numbers. □

INTERNATIONAL VARSITY MATHS OLYMPIAD

A team of UCT mathematics students achieved high ranking at the International Mathematics Competition for University Students (IMC), held in Blagoevgrad, Bulgaria, in August.

The IMC began in 1994. It is an annual competition for students completing their first, second, third or fourth year of university education and consists of two 5-hour contest papers, with each paper consisting of just five problems from the fields of Algebra, Analysis (Real and Complex), Geometry and Combinatorics. The problems are all very challenging.

This year's competition was jointly organized by University College London and the American University in Bulgaria, Blagoevgrad, and took place from 6th to the 12th of August. A total of 321 students from 72 universities took part.

This was the first time that UCT had entered a team: first-years Robert Spencer and Dylan Nelson, second-year student Sean Wentzel and Liam Baker, a maths honours student. No strangers to international competitions, in their schooldays they had all represented South Africa at the International Mathematical Olympiad and International Olympiad of Informatics more than once. Dr Henri Laurie of the Department of Mathematics and Applied Mathematics, led the team, which was sponsored by the Faculty of Science and the Department of Mathematics and Applied Mathematics UCT.

The UCT results were excellent. Sean and Liam were ranked overall 30th and 33rd respectively, and were awarded first prizes and won gold medals. Robert achieved an Honourable Mention.

Teams can be of any size, but the team score is computed by adding the top three scores and the average of all the other scores. By this measure, two Russian universities topped the rankings: the Moscow Institute of Physics and Technology came first with 257.83 points, followed by the Lomonosov Moscow State University on 232.2. On their first outing UCT achieved an excellent 18th place, ahead of top universities such as Yale (25th) and Cambridge (32nd).

This was an excellent start, and it is hoped to make UCT's participation in the IMC an annual event. □

THE CRYPTOPOCALYPSE

The modern world relies heavily on some clever mathematics to ensure that confidential messages cannot be read by spies, whether in the military, diplomatic or commercial worlds. Cryptographers have devised wickedly clever codes which are often said to be unbreakable.

One such code is called RSA, after its inventors, Israeli mathematicians Ron Rivest, Adi Shamir and Leonard Adleman, who first publicly described the algorithm in 1977. Clifford Cocks, an English mathematician, had developed an equivalent system in 1973, but it wasn't declassified until 1997.

A user of the RSA system creates a number N which is the product of two large prime numbers p and q. The number N is published, and is used to encrypt the message. The trick is that in order to decrypt the message, the two separate prime numbers p and q must be known. The security of the RSA system relies upon the difficulty of finding the prime factors of large numbers. Even with the fastest computers, finding the prime factors of a 200-digit number can take years of trial and error.

Cryptographers are nevertheless getting worried.

- Computers are getting faster and better networked
- Fundamental advances in mathematics are producing better algorithms for finding the prime factors of large numbers.
- Edward Snowden's leaked revelations suggest that mathematicians in the USA National Security Agency are cracking many codes thought to be invincible.

While advances in computing hardware and software are in some ways predictable (Moore's Law says that processing power for computers will double every two years), fundamental advances in mathematics are not.

The threat to the security of the RSA algorithm is real, and it is expected that the 1024-bit RSA algorithm will soon be crackable. Companies are advised to ensure that they can readily upgrade to 2048-bit RSA. Better still, they should switch to elliptic-curve cryptography (ECC), an esoteric branch of number theory that appears to be safe from attack at present. But if somebody builds a quantum computer (which may happen in the next ten years), even ECC will be vulnerable.

An IT developer in a big software company specialising in secure banking systems commented: "Most IT companies, including companies like ours where security is critical and widely used, are just consumers of industry-agreed cryptographic algorithms. It's not an area where companies can afford to be innovators. It's more an area for governments, a few specialist security companies, and companies like Google, Microsoft etc. who are big enough to have their own world-class research labs. Unless you're really good, and publish work for wide review in the industry, any type of cryptography you invent or customize for yourself is likely to be insecure in ways you didn't expect." □

ANSWERS

Negative Thoughts (page 7):
Anybody born in 2070 will be able to say that they will be 46 years old in $2116 = 46^2$. They could also claim to have been -45 years old in $2025 = (-45)^2$. In general, anybody born in the year $n(n+1)$ will be able to claim "I will be $n+1$ years old in year $(n+1)^2$, since $n(n+1)+(n+1) = n^2+2n+1 = (n+1)^2$. They can also claim to have been $-n$ years old in the year n^2, since $n(n+1) - n = n^2$.

Quick Quiz (page 9):
1. 4100 (not 5000) 2. Mary (not Nunu) 3. 12111 (not 11111).
How many did you get right? Be honest!

Varsity Test (page 18):
EACCA DDDED ABEEB ECCBE DCAAD

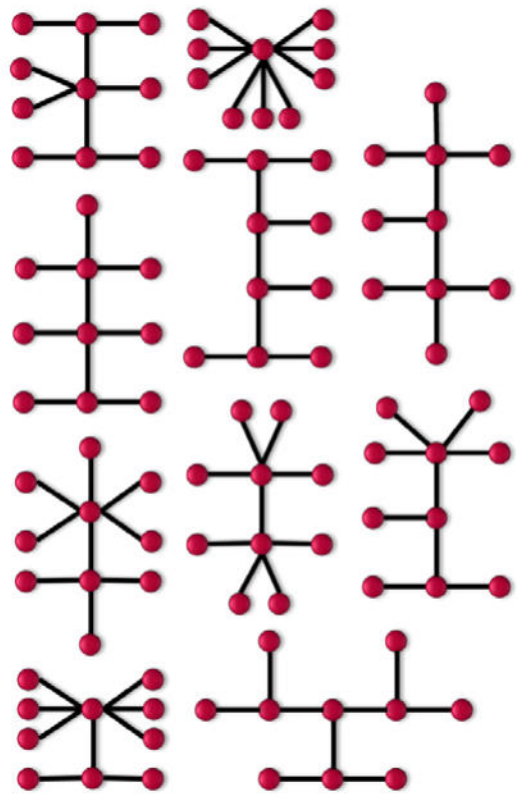
Interprovincial Mathematics Olympiad (page 22):
Junior Individual paper: DDAAD EBCBE AEBCA
Senior Individual paper: EDCBB DBACD CCAED

Junior Team paper:
(1) A, B, C, E (2) 192 (3) 5 (4) CARL GAUSS = (9245, 12877) or (9234, 12866) or (7023, 10655) or (9201, 12833) (5) 36229
(6) $10^2 + 11^2 + 12^2 = 13^2 + 14^2 + 12^2$, (7) $2\sqrt{\pi^2+4}$, (8) 125000, (9) 13°, (10) 16

Senior Team paper:
(1) 9 (2) $8\sqrt{3}$ (3) CARL GAUSS = $(9245, 12877)$ or (9234, 12866) or (7023, 10655) or (9201, 12833) (4) 125000 (5) 1 (6) 2, 3, 4, 6, 8, 12, 24
(7) (1, 1) and (16, 11) (8) n^2 (9) 132 (10) 5

mathematical digest

No. 174 January 2014



Irreducible Trees

mathematical digest

Number 174 **January 2014**

'Tis told by one whom stormy waters threw,
With fellow-sufferers by the shipwreck spared,
Upon a desert coast, that having brought
To land a single volume, saved by chance,
A treatise of Geometry, he wont,
Although of food and clothing destitute,
And beyond common wretchedness depressed,
To part from company and take this book
(Then first a self-taught pupil in its truths)
To spots remote, and draw his diagrams
With a long staff upon the sand, and thus
Did oft beguile his sorrow, and almost
Forget his feeling.

(From *The Prelude*, by William Wordsworth)

Department of Mathematics and Applied Mathematics
University of Cape Town

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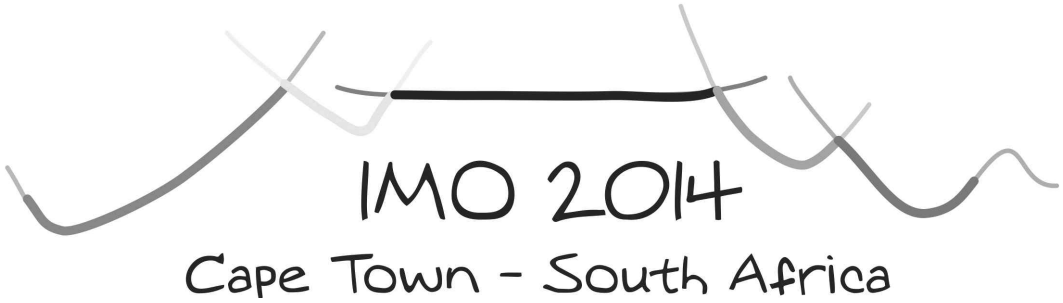
Cover: The cover shows all non-isomorphic homeomorphically irreducible trees of degree 10. See page 16 for a translation!

IMO COUNTDOWN

The 55th International Mathematical Olympiad (IMO) will take place in Cape Town from 3rd to 13th July 2014. Invitations have gone out to over one hundred countries to send teams of their top six high school mathematicians to take part in the biggest, the most prestigious and undeniably the toughest of all the scientific Olympiads for high schools.

First held in 1959 in Romania, the IMO has taken place in every continent except Africa. African participation in the event has been disappointing. Before South Africa's first participation in 1992, only countries on the continent's Mediterranean coast had taken part: Tunisia, Morocco and Algeria. Mozambique joined in 2004, followed by Nigeria (2006), Benin, Zimbabwe and Mauritania (2009), Ivory Coast (2010) and Uganda (2012). Special efforts are being made to encourage participation on IMO2014 by more African countries.



The IMO is the most challenging intellectual contest in the world. Two papers are written, each consisting of just three mathematical problems with what appears to be a generous $4\frac{1}{2}$ hours to crack them. However, the problems are very challenging, even though they may take only a few lines to state and are phrased in the simple language of school mathematics.

Preparations for IMO2014 are well in hand. A Problems Committee of international experts has been appointed to draw up a short list of problems for the contest, with the Jury (the team leaders of the participating countries) making the final selection. Then the papers have to be translated into over 50 languages. After the papers have been written, the team leaders mark them, under the supervision of 50 Coordinators (experienced Olympiad mathematicians, both South African and international), who ensure that the marking is consistent and fair. The Jury then makes the final decisions on the award of Gold, Silver and Bronze medals.

The IMO will take place on the campus of the University of Cape Town. It is expected to be the biggest IMO ever. □

A THIRD ROUTE TO ROOT TWO

A. F. Beardon

University of Cambridge

In *Mathematical Digest* **173** (October 2013) an article appeared with the title "Two Routes to Root Two". Here is a third route.

Suppose that we can solve the equation $X^2 - 2Y^2 = 1$, with X and Y positive integers. Then $X^2 > 2Y^2$ so that

$$0 < \frac{X}{Y} - \sqrt{2} < \left(\frac{X}{Y} - \sqrt{2}\right)\left(\frac{X}{Y} + \sqrt{2}\right) = \frac{X^2}{Y^2} - 2 = \frac{1}{Y^2},$$

and this shows that if Y is large then X/Y is a good rational approximation to $\sqrt{2}$. So how can we solve the equation $X^2 - 2Y^2 = 1$?

A little trial and error gives a solution $(X, Y) = (3, 2)$, but as Y is small, X/Y $(= 3/2)$ is not a good approximation to $\sqrt{2}$. However, there is a way to generate an infinite number of solutions in which the terms Y increase to as much as we like, and where the ratios X/Y converge to $\sqrt{2}$.

Suppose that X, Y, U and V are positive integers with $X^2 - 2Y^2 = 1$, and $U^2 - 2V^2 = 1$. Now define the matrices P and Q by

$$P = \begin{pmatrix} X & 2Y \\ Y & X \end{pmatrix}, \quad Q = \begin{pmatrix} U & 2V \\ V & U \end{pmatrix}.$$

Then

$$PQ = \begin{pmatrix} XU + 2YV & 2[XV + YU] \\ XV + YU & XU + 2YV \end{pmatrix},$$

and the coefficients of this matrix also satisfy the same equation, for

$$\begin{aligned} (XU + 2YV)^2 - 2(XV + YU)^2 &= \det(PQ) \\ &= \det(P)\det(Q) \\ &= (X^2 - 2Y^2)(U^2 - 2V^2) \\ &= 1. \end{aligned}$$

It follows from this that if we write

$$P^n = \begin{pmatrix} X_n & 2Y_n \\ Y_n & X_n \end{pmatrix},$$

then, for all n, $X_n^2 - 2Y_n^2 = 1$.

Now we know that $(X, Y) = (3, 2)$ is a solution so, for example, if we use this solution we obtain

$$P = \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}, \quad P^4 = \begin{pmatrix} 577 & 816 \\ 408 & 577 \end{pmatrix}, \quad P^6 = \begin{pmatrix} 19601 & 27720 \\ 13860 & 19601 \end{pmatrix}.$$

In fact, 577/408 and 19601/13860 are approximations to $\sqrt{2}$ that are correct to five and eight decimal places, respectively.

Of course, we can use the same method for any equation of the form $X^2 - NY^2 = 1$, for if N is not a perfect square then $\sqrt{N}$ is irrational, its decimal expansion is infinite and non-repeating, and the problem is to find a good rational approximation to $\sqrt{N}$.

The earliest evidence we have of good approximations to the irrational square root of an integer is from Greece and India. For example, Baudhâyana (850 BC) gave 577/408 as an approximation to $\sqrt{2}$, and Archimedes (250 BC) gave 1352/780 as an approximation to $\sqrt{3}$; both of these are accurate to five places of decimals.

How were these obtained? Perhaps we shall never know, but surely one possible clue is that

$$577^2 - 2 \times 408^2 = 1, \quad 1352^2 - 3 \times 780^2 = 1.$$

As a second example, consider the equation $X^2 - 7Y^2 = 1$. A little experimentation provides the solution $(X_1, Y_1) = (8, 3)$, and in this case we find that

$$M = \begin{pmatrix} 8 & 21 \\ 3 & 8 \end{pmatrix}, \quad M^6 = \begin{pmatrix} 8193151 & 2167690 \\ 3096720 & 8193153 \end{pmatrix},$$

so that $8193151^2 - 7 \times 3096720^2 = 1$. In fact, 8193151/3096720 is a rational approximation to $\sqrt{7}$ that is correct to fourteen places of decimals. $\square$

LIMERICK

It filled Galileo with mirth
To watch his two rocks fall to earth.
He was heard to proclaim
"Their rates are the same
And quite independent of girth!"

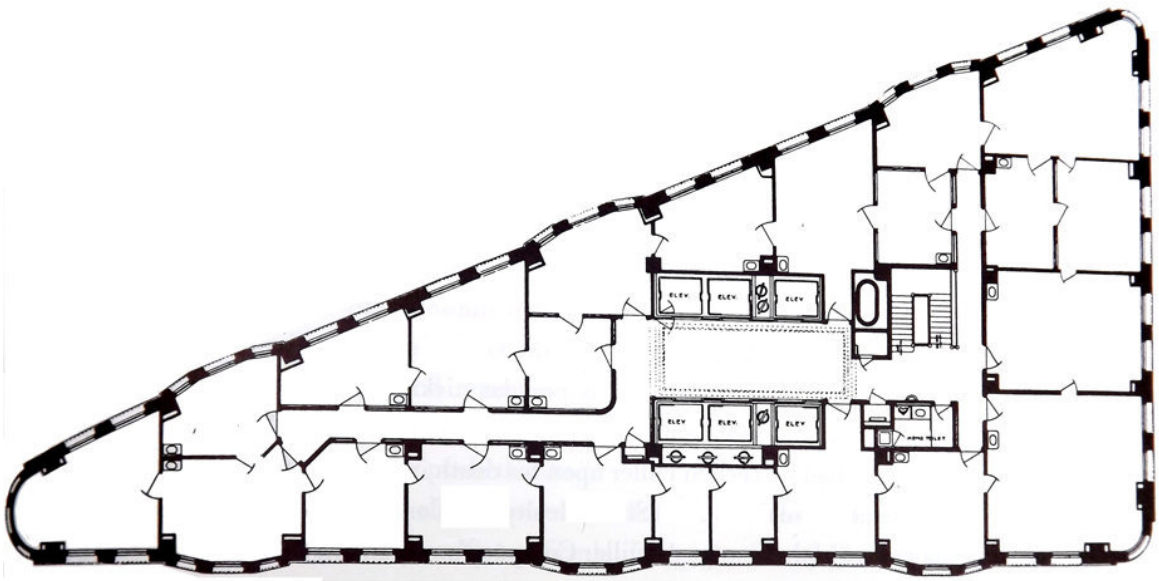
$\square$

PYTHAGORAS AND THE FLATIRON

The Flatiron Building is one of the landmarks of New York. Built in 1902, the Fifth Avenue building is triangular in shape: its footprint is a 5-12-13 triangle.

So it was appropriate that, on the 5th day of the 12th month of the 13th year of the century (5-12-13) the Flatiron Building celebrated its Pythagorean shape when thousands of New Yorkers with glow sticks lit up the building.

The event was organized by MoMath – the new National Museum of Mathematics in New York. It was the biggest mathematics demonstration ever seen in New York, and it will not be for another 92 years (on 13 December 2105 – that's 13-12-05) that the date will again be given by this Pythagorean triple.



□

TRIANGULAR-SQUARE NUMBERS

A. F. Beardon

University of Cambridge

A *square number* is a positive integer of the form m^2, and a triangular number is a positive integer of the from $\frac{1}{2}n(n+1)$, and these, together with hexagonal numbers, appear in the article in "Hexagonal Numbers Again", in *Mathematical Digest* **173** (October 2013).

The great Swiss mathematician Leonhard Euler (1707-1783) listed all positive integers that are simultaneously square and triangular numbers.

To find all such numbers we need to find all integers m and n that satisfy the equation $\frac{1}{2}n(n+1) = m^2$ or, equivalently, $X^2 - 8Y^2 = 1$, where $X = 2n+1$ and $Y = m$. We can find solutions of this equation using the method described in "A Third Route to Root Two" (page 2).

Clearly $(X, Y) = (3, 1)$ is a solution and we have

$$M = \begin{pmatrix} 3 & 8 \\ 1 & 3 \end{pmatrix}, \quad M^2 = \begin{pmatrix} 17 & 48 \\ 6 & 17 \end{pmatrix}, \quad M^8 = \begin{pmatrix} 665857 & 1883328 \\ 235416 & 665857 \end{pmatrix}.$$

Thus, for example, if $2n + 1 = X = 17$ and $m = Y = 6$, then we obtain the square and triangular number 36 with

$$6^2 = 36 = \tfrac{1}{2}(8 \times 9).$$

To obtain a (much) bigger square-triangular number we could use the fact that $665857^2 - 8 \times 235416^2 = 1$.

Interestingly, Euler also proved that while a triangular number can be a square of an integer, no triangular number except 1 is a cube, or a fourth power, of an integer. □

BOTTLE PROBLEM

Place ten milk bottles in a 5×5 crate so that no row, column or diagonal contains more than three bottles.

The answer is on page 29. □

COMPOUND INTEREST IN BABYLON

How long will it take an investment, earning interest at 20% per annum, to double in value?

This is a standard problem in compound interest, which is quickly solved with logarithms. If the number of years is n, then $1.2^n = 2$. Taking logs gives $n = \dfrac{\log 2}{\log 1.2} = 3.80$, to two decimal places.

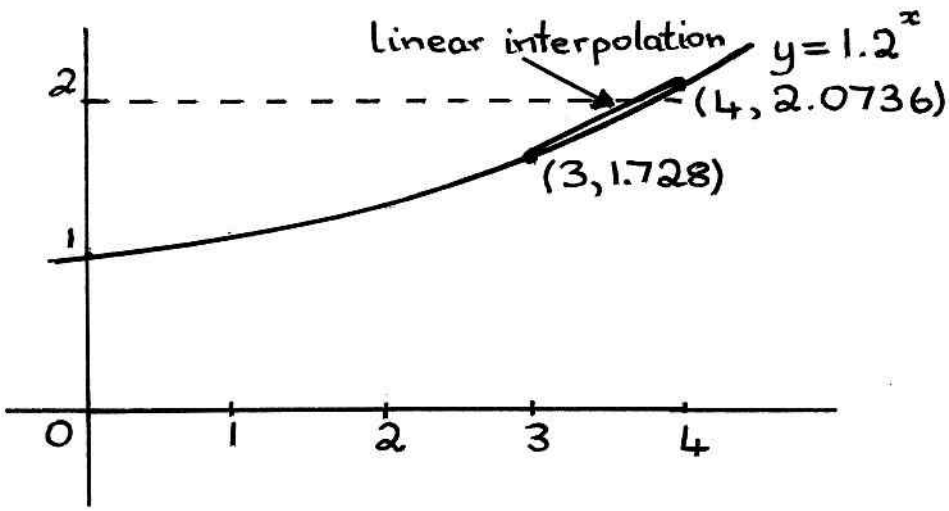
This compound interest problem can be traced back nearly four thousand years, to Larsa, the ancient capital of Babylonia on the Euphrates river in present-day Iraq. This is the biblical Ellasar, mentioned in the Bible (Genesis, chapter 14, verse 1).

The problem was one of five on a clay tablet, which is now in the Louvre in Paris. Archaeologists and experts in ancient languages have been able to read the tablet. The answer given (3.787 years) turns out to be remarkably accurate.

How did the Babylonian mathematicians solve the problem? They did not know about logarithms, which were devised only thousands of years later (in the early 17th Century, by the Scottish mathematician John Napier). Though the clay tablet gives no explanation of the method used, it is possible to make a shrewd guess at how it was done.

The problem is to solve the equation $1.2^x = 2$. Since $1.2^3 = 1.728$, the solution is bigger than 3, and since $1.2^4 = 2.0736$ the solution is smaller than 4. So the solution lies between 3 and 4.

Now look at the graph of $y = 1.2^x$.



The straight line joining the points $(3, 1.728)$ and $(4, 2.0736)$ lies very close to the graph. It has slope $\dfrac{2.0736 - 1.728}{4 - 3} = 0.3456$ and equation

$$y - 1.728 = 0.3456(x - 4).$$

Setting $y = 2$ and solving the equation for x gives $x = 3.787$, which is the solution given on the clay tablet. It is therefore reasonable to assume that the anonymous Babylonian scribed obtained the solution in exactly this way.

This method (of approximating a graph by a straight line segment) is known as *linear interpolation*. The method has been found on many other Babylonian mathematical texts, especially astronomical calculations. □

THE 2014 SOUTH AFRICAN MATHEMATICS OLYMPIAD

The South African Mathematics Olympiad (SAMO) is the biggest Olympiad in the country, with more than 80 000 participants in 2013.

The Olympiad is in three rounds. The first round is at three levels: grade 8, grade 9 and grades 10 to 12. Each paper consists of multiple-choice problems, and those who achieve over 50% progress to round two.

The second-round problems are more challenging, but require answers only. Here there are two levels: grades 8 and 9, and grades 10 to 12. The top hundred at each level qualify for the third round, a paper consisting of real Olympiad problems: tough questions requiring insight, ingenuity, and logically rigorous proofs.

Important dates:
Closing date for entries: 7 February
First round: 13 March
Second round: 13 May
Third round: 10 September

Contact information:
South African Mathematics Foundation
Tel + 27 (0)12 392 9362/24/42/72
E-mail: info@samf.ac.za
Website: www.samf.ac.za □

SAMF
SOUTH AFRICAN MATHEMATICS FOUNDATION

CALCULATING LOGARITHMS

The *logarithm* of a positive real number A to the base a, written $\log_a A$, is the power to which the base must be raised to give A. So

$$a^{\log_a A} = A.$$

In that definition it must be assumed that a is a positive real number, and not 1.

So we have the following:

$$\log_2 8 = 3, \quad \text{since } 2^3 = 8 \qquad\qquad \log_3 81 = 4, \quad \text{since } 3^4 = 81$$

$$\log_4 2 = \frac{1}{2}, \quad \text{since } 4^{\frac{1}{2}} = 2 \qquad\qquad \log_8 4 = \frac{3}{2}, \quad \text{since } 8^{\frac{3}{2}} = 4$$

When logarithms were invented, they were used primarily for calculations. Because our counting system is base 10, logarithms to the base 10 were the main focus. As a result, when the base is omitted, it is assumed that it is 10.

The logs of powers of 10 are easy to calculate. Straight from the definition we have:

$$\log 10 = 1, \qquad \log 100 = 2, \qquad \log \tfrac{1}{10} = -1, \qquad \log 1 = 0, \qquad \log 1000 = 3$$

What about logarithms of other numbers, which are not simple powers of 10? What are the values of $\log 2, \log 3, \log 4$, etc.

With just a little numerical ingenuity we can calculate logs of these numbers (at least to an accuracy of two decimal places).

In what follows, the symbol "$\doteq$" means "approximately equal to" — usually two decimal place accuracy.

Let's start with the calculation of $\log 2$. The key observation is that

$$2^{10} = 1024 \doteq 1000 = 10^3.$$

So $10 \log 2 \doteq 3$, giving $\log 2 \doteq 0.30$ (which turns out to be accurate to two decimal places).

Having calculated $\log 2$, we now tackle $\log 3$. Noting that

$$3^5 = 243 \doteq 250 = \frac{1000}{4}$$

we deduce that

$$5 \log 3 \doteq \log 1000 - \log 4 = 3 - 2 \log 2$$

from which it follows that $\log 3 \doteq 0.48$ (which is again accurate to two decimal places).

The next calculation is easy:

$$\log 4 = 2 \log 2 \doteq 0.60$$

and so is the next:

$$\log 5 = \log(10/2) = \log 10 - \log 2 \doteq 0.70$$

and the next

$$\log 6 = \log 2 + \log 3 \doteq 0.78.$$

There is another way of calculating $\log 6$, based on the observation that

$$6^9 = 10\ 077\ 696 \doteq 10^7$$

from which it follows that

$$\log 6 \doteq \frac{7}{9} \doteq 0.78.$$

To calculate $\log 7$, we note that

$$7^2 = 49 \doteq 50$$

giving

$$2 \log 7 \doteq \log \frac{100}{2} = \log 100 - \log 2 = 2 - 0.03 = 1.70$$

and hence $\log 7 \doteq 0.85$.

The logarithms of $8 = 2^3$ and $9 = 3^2$ should now give no trouble (they are 0.90 and 0.96, respectively). And $\log 10 = 1$, of course.

Since $9 \times 11 = 99 \doteq 100$, $\log 11 \doteq 2 - 2 \log 3 \doteq 1.04$.

With $\log 12 = 2 \log 2 + \log 3 \doteq 1.08$ giving no trouble, our next target is $\log 13$. Here there is an opportunity for nice little bit of numerical ingenuity: the observation that

$$7 \times 11 \times 13 = 1001 \doteq 1000.$$

So $\log 13 \doteq 3 - \log 7 - \log 11 \doteq 1.11$

Passing quickly over the obvious calculations of the logarithms of 14, 15 and 16, we are next confronted by $\log 17$, and once again a numerical coincidence turns out to be useful:

$$17^2 = 289 \doteq 288 = 2^5 \times 3^2$$

from which we get $\log 17 \doteq 1.23$.

We've done enough to leave further calculations left to the reader. $\square$

DOUBLETS

Doublets is a word game invented by Lewis Carroll on Christmas Day in 1877, and has been popular ever since. This is how Carroll described it in the magazine *Vanity Fair*:

> The rules of the Puzzle are simple enough. Two words are proposed, of the same length; and the Puzzle consists in linking these together by interposing other words, each of which shall differ from the next word in one letter only. That is to say, one letter may be changed in one of the given words, then one letter in the word so obtained, and so on, till we arrive at the other given word. The letters must not be interchanged among themselves, but each must keep to its own place. As an example, the word 'head' may be changed into 'tail' by interposing the words 'heal, teal, tell, tall'. I call the given words 'a Doublet', the interposed words 'Links', and the entire series 'a Chain', of which I here append an example:
>
> HEAD → HEAL → TEAL → TELL → TALL → TAIL.
>
> It is, perhaps, needless to state that it is de rigueur that the links should be English words, such as might be used in good society."

Vladimir Nabokov referred to the game as "word golf" in his novel *Pale Fire* and noted that WORD can be changed to GOLF in three links

WORD → WOLD → GOLD → GOLF

and there is a chain from HATE to LOVE requiring only three links:

HATE → LATE → LAVE → LOVE.

It's harder to find a chain from TRIG into GEOM, but there is one of six links:

TRIG → TRIM → TRAM → TEAM → TERM → GERM → GEOM.

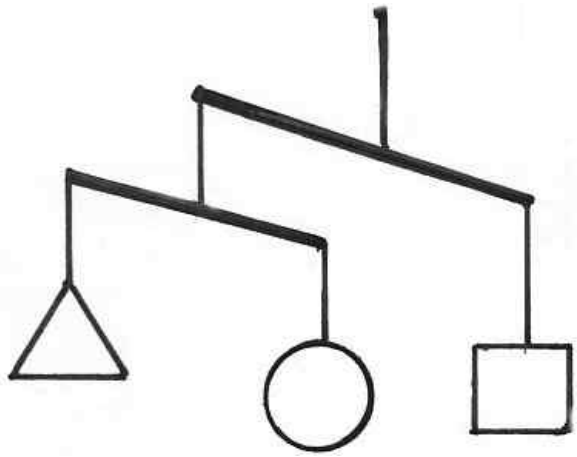
Can you change HAND to FOOT, and navigate SHIP into DOCK?

Six-letter doublets are often impossible, while three-letter doublets can be easy. (Lewis Carroll found that he needed six links to evolve APE into MAN, but it can be done in four. How?)

Answers are on page 29. □

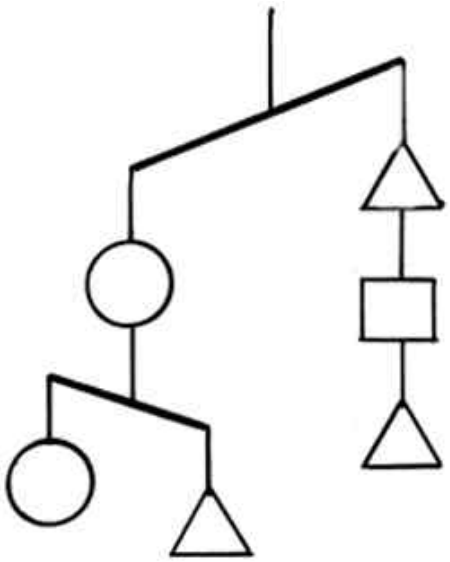
IMBALANCE PROBLEMS

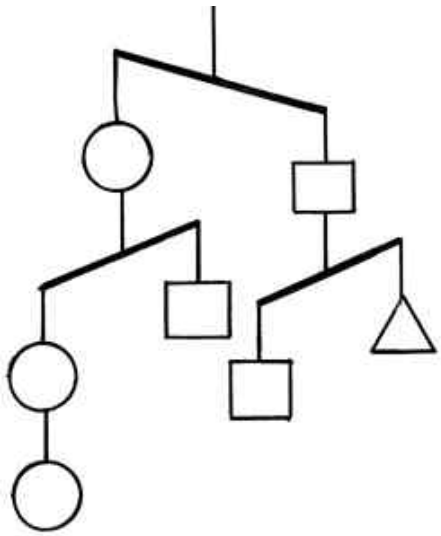
Imbalance Problems were invented by Paul Salomon, who teaches mathematics at the John Burroughs school in St Louis, Missouri, USA.



In the diagram, the slanted lines are thin rods suspended on strings from their centres, and objects are suspended from the ends of the rods. The rods are not horizontal, showing that the objects have different weights, but two objects with the same shape have the same weight. The problem is to work out the relative weights of the objects.

In this simple problem, it is easy to see that the square object is heaviest, followed by the circle, and the triangle is the lightest.

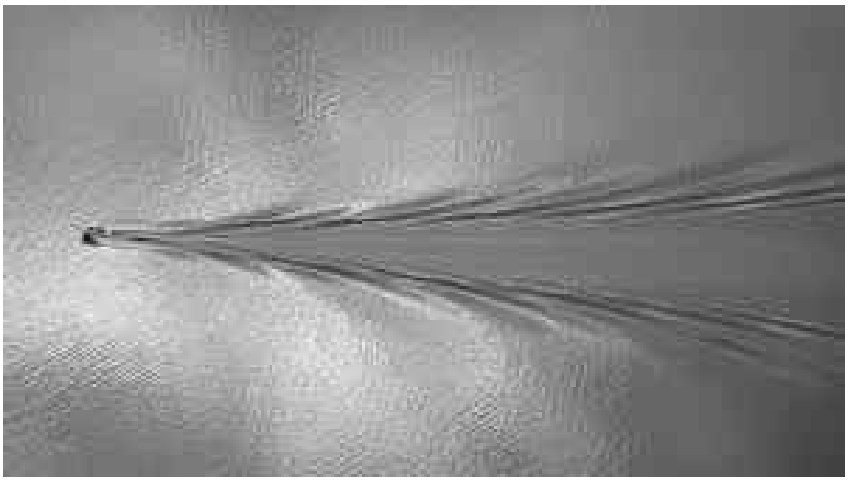




Try to solve the problems (the answers are on page 29) and then invent your own Imbalance Problems and send them to the Editor. The best problems will be published. □

THE KELVIN WAKE

When a ship moves through a calm sea it displaces the water. Since water is incompressible, the ship causes a V-shaped wave which moves outward from its source until it dissipates by friction or dispersion. The study of wakes has important implications for ship design, and for understanding how the banks of navigable rivers can get eroded by the passage of ships.



Wakes were first studied by William Thomson, better known as Lord Kelvin, the famous 19th Century British physicist. He found that the faster the speed of the ship, the faster the wake widens. As a consequence, his calculations showed that the wake angle is a constant 19.47°, independent of the speed of the ship. To be precise, it is the angle whose sine is $\frac{1}{3}$. And it's not just ships that generate this exact wake angle: even a duck paddling across a village pond shows the same wave pattern.

But sometimes narrower wake angles are observed, and these deviations from Kelvin's theory were put down to special situations like shallower water. Recently two French physicists Marc Rambaud and Frédéric Moisy have come up with a more complete analysis.

They went to the Google Earth database and searched for images of ships at sea. They measured the wake angle and length of each ship, and estimated its velocity using the wavelength of the waves of the ship's wake. They then plotted the wake angle against the *Froude number*, a dimensionless physical quantity given by the formula

$$Fr = \frac{v}{\sqrt{gL}} \ ,$$

where v is the velocity of the ship, g is the acceleration due to gravity and L is the waterline length of the ship.

Rambaud and Moisy found that for $Fr < \frac{1}{2}$ the wake angle is constant, but for larger values the wake angle decreases as Fr increases.

The Froude number is analogous to the Mach number for supersonic aircraft, which is the ratio of the velocity of an aircraft to the velocity of sound. The tapering of the Mach cone (the shock wave) for supersonic aircraft resembles the narrowing of the wake of a ship. □

KAPREKAR'S PROCESS

Take any three different digits, arrange them in decreasing and increasing order, and subtract the smaller number from the larger, getting another three digit number. Repeat the process on the result.

For example, starting with 1, 8 and 9, you get $981 - 189 = 792$. Repeat the process: $972 - 279 = 693$. Do it again: $963 - 369 = 594$. But now a repetition gives the same three digits: $954 - 459 = 495$, and the process repeats.

This is known as Kaprekar's Process, discovered by D.R. Kaprekar in 1955.

Can you see why you always end up with the digits 4, 5 and 9? Starting with three digits $a > b > c$, the Kaprekar process will give $abc - cba = d9e$, where $d = a - c - 1$, so that $d - e < a - c$. So the largest and smallest of the digits move closer together, and very quickly you get into the cycle $954 - 459 = 495$.

If you carry out this process with four-digit numbers (not all the same — that's a very uninteresting case), you end up with 6174. For example, starting with 6543, you get in succession:

$6543 - 3456 = 3087 \to 8730 - 0378 = 8352 \to 8532 - 2358 = 6174 \to 7641 - 1467 = 6174$.

Here Kaprekar's process can take a little longer than in the three-digit case, and it is not easy to see why you will always end up with 6174.

When Kaprekar's process is applied to five digit numbers, the situation becomes more complicated, and you have to resort to a computer. It turns out that the process enters in one of three possible cycles:
$\{53955, 59994\}$, $\{71973, 83952, 74943, 62964\}$, $\{75933, 63954, 61974, 82962\}$.
We leave it to the energetic reader to investigate six-digit numbers. □

MATHEMATICS IN OXFORD

In 1868 an Oxford mathematician proposed the development of a School of Mathematics.

The main features of his proposal were:

1. A very large room for calculating Greatest Common Measure. To this a small one might be attached for Least Common Multiple: this, however, might be dispensed with.
2. A piece of open ground for keeping Roots and practising their extraction: it would be advisable to keep Square Roots by themselves, as their corners are apt to damage others.
3. A room for reducing Fractions to their Lowest Terms. This should be provided with a cellar for keeping the Lowest Terms when found, which might also be available to the general body of Undergraduates, for the purpose of "keeping Terms."
4. A large room, which might be darkened, and fitted up with a magic lantern for the purpose of exhibiting Circulating Decimals in the act of circulation. This might also contain cupboards, fitted with glass-doors, for keeping the various Scales of Notation.
5. A narrow strip of ground, railed off and carefully levelled, for investigating the properties of Asymptotes, and testing practically whether Parallel Lines meet or not: for this purpose it should reach, to use the expressive language of Euclid, "ever so far."

Who was the mathematician? The answer is on page 29. □

A NUTTY PERCENTAGE

Reader David Allison alerted us to a consumer column in the *Cape Times* (4 November 2013) in which the deceptive repackaging of a brand of supermarket Pecan Nuts was featured. The former 100g packet was reduced in weight to 75g, and the price was unaltered. The journalist commented that the price had therefore gone up by 25%. Was that right? See page 30 for the answer. □

THE SQUARE ROOT OF THREE

David Feinberg

I fear that I will always be
A lonely number like root three.

A three is all that's good and right,
Why must my three keep out of sight?

Beneath a vicious square root sign,
I wish instead I were a nine.

For nine could thwart this evil trick,
With just some quick arithmetic.

I know Ill never see the sun,
As 1.7321.

Such is my reality,
A sad irrationality.

When hark! What is this I see?
Another square root of a three

Has quietly come waltzing by.
Together now we multiply

To form a number we prefer,
Rejoicing as an integer.

We break free from our mortal bonds
And with the wave of magic wands

Our square root signs become unglued
Your love for me has been renewed.

David Feinberg, professor of computer science at Carnegie Mellon, wrote this poem while at Randolph High School, and published it in the school's literary magazine.

Later, two of his classmates were directing the movie *Harold and Kumar Escape From Guantanamo Bay*, and obtained Feinberg's permission to include it in a pivotal scene. The poem is recited by Kumar (Kal Penn) in a romantic reconciliation at the end of the story. □

HUNTING TREES

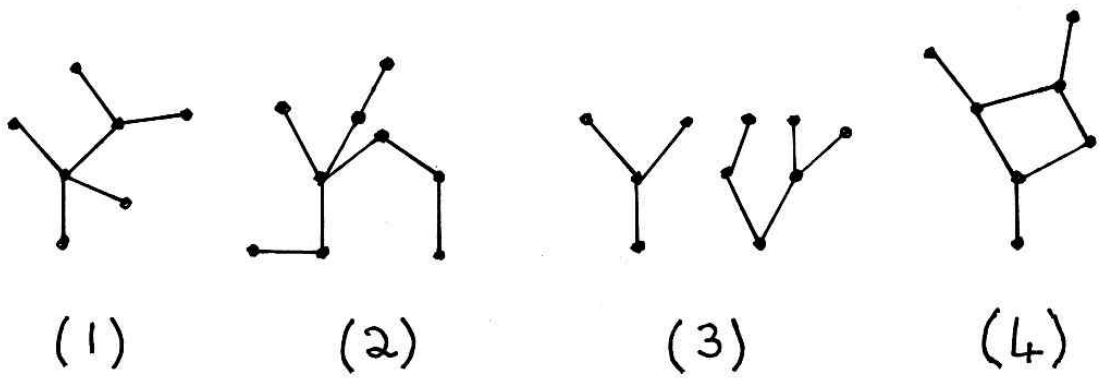
In the 1998 movie *Good Will Hunting*, Matt Damon plays the role of an untutored mathematical genius employed as a janitor in the Massachusetts Institute of Technology. The movie is noted for the amount of genuine mathematics it contains, making the story of the troubled genius seem authentic.

While doing his evening routine of mopping the floors, Will Hunting finds a problem left on a board for students to work on. It is said to be a problem that took MIT professors two years to solve, but Will Hunting cracks it without difficulty. What was the problem?

Succinctly stated, it amounts to finding all non-isomorphic homeomorphically irreducible trees of degree 10. That sounds sufficiently scary, but it can be explained in simple words.

A *tree* is a connected diagram of vertices (dots) joined up by edges (lines), without any circuits. That amounts to saying that for any two vertices, there is exactly on path from one to the other along the edges. The *degree* of a tree is simply the number of vertices.

In the examples below, (1) and (2) are trees, but (3) is not (it is disconnected), nor is (4) (it has a circuit).



Two trees with the same arrangement of vertices and lines, but with different lengths and angles, and differently orientated in space, are not considered different: they are *isomorphic*. We are looking for trees that are *non-isomorphic*: they have an essentially different layout.

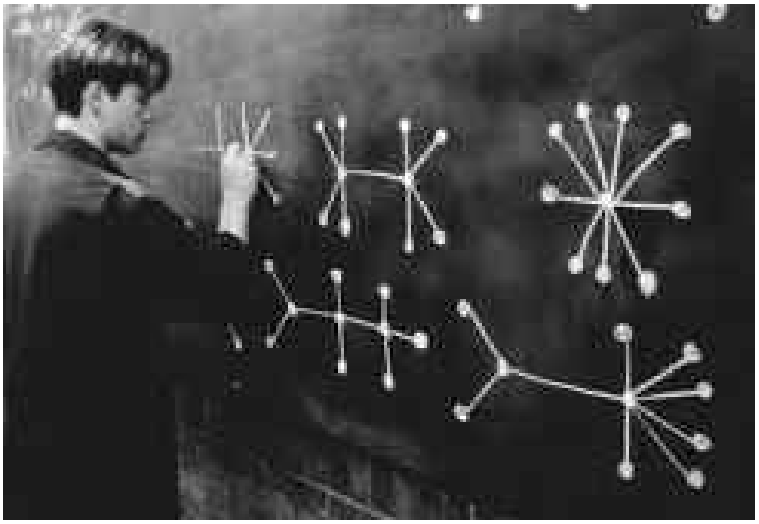
Homeomorphically irreducible means that the tree has no vertex with exactly two lines emanating from it. Tree (1) above is irreducible but (2) is not.

Let us denote by $N(n)$ the number of non-isomorphic homeomorphically irreducible trees of degree n.

It is not difficult to see that $N(1) = N(2) = N(4) = N(5) = 1$ and that $N(3) = 0$.

Then it gets a little harder, but it is still not too difficult to work out that $N(6) = N(7) = 2, N(8) = 4$ and $N(9) = 5$.

The problem in *Good Will Hunting* was to calculate $N(10)$. In the movie, Will Hunting is interrupted when he has drawn eight trees.



What is the value of $N(10)$? Can you find all non-isomorphic homeomorphically irreducible trees of degree 10? The answer is on the cover. □

EQUATIONS

Solving the equations

$$513x + 487y = 8513$$

$$487x + 513y = 8487$$

is, in theory, straightforward: just multiply the first by 513 and the second by 487, then subtract. That step eliminates y and gives an equation from which you can work out the value of x. Now all you have to do is substitute that value of x into either of the two original equations to find y.

That amount of calculation, even with a calculator, is most unattractive. However, it is possible to solve the equations in your head, without writing anything down. Can you see how? The answer is on page 29. □

MARITAL PROBLEMS

Marital Problem 1:
Ahmed is looking at Bongani and Bongani is looking at Carol. Ahmed is married and Carol is unmarried. Prove that a married person is looking at an unmarried person.

Proof
If Bongani is married, then he (a married person) is looking at Carol, who is unmarried.
If Bongani is unmarried, then Ahmed, who is married, is looking at an unmarried person, Bongani.

Marital Problem 2:
Ahmed is looking at Bongani, Bongani is looking at Carol and Carol is looking at Dave. Ahmed is married and Dave is unmarried. Prove that a married person is looking at an unmarried person.

Proof
Suppose Carol is married. Then a married person (Carol) is looking at an unmarried person (Dave).
Suppose Carol is unmarried. Then we may just apply the result in Marital Problem 1.

Marital Problem 3:
Ahmed is looking at Bongani, Bongani is looking at Carol, Carol is looking at Dave and Dave is looking at Ethan. Prove that a married person is looking at an unmarried person.

Proof
Suppose Dave is married. Then a married person (Dave) is looking at an unmarried person (Ethan).
Suppose Dave is married. Then we may just apply the result in Marital Problem 3.

It can now be seen that the Marital Problem can be stated and solved for any line of n people looking at each other as described, with the first person married, and the last unmarried. Somewhere down the line there has to be a married person looking at an unmarried person.

We have seen above that the result is true when $n = 3$, or 4, or 5. (It is, of course, obvious that the result is true when $n = 2$.

We now suppose that the result is true for some natural number k.

If we consider a line of $k + 1$ people looking at each other, with the first person married and the last unmarried, then

EITHER the last but one (in position k) is married, in which case we have a married person (the last but one) looking at an unmarried person (the last),

OR the last but one person is unmarried, in which case we know from the established case of a line of k people that there is a married person looking at an unmarried person.

This method of proof is known as *mathematical induction*.

There is a simple geometric way of solving the marital problem.

If we have a line of people $A, B, C, \cdots, Z$ looking at each other as described, with A married and Z unmarried. Label each person with 0 if married and 1 if unmarried. We then have a string of 0's and 1's, starting with a 0 (A is married) and ending with a 1 (Z is unmarried). Go down the line until you find the first 1. Then the previous number was 0, and that married person is looking at an unmarried person. □

MIXING DRINKS

If you were to mix a litre of pure alcohol (C_2H_6O) with a litre of pure water (H_2O) you would not get two litres of 50% proof alcohol, but 1.92 litres. This may be surprising to barmaids, but is a well-known phenomenon in the chemistry laboratory.

What is more surprising is that the weight of the mixture is *more* than the total weights of the separate litres of water and alcohol. How can that be? Surely the basic law of conservation of matter implies that you can't create extra matter?

Can you explain the paradox? If not, turn to page 30. □

PROGRAMMING OLYMPIAD RESULTS

The results of the 2013 Programming Olympiad were announced at a banquet in Cape Town on Monday 30 September 2013.

Fifteen of the top high school programmers out of an initial entry of 4747 had spent the weekend in a two-day programming competition. Peter Waker, the Manager of the Computer Olympiad, pointed out a remarkable feature: only six of the finalists were in grade 12. He commended the finalists, saying "To be even number 15 out of 4 747 entries is a remarkable achievement. Watch these youngsters; they will create the future."

Gold Medal:
Guy Paterson-Jones (grade 12, Bishops)

Silver Medals
Yaseen Mowzer (grade 9, Fairbairn College)
Shaylan Lalloo (grade 12, Pearson High School)

Bronze Medal
Ulrik de Muelenaere (grade 10, Waterkloof High School)
Darren Roos (grade 11, Pretoria Chinese School)
Robin Visser (grade 11, St George's Grammar School)

Runners-up
Jonathan Alp (grade 11, Parklands College)
Matthew Cherry (grade 12, St Dominic's Priory)
Frederik du Toit (grade 10, Centurion High School)
Pieter Kok (grade 10, Centurion High School)
Demetre Liakos (grade 11, Selly Park Secondary School)
Jan de Wet Linde (grade 12, Jim Fouché High School)
Kiuran Naidoo (grade 12, Clifton School)
Thomas Orton (grade 11, Bishops)
Ryno Swart (grade 12, Durbanville High School)

Computer Olympiad
South African Computer Olympiad: a project of the Computer Society of South Africa.

The 2014 Computer Olympiad starts with a Talent Search, held in schools on Tuesday 11 March.

The first round of the Applications Olympiad will be held in schools on Friday 9 May, with the finals in Cape Town on Tuesday 1 July and the Awards Ceremony on Wednesday 2 July.

The first round of the Programming Olympiad will be held in schools on Friday 1 August, with the finals in Cape Town on Saturday 11 and Sunday 12 October and the Awards Ceremony on Monday 13 October.

For full details, contact:

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THIRTY DAYS HATH SEPTEMBER

Have you ever been caught by a smartypants who asks you: "How many months of the year have 30 days?"

The natural response is to recite quickly to yourself the little nursery rhyme

> Thirty days hath September
> April, June and November.
> All the rest have thirty-one,
> Excepting February alone,
> Which hath but twenty-eight days clear
> And twenty-nine in each Leap Year.

You then give the answer: four.

Your smartypants friend cries triumphantly: "Wrong! Eleven months of the year have thirty days in them."

Here's a way of getting your own back if caught out by this catch question. Just say: "Well, if you are going to interpret 'thirty' as 'thirty or more' in your question, then you must accept my answer of 'four' as 'four or more', which is correct. □

SOLAR POWER MATHS

The Sun is the source of life on the Earth, pouring energy onto our planet at the rate of 1.4 kilowatts per square metre. If all that energy could be harnessed, our power problems would be no more.

The solution is, however, not that simple. For a start, that figure of 1.4kW/m^2 applies to the top of the Earth's atmosphere. Much of the Sun's power is lost in passing through the atmosphere, blocked by dust particles or reflected by clouds.

In higher latitudes, the Sun's rays pass through a thicker layer of air than at the equator, so more energy is lost. Then there is the Earth's rotation, which means that for half the time the Sun does not shine, and some countries are less sunny than others. In South Africa we have an average of about 2500 hours of sunshine per year.

Every country has an "availability factor" k_a, which describes what proportion of the Sun's energy is available. In South Africa $k_a = 0.16$, a relatively high figure. That means that the average solar radiation per square metre in South Africa is $1400 \times 0.16 = 224$ watts.

How can solar energy be captured and used? One way is to use the Sun's rays to heat water. Solar panels on the roofs of houses are now a common sight and can bring down the cost of hot water in the home by a significant amount. However, a constant and cheap hot water supply will not run domestic appliances such as the fridge, TV or microwave oven. Nor is it of any use in charging a cellphone or electric toothbrush. For all those purposes solar radiation must be converted to electricity.

This can be done by using panels of photovoltaic cells, which convert sunlight to electicity directly. This process is neither efficient nor cheap.

The currently level of efficiency of photovoltaic cells is about 15%. A solar panel of 1 m^2 in South Africa will generate $224 \times 0.15 = 34.6$ watts. That means that if you want a constant power supply to your home of 1 kW you will need need $1000 \div 34.6 = 29.8$ square metres of solar panels. That will be enough for low-power appliances such as a fan, radio or cellphone charger, but not nearly enough for cooking.

How much will it cost to set up a house for full solar-powered electricity? Apart from the solar panels, you will need an inverter (to convert the solar-generated DC current to AC) and a bank of batteries to store energy so

you can watch TV after sunset. Inverters and batteries cause further losses of efficiency, and any engineer will recommend a safety margin of 20% on the installation specifications. Overall, the bill may run up to R250 000. Bearing in mind that the equipment will need maintenance and has a finite life-span, the cost of running a normal household on solar power may be around ten times the cost of power from the Eskom grid. □

APPLICATIONS OF MATHEMATICS

In universities, mathematics is taught mainly to men who are going to teach mathematics to men who are going to teach mathematics to ... . Sometimes, it is true, there is an escape from this treadmill. Archimedes used mathematics to kill Romans, Galileo to improve the Grand Duke of Tuscany's artillery, modern physicists (grown more ambitious) to exterminate the human race. It is usually on this account that the study of mathematics is commended to the general public as worthy of State support.

(Bertrand Russell, in *Human Society in Ethics and Politics*, 1954) □

BAR JOKES

A neutrino walks into a bar and asks how much the drinks cost. The barman replies "For you, no charge."

A chemist walks into a bar and asks for a glass of hijklmno. The barman thinks "H to O" and gives him a glass of water.

A physicist walks into a bar, and calls out: "Ten drinks for everybody in the house!" The barman comments: "Now that's an order of magnitude."

An infinite number of mathematicians walk into a bar. The first orders one pint of beer, the second orders half a pint, the third orders a quarter of a pint, and so on. The barman waits until all the orders are in, then puts two pints on the counter, saying "You mathematicians clearly don't know your limits!" □

VARSITY TEST

Varsity readiness tests are now a regular feature of MATHEMATICAL DIGEST. They are designed to highlight those parts of school maths that are of particular importance for students planning to study mathematics at university level. These include:

- arithmetical ability (without needing a calculator)
- algebraic skills
- thorough knowledge of trig formulas, and how to derive them when you don't have a formula sheet
- geometrical and three-dimensional perception
- understanding of logic (proofs, counter-example and contradiction)

The present Senior Certificate examination is, unfortunately, not very challenging when it comes to these areas, and experience has shown that even an A Symbol for SC Mathematics in these inflationary times is no guarantee of success when it comes to tackling varsity maths. There is, of course, rather more to university maths courses than multiple-choice problems, but any difficulty you might have in tackling the problems below will highlight a weakness in your mathematical background that may cause you trouble in a first-year university maths course.

1. $\left(1+\dfrac{1}{a-1}\right)\left(1+\dfrac{1}{a}\right)\left(1+\dfrac{1}{a+1}\right)$ is equal to

   (A) $1+\dfrac{1}{a+2}$ (B) $1+\dfrac{3}{a-1}$ (C) $1+\dfrac{1}{3a}$ (D) $1+\dfrac{1}{a^3-a^2}$ (E) $1+\dfrac{2}{a}$

2. $7^4 - 2\times 42^2 + 6^4$ is equal to

   (A) 169 (B) 179 (C) 189 (D) 199 (E) 209

3. For which values of k does the equation $\dfrac{x^2+x}{x^2+x-2} = k$ have no solution?

   (A) $\frac{1}{9} < k \leq 1$ (B) $1 \leq k < 3$ (C) $0 < k \leq 3$ (D) $-1 \leq k \leq 2$
   (E) $\frac{1}{3} < k \leq 2$

4. The area of a right-angled triangle is 10 cm^2 and its perimeter is 20 cm. What is the length of the hypotenuse of the triangle?

   (A) $7\frac{1}{2}$ (B) 8 (C) $8\frac{1}{2}$ (D) 9 (E) $9\frac{1}{2}$

5. The fourth power of $\sqrt{1+\sqrt{1+\sqrt{1}}}$ is

(A) $2+\sqrt{3}$ (B) $\frac{1}{2}(7+3\sqrt{5})$ (C) $1+2\sqrt{3}$ (D) 3 (E) $3+2\sqrt{2}$

6. If $a=\log_8 225$ and $b=\log_2 15$, then

(A) $a=b/2$ (B) $a=2b/3$ (C) $a=b$ (D) $b=a/2$ (E) $a=3b/2$

7. If r is a root of the equation $r^2+6r+3=0$, what is the value of $(r+2)(r+4)$?

(A) 2 (B) 3 (C) 4 (D) 5 (E) 6

8. In triangle ABC, $\angle A=50^\circ$. The bisectors of $\angle B$ and $\angle C$ meet in O, inside the triangle. Then $\angle BOC$ is equal to

(A) 100° (B) 105° (C) 110° (D) 115° (E)120°

9. $(a^2+b^2)(c^2+d^2)$ is equal to

(A) $(ac-bd)^2+(bc+ad)^2$ (B) $(ad-bc)^2+(ab+cd)^2$

(C) $(ab-cd)^2+(ac+bd)^2$ (D) $(ac-bd)^2+(ab+cd)^2$

(E) $(ab-cd)^2+(ab+cd)^2$

10. If $\cos 2x=\frac{1}{8}$ and $0^\circ<x<90^\circ$, then $\cos x$ is equal to

(A) $\frac{1}{4}$ (B) $\frac{1}{2}$ (C) $\frac{1}{16}$ (D) $\frac{2}{3}$ (E) $\frac{3}{4}$

11. The solution of the inequality $3^{2x+1}+3<3^{x+2}+3^x$ is the interval

(A) $(-1,1)$ (B) $(1,\frac{3}{2})$ (C) $(-\frac{1}{2},3)$ (D) $(\frac{1}{2},3)$ (E) $(1,2)$

12. In triangle ABC, D, E and F lie on sides AB, BC and CA, respectively, so that $BD=DE$ and $CE=CF$. If $\angle A=80^\circ$, what is the size of $\angle DEF$?

(A) 70° (B) 50° (C) 40° (D) 100° (E) 80°

13. Find all values of x such that $-90^\circ\le x\le 270^\circ$ and $2\sin x\cos x=\sin x$.

(A) $-60^\circ, 0^\circ, 60^\circ, 180^\circ$
(B) $-60^\circ, -30^\circ, 0^\circ, 60^\circ, 180^\circ$
(C) $0^\circ, 60^\circ, 150^\circ, 180^\circ$
(D) $-60^\circ, 0^\circ, 60^\circ, 120^\circ, 180^\circ$
(E) $-30^\circ, 0^\circ, 30^\circ, 120^\circ, 180^\circ$

14. A boy runs from his home to school at 9 km/h, then walks back home at 5 km/h. What was his average speed, to one decimal place, for the trip to school and back?

(A) 6.3 km/h (B) 6.4 km/h (C) 6.6 km/h (D) 6.9 km/h (E) 7.1 km/h

15. If $(3x+1)(4-x)(2+7x)$ is multiplied out to the form ax^3+bx^2+cx+d, then $a+b+c+d$ is equal to

(A) 108 (B) 144 (C) 96 (D) 104 (E) 98

16. Which of the following is a factor of the quadratic $6x^2-7x-24$?

(A) $x+12$ (B) $x-3$ (C) $6x+1$ (D) $x-24$ (E) $3x-8$

17. The set of all real numbers x such that $x^2<5x+24$ is the interval

(A) $(-3,8)$ (B) $(1,7)$ (C) $(3,8)$ (D) $(-8,-3)$ (E) $(-2,4)$

18. How many negative integers satisfy the inequality $6x^2-29x-65<0$?

(A) none (B) 1 (C) 5 (D) 8 (E) infinitely many

19. A right-angled isosceles triangle has an area of 8. The perimeter of the triangle is

(A) $8+4\sqrt{2}$ (B) $4+4\sqrt{2}$ (C) 16 (D) 12 (E) $8\sqrt{2}$

20. Triangle ABC is right-angled at B. If $AB=5n$ and $BC=12n$, where n is a positive integer, which of the following is not a possible length for AC?

(A) 39 (B) 69 (C) 91 (D) 104 (E) 143

21. $\sqrt[3]{27^{27x^{27}}}$ is equal to

(A) 27^{3x^9} (B) 27^{3x^3} (C) 27^{27x^9} (D) $27^{9x^{27}}$ (E) $27^{3x^{27}}$

22. $\log_2 500$ is equal to

(A) $4+2\log_2 5$ (B) $1+3\log_2 5$ (C) $3+2\log_2 10$
(D) $3+2\log_2 5$ (E) $2+3\log_2 5$

23. If $f(x)=\dfrac{2x+1}{3x+2}$ and $g(x)=\dfrac{x-2}{2x-3}$ then $g(f(x))$ is equal to

(A) $\dfrac{5x-7}{11x-12}$ (B) $\dfrac{4x-5}{7x-11}$ (C) $\dfrac{3x-7}{7x-11}$ (D) $\dfrac{3x-8}{5x-11}$ (E) $\dfrac{4x+3}{5x+4}$

24. The parabola with equation $y=x^2-3x+2$ is translated 2 units to the right and 3 units up. The x-coordinate of the point of intersection of the two parabolas is

(A) $\frac{5}{2}$ (B) -3 (C) $\frac{13}{4}$ (D) $\frac{7}{4}$ (E) -2

25. When $(3x-2)(2x+5)(x+3)$ is multiplied out, the coefficient of x is

(A) 29 (B) 33 (C) 17 (D) 23 (E) 31

Answers are on page 30. □

MAGIC RECTANGLES

A *magic rectangle* is an $m \times n$ array of the positive integers from 1 to $m \times n$ such that the numbers in each row have a constant sum and the numbers in each column have a constant sum. These two sums will be different if the rectangle is not a square.

An example of a 3×5 magic rectangle is shown below, with row sum 40 and column sum 24.

6	7	8	9	10
13	3	1	11	12
5	14	15	4	2

When can the integers from 1 to 24 be arranged to make a magic rectangle? The grid to be filled must be either 2×12, 3×8 or 4×6. Can you put the numbers from 1 to 24 into each of these three grids to make magic rectangles?

See page 30 for answers. □

PYTHAGOREAN DATES

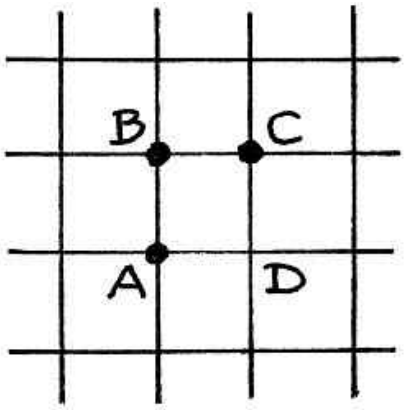
A *Pythagorean date* is one formed of the three numbers of a Pythagorean triple. The most recent example was 5-12-13 (see *Pythagoras and the Flat-iron*, page sss).

When will the next Pythagorean date occur? The numbers in a Pythagorean triple can be used in any order.

The answer is on page 30. □

FROG JUMPS

Frog Jumps is a game played on a grid of squares. It starts with three frogs (counters) placed at the vertices of one of the squares, with the fourth vertex vacant.



Frogs may jump over each other according to a simple rule, "reflection in another frog". What that means is that a frog at vertex P may jump over another frog at vertex Q to a vacant vertex R, where PQR is a straight line of vertices and $PQ = QR$. A frog that is jumped over is not removed.

The object of the game is to jump the three frogs over each other from one grid point vertex to another in such a way that eventually one of them lands on the vacant vertex D, and to do this in the smallest number of jumps. Can you do it? The answer is on page 30. □

GUILTY PARTY

Beane, Gorne and Dunnit is a firm of attorneys well-known for their dishonest ways.

When a client made indignant enquiries about the disappearance of money in a trust fund administered by the partnership, and accused them of theft, he got three responses:

Beane said: "I didn't do it."
Gorne said: "Oh yes you did!"
Dunnit said: "Beane is innocent."

The client learnt that two of these statements were lies, and the third true.

Who stole the money?

The answer is on page 31.

ERRATUM

In *Mathematical Digest* **173** (October 2013), question 1 of the Varsity Test (page 18): $\sin 15^\circ \cos 75^\circ$ should be $\sin 15^\circ \cos 15^\circ$. □

ANSWERS

Bottle Problem (page 5):

o	x	x	o	x
x	x	o	x	o
o	x	o	x	x
x	o	x	x	o
x	o	x	o	x

Doublets: (page 11):
HAND → BAND → BOND → FOND → FONT → FOOT
SHIP → SLIP → SLAP → SOAP → SOAK → SOCK → DOCK
APE → OPT → OAT → MAT → MAN

Imbalance Problems (page 11):
In each problem, the weights of the circle, square and triangle are denoted by a, b and c, respectively.
1. $b > c > a$
2. $c > b > a$
3. $c > a > b$

A Nutty Percentage (page 14):
The journalist was wrong. While the weight reduction of the packet of nuts was indeed 25%, the price increase was more than that.

The price of the packet of nuts was not mentioned in the report, so we suppose it was P. Then the price per gram of the nuts went from $P/100$ to $P/75$. So the price increase was $P/75 - P/100$ and the percentage increase was $\dfrac{P/75 - P/100}{P/100} \times 100$ which comes out at $33\frac{1}{3}\%$.

Mathematics in Oxford (page 14):
The mathematician was the Christ Church don, Charles Lutwidge Dodgson, better known as Lewis Carroll.

Hunting Trees (page 16):
The answers are on the cover.

Equations (page 17):
The key observation is that $487 + 513 = 1000$. So if you add the two equations you get $x + y = 17$ and if you subtract them you get $x - y = 1$. So $x = 9$ and $y = 8$.

Mixing Drinks (page 19):
Mixing 1 litre of alcohol and 1 litre of water to give 1.92 litres of mixture means than the mixture is now displacing 80 millilitres of air less than the two separate litres, and 80 millitres of air weighs about a tenth of a gram. So the denser mixture experiences less buoyancy from the air in which it is weighted, and will weigh 0.1g more. (Of course, if the mixture was weighed in a vacuum, there would be no increase of weight.)

Varsity Test (page 24):
BAABE BDDAE ABABA EABAB DEECD

Magic Rectangles (page 27):
Making a 2×12 Magic rectangle is not difficult. Since the sum of the numbers from 1 to 24 is $\frac{1}{2}(24)(24+1) = 12 \times 25 = 300$, the column sum would have to be 25. One possibility is shown below.

1	23	3	21	5	19	7	17	9	15	11	13
24	2	22	4	20	6	18	8	16	10	14	12

In fact, all the other solutions can be obtained by permuting the columns of this rectangle. There are therefore 12! solutions, or more than 4.79 million.

Cutting the rectangle above into two 2×6 pieces and putting one on top of the other quickly gives a 4×6 Magic Rectangle.

1	23	3	21	5	19
24	2	22	4	20	6
7	17	9	15	11	13
18	8	16	10	14	12

Given that there are millions of solutions in the 2×12 and 4×6 cases, it may come as a surprise that it is not possible to make a 3×8 Magic Rectangle. Why not? Just think of the column sums. Since the sum of the numbers from 1 to 24 is 300 (as noted above), each of the 8 column sums would have to be $300 \div 8$, which is not an integer.

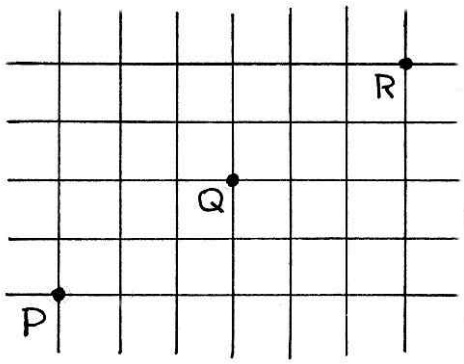
Pythagorean Dates (page 27):
There are no Pythagorean dates in 2014. The next Pythagorean date will be 17 August 2015, or 17-8-15 (since $8^2 + 15^2 = 17^2$). Noting that $9^2 + 12^2 = 15^2$, we can find two other Pythagorean dates next year: 12-9-15 and 9-12-15.

Guilty Party (page 28):
Beane stole the money and lied, Gorne told the truth and Dunnit lied.

Frog Jumps (page 28):
The task is impossible, no matter how many jumps are made.

Suppose that the grid points are labelled in the normal way, and that the three frogs are initially at $(0,0)$, $(1,0)$ and $(0,1)$. The object of the game is to land a frog at $(1,1)$.

Suppose that in general a frog at $P(a,b)$ jumps over a frog at $Q(c,d)$. It will then land at the grid point $R(a+2(c-a), b+2(d-b))$.



The important point to note is that the coordinates of the landing point have the same parity as the coordinates of the initial point, since the horizontal and vertical components of the jump are both even.

The three frogs start of at points $(0,0), (0,1)$ and $(1,0)$, with parity patterns (even, even), (even, odd) and (odd, even), respectively, and however they jump they will always land at points with the same parity pattern. However, the target point has parity pattern (odd, odd). So none of the three frogs can ever land there. □

THE TWIN PRIME CONJECTURE

Do there exist infinitely many pairs of prime numbers differing by two? Examples are easy to find: (3, 5), (5, 7), (11, 13), (17, 19), etc. There are infinitely many primes, but are there infinitely many prime pairs? This is the famous *twin prime conjecture*, which has been around for many years.

In April 2013 Yitang Zhang announced that he had a proof that there is an integer N, such that there are infinitely many pairs of primes that differ by N, and that N is at most 70 million. Terence Tao subsequently proposed a Polymath project collaborative effort to reduce Zhang's bound of 70 million, and in October the figure was reduced to 4680. □

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Lancelot Hogben

mathematical digest

Number 175 **April 2014**

If God exists and if He really did create the world, then, as we all know, He created it according to the geometry of Euclid and the human mind with the conception of only three dimensions in space. Yet there have been and still are geometricians and philosophers, and even some of the most distinguished, who doubt whether the whole universe, or to speak more widely, the whole of being, was only created in Euclid's geometry; they even dare to dream that two parallel lines, which according to Euclid can never meet on earth, may meet somewhere in infinity. I have come to the conclusion that, since I can't understand even that, I can't expect to understand about God.

Fyodor Dostoyevsky, in *The Brothers Karamazov*

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Cover: Lancelot Hogben's portrait, by Irma Stern, is in the Irma Stern Museum of the University of Cape Town.

LANCELOT HOGBEN

Lancelot Hogben (1895-1975) was the author of the hugely successful book *Mathematics for the Million*. He was born in Portsmouth and studied physiology at Trinity College, Cambridge, graduating in 1915. During World War I he was a conscientious objector, which landed him in Wormwood Scrubs prison. The harsh conditions caused his health to collapse, and he was released in 1917.

In 1918 he married the mathematician, statistician, and feminist Enid Charles, with whom he had two sons and two daughters.

In 1927 Hogben accepted the position of Professor of Zoology at the University of Cape Town. His speciality was endocrinology. While studying the African clawed frog *Xenopus laevis*, better known locally as the platanna, he discovered that female platannas, when injected with urine from a pregnant woman, ovulated within hours. This was the origin of the Hogben Pregnancy Test, which was for many decades the standard pregnancy test all over the world.

Hogben's research left a lasting impression on the history of biology. The platanna is now one of the most widely used model organisms in biological research.

While at UCT, Enid earned a Ph.D. in physiology. Lancelot found the job in South Africa attractive, but his dislike of the country's racial policies drove him to leave after only two years.

In 1937 he wrote *Mathematics for the Million*, which became an international best-seller, receiving plaudits from none other than Albert Einstein ("It makes alive the contents of the elements of mathematics") and H.G. Wells ("A great book, a book of first class importance"). It was followed by another best-seller, *Science for the Citizen*.

Hogben was fascinated with language, and edited Frederick Bodmer's book *The Loom of Language*. He proposed an international language which he called *Interglossa*, based on Greek and Latin roots together with a Chinese-like syntax.

Lancelot Hogben died in 1975. *Mathematics for the Million* lives on. □

TERENCE TAO: IMO PATRON

Professor Mike Clapper
Executive Director
International Mathematical Olympiad Foundation

We are proud to announce that Terence (Terry) Tao has agreed to accept an appointment as Patron of the IMO (International Mathematical Olympiad) Foundation. The foundation exists to promote the IMO and to provide support for host countries and participants.

There could be no more appropriate patron than Terence Tao. His position as one of the world's leading mathematicians is well-known and he has always been a strong supporter of the IMO and related activities. Terence competed in the IMO in 1986 and was the youngest ever gold medallist, at the age of 12 in 1988.

Born in Adelaide in 1975 of Chinese parents, Terence was educated at Blackwood High until entering Flinders University in 1989, completing his B.Sc. (Hons) in 1991 and his M.Sc. in 1992. He then began his doctorate at Princeton University under Elias Stein, completing this in 1996. He was made an Assistant Professor at UCLA in 1996 and a Full Professor in 2000 (the youngest ever appointed by UCLA). Terence has received numerous awards and prizes, including the Salem Prize in 2000, the Bochner Prize in 2002, the Fields Medal and SASTRA Ramanujan Prize in 2006, the MacArthur Fellowship and Ostrowski Prize in 2007, the Waterman Award in 2008, the Nemmers Prize in 2010, and the Crafoord prize in 2012.

Terence Tao currently holds the James and Carol Collins chair in mathematics at UCLA, and is a Fellow of the Royal Society, the Australian Academy of Sciences (Corresponding Member), the National Academy of Sciences (Foreign member), and the American Academy of Arts and Sciences. His areas of research include harmonic analysis, PDE, combinatorics, and number theory. Some of his best-known results include the Green-Tao theorem which states that there are arbitrarily long arithmetic progressions of prime numbers and his more recent result that every odd

number greater than 1 is the sum of at most five primes, a significant step towards proof of the Goldbach conjecture.

Professor Tao says, "I have very fond memories of taking part in IMO. Like any other school sporting event, there is a level of excitement in participating with peers with similar interests and talents in a competitive activity. The opportunity to travel nationally and internationally is an experience I strongly recommend for all high-school students. Taking part in IMO can be a life-changing event for young, gifted mathematicians. I therefore wholeheartedly support the IMO foundation." □

SONNET

Poets will recognize the following as a sonnet in iambic pentameter, though not following either the Shakespearean or Italian forms. What will mathematicians make of it?

Now I defy a tenet gallantly
Of circle canon law: these integers
Importing circles' quotients are, we see,
Unwieldy long series of cockle burs
Put all together, get no clarity;
Mnemonics shan't describeth so reformed
Creating, with a grammercy plainly,
A sonnet liberated yet conformed.
Strangely, the queer'st rules I manipulate
Being followed, do facilitate
Whimsical musings from geometric bard.
This poesy, unabashed as it's distressed,
Evolved coherent–a simple test,
Discov'ring poetry no numerals jarred.

See page 31 for a mathematical explanation. □

FINANCIAL MATHS

A credit card has three dimensions: length, width and debt. □

FROM PROSTHAPHAERESIS TO LOGARITHMS

Trigonometry was developed as a mathematical tool for astronomy. Astronomers have from the earliest times been faced with performing massive computations, with numbers of 15 digits or more not uncommon. Addition and subtraction of such numbers present no great problem, but multiplication is another story.

Mathematicians, particularly those who were also astronomers, were looking for an easier way than long multiplication. They had by this time developed extensive tables of the trig functions, and were familiar with trig identities such as

$$\cos A \cos B = \frac{1}{2}(\cos(A+B) + \cos(A-B))$$

which had been discovered in the 10th Century by the Arab astronomer Ibn Yunus. This identity is obtained by adding the two standard addition formulae

$$\cos(A+B) = \cos A \cos B + \sin A \sin B$$
$$\cos(A-B) = \cos A \cos B - \sin A \sin B.$$

and dividing by 2.

It is not clear who first hit upon the idea of *prosthaphaeresis*, an impressive word cobbled together from the Greek words for addition and subtraction. Copernicus refers to it in his work *De Revolutionibus Orbium Coelestium*, published in 1543, and the Danish astronomer Tycho Brahe (1546–1601) used prosthaphaeresis extensively.

How does it work? The key observation is that the left hand side of Ibn Yunus' identity is a product, and the right hand side is a sum.

Suppose we want to calculate 0.6756×0.8729.

From trig tables,

$$0.6756 = \cos 47°30' \text{ and } 0.8729 = \cos 29°12',$$

so we put $A = 47°30'$ and $B = 29°12'$.

From trig tables again,

$$\cos(A+B) = \cos 76°42' = 0.2300 \text{ and } \cos(A-B) = 18°18' = 0.9494.$$

Putting all this in Ibn Yunus' identity gives

$$0.6756 \times 0.8729 = \frac{1}{2}(0.2300 + 0.9494) = 0.5897.$$

Prosthaphaeresis converts a multiplication problem to one of addition and subtraction, and was the forerunner of the logarithmic method.

Logarithms were invented by the Scottish mathematician John Napier (1515-1617), Laird of Merchiston (now a suburb of Edinburgh).

Napier heard of prosthaphaeresis in a curious way. In 1590 James VI of Scotland, later to become James I of England as well, set out to Denmark to marry Princess Anne, the daughter of King Frederick of Denmark. A storm forced their ship to take refuge on Tycho Brahe's island of Hven. During the party's stay on the island, John Craig, court physician to James I, learned of the method of prosthaphaeresis, and communicated it to Napier on his return.

Napier had long been interested in computational techniques, and the news of Tycho Brahe's method spurred him on. In 1614 he published his pioneering work *Mirifici Logarithmorum Canonis Descriptio*. Napier's logarithms were not quite like those in use today, and Henry Briggs, an Oxford mathematics professor, later collaborated with Napier to produce tables of logarithms to base 10 which were indispensable for doing calculations for the next 300 years.

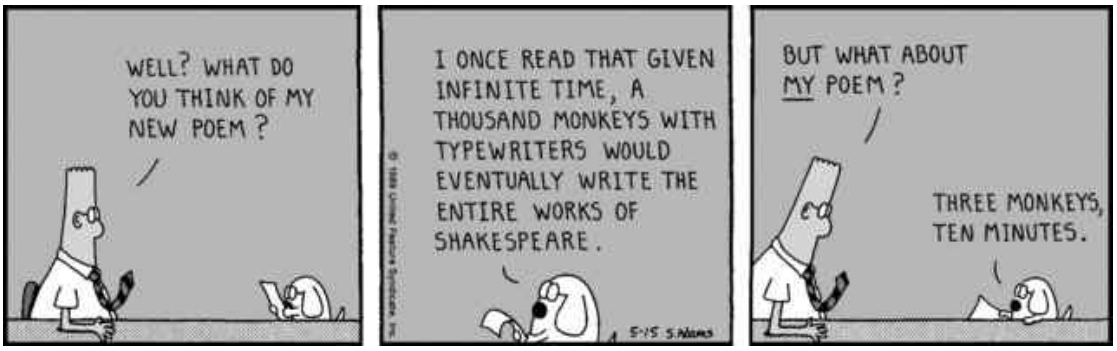
Today nobody would buy a book of log tables or tables of trig functions. They have been made obsolete by the calculator, just as the log tables of Napier and Briggs made prosthaphaeresis obsolete. But in the same way that Ibn Yunus' identity has never been forgotten, though its use for computational purposes is today irrelevant, so the logarithmic function will always be an indispensable tool in mathematics. The pH of your swimming pool, the decibel level of the motorbike of the teenager next door and the Richter scale for measuring earthquakes, are all logarithmic. □

THE STENOGRAPHIC MONKEYS

In 1913 the French mathematician Emile Borel conducted a thought experiment. Imagine, he said, a million monkeys in front a million typewrites, hitting the keys at random. How long would it be before they had typed out all the books in every library in the world?

The English astronomer Arthur Eddington suggested that the successful production by the monkeys of all the books in the British museum was more likely than all the molecules moving randomly in a container suddenly finding themselves all on one side of the container.

The idea has been seized upon enthusiastically by many humorists. It featured in the quirky Dilbert comic strip.



After a meeting of the British Association, at which the stenographic monkeys were mentioned, the verses below appeared in the magazine *Punch*.

Life is brief but art is longer
So the sages say in sooth
Nothing could be worse or wronger
Than to doubt this ancient truth.

Give me half a dozen monkeys –
Set them to the lettered keys.
And instruct these simian flunkeys
Just to hit them as they please.

Lo, the anthropoid plebeians
Toiling at their careless plan
Would in course of countless aeons
Duplicate the lore of man.

Thank you, thank you men of science,
Thank, you, thank you British Ass!
I for one have placed reliance
On the tidbits that you pass.

And this season's nicest chunk is
Just to sit and think of those
Six imperishable monkeys
Typing in eternal rows.

(Lucio, quoted by Warren Weaver, in *Lady Luck*) □

AARDVARKS AND ZEBRAS

Annie, Bobby, Connie, Donny and Ettie play a game in which each is either an aardvark or a zebra. Aardvarks always tell the truth, while zebras always lie.

Annie says that Bobby is an aardvark.
Connie says that Donny is a zebra.
Ettie says that Annie is not a zebra.
Bobby says that Annie is not a zebra.
Bobby says that Connie is not an aardvark.
Donny says that Ettie and Annie are different kinds of animals.

Who's who in the zoo? The answer is on page 31. □

MATRIC AND THE IMO

Matric results were in the limelight at the beginning of the year, and amid the euphoria of students scoring multiple A's it was forgotten that in real terms our school maths and science results lie at the bottom of world rankings. On average South African mathematics education is terrible. There are, of course, some very good schools, but not very many, and when the results from the international tests are analysed it turns out that even our top independent and former Model C schools, with all their wealth and resources, are not much better than average in international terms. Though thousands of students are now boasting A's for matric maths, many of them are really only average.

In July South Africa will be the venue for the International Mathematical Olympiad (IMO), the oldest and toughest of the world scientific Olympiads for high schools. Over a hundred countries will take part, with teams of up to six. Six hundred teenagers from a hundred countries, representing the next generation of the world's top mathematicians, will write two gruelling problem-solving papers over two days. How will South Africa's team fare against the best in the world?

How have they done in the past?

The IMO began in 1959 and during the long apartheid years South Africa was not able to take part. Then came 1990 and the release of Nelson Mandela. The 1991 IMO was held in Uppsala, and Sweden, a major supporter of the anti-apartheid struggle, was now prepared to invite an Observer from South Africa to attend.

The next year saw South Africa's full participation in the IMO in Moscow. We performed dismally and were ranked last. It was not the fault of the team: their subsequent successful careers in academia, engineering, finance and IT showed that they were top quality students. The problem was that they were just not ready for an IMO.

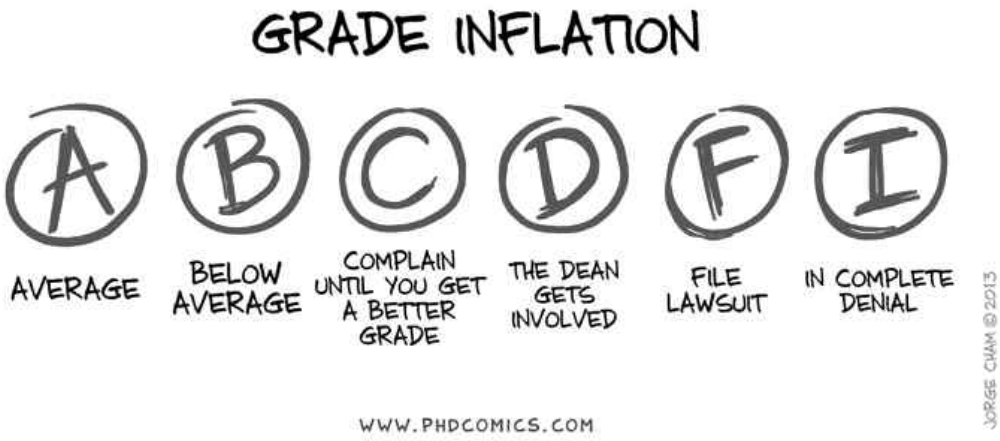
To prepare teams for future IMOs we had to set up a nationwide talent search and training programme, with monthly assignments, camps and IMO-level selection tests. The rigorous programme paid off. At the IMO in Hong Kong in 1994 we won three medals and shot up the rankings to 27th place, ahead of most Western European countries. Since then we have won a total of 46 IMO medals: 1 Gold, 9 Silver and 36 Bronze medals, with

43 Honourable Mentions (students who did not win a medal, but solved at least one problem perfectly).

Success in an IMO requires original thinking and a deep knowledge and understanding of school maths. Just knowing the basic stuff and how to apply it in simple predictable problems is not good enough. Senior certificate maths papers are shallow and superficial. Around the country, hundreds of boys and girls celebrated matric maths marks in the high nineties, and perfect scores of 100% were not uncommon. How would these students perform at an IMO?

The difficulty of IMO problems can be illustrated by a simple numerical comparison. An IMO paper of $4\frac{1}{2}$ hours consists of just three problems. The problems are original, having been composed by top mathematicians from all over the world especially for the event. Though phrased in the familiar language of school algebra and geometry, they are nevertheless fearsomely difficult. While a typical 5-mark problem in a 3-hour matric paper can be knocked off in 5 minutes by a routine application of a well-drilled technique, an IMO problem will take an average of 90 minutes for its solution. Five minutes for a matric problem, ninety for an IMO problem. That comes out at eighteen times more difficult. Without special training, our straight-A students would score straight zeros at an IMO.

In July this year the world's top teenage mathematicians will gather at UCT to take part in the 55th International Mathematical Olympiad. Six of them will be South African, and they will be selected in May after an eighteen-month training programme. Their success will rely on three factors: their natural talent for mathematics, their commitment to an intensive training programme, and provision of expert coaching. On average our maths education is terrible. The IMO will show how good our very best students can be. □



THE EARLIEST HEBREW GEOMETRY

The Israelites of biblical times developed no mathematics, and the earliest Hebrew treatise on geometry dates from about 150 A.D. Entitled *Mishnat ha-Middot* ("Treatise of Measures"), it is conjectured to have been written by Rabbi Nehemiah, an influential scholar and teacher who was intimately concerned with the rabbinical interpretation of biblical law. As a mathematician, Nehemiah was concerned with calendar calculations and with the measurements of the Ark of the Covenant and the Tabernacle.

The emphasis on *Mishnat-ha-Middot* is on the calculation of lengths, areas and volumes of various figures. Rules of calculation are given, illustrated by numerical examples, but no proofs of the formulas used are provided.

One important problem was the determination of the circumference and area of a circle, given its diameter. There is a well-known record in the Bible (I Kings, chapter 7, verse 23) which shows that it was known in the days of King Solomon that π was approximately equal to 3.

> And he made a molten sea, ten cubits from one brim to the other: it was round all about, and his height was five cubits: and a line of thirty cubits did compass it round about.
>
> (King James translation)

An almost identical description of the "molten sea" (or "brazen sea" – a large circular basin used for priestly ablution) is given in II Chronicles, chapter 4, verse 2.

Nehemiah was better informed on the measurement of a circle than the practical temple-builders of King Solomon's time. He describes a method of determining the area of a circle as follows:

> How is it with the circle? Let him multiply the thread into itself, and throw away one-seventh and half of one-seventh; the rest is the area.
>
> For instance, if the thread is 7, its multiplication into itself is 49; a seventh and a half a seventh of it is $10\frac{1}{2}$; the area is thus $38\frac{1}{2}$.

By "thread" Nehemiah clearly means "diameter". If we denote the diameter by d and the area by A, Nehemiah's method is given by the formula

$$A = d^2 - \frac{1}{7}d^2 - \frac{1}{14}d^2 = \frac{11}{14}d^2.$$

If instead of the diameter we use the radius r (so $d = 2r$), we have

$$A = \frac{11}{14}d^2 = \frac{11}{14}(4r^2) = \frac{22}{7}r^2,$$

which amounts to using $3\frac{1}{7}$ as the value of π.

That's a little too much. Over 300 years earlier Archimedes had determined that

$$3\frac{10}{71} < \pi < 3\frac{1}{7}.$$

So Nehemiah's value of π was not original, nor are his other results. He was clearly well schooled in Greek Mathematics. *Mishnat ha-Middot* resembles the work of Hero of Alexandria (100 A.D.) and al-Khwarizmi (800 A.D.) and is seen as a link between Greek and Islamic mathematics. □

BELPHEGOR'S PRIME

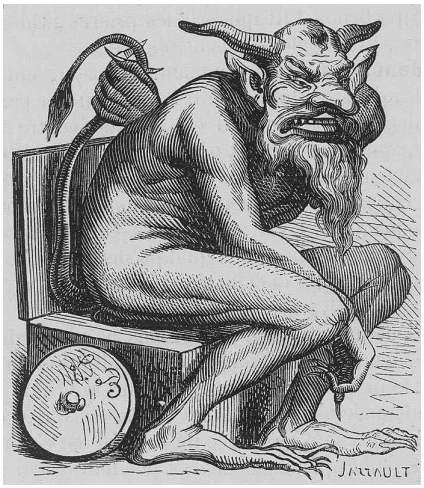
It is obvious that the number

1000000000000066600000000000001

is palindromic: it reads the same from either end. What is far less obvious, and a real surprise, is that it is a prime number.

Look more closely, for this prime number has particular numerological significance. In the middle is the Number of the Beast, 666, flanked by two strings of 13 unlucky zeros. For those reasons, the number is named after one of the seven Princes of Hell, Belphegor.

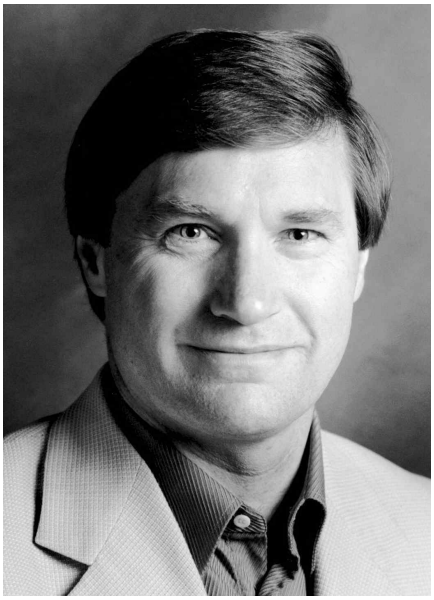
Demonologists say that Belphegor is responsible for encouraging one of the Seven Deadly Sins, sloth. He seduces the unwary by suggesting to them ingenious inventions to make them rich without any effort. In mathematics, he tempts lazy students to focus only on old exam papers without understanding the work properly. His power is particularly strong in the month of April. □



IMO CELEBRITY LECTURES

A new feature of this year's International Mathematical Olympiad will be a programme of "Celebrity Lectures" sponsored by the IMO Foundation. Two internationally famous mathematicians will take part in the programme.

John Barrow is a renowned cosmologist, theoretical physicist, and mathematician. He is currently Research Professor of Mathematical Sciences at the University of Cambridge, and has over 480 research publications to his credit. Barrow's public lectures have included presentations at 10 Downing Street, Windsor Castle and the Vatican. He has written a number of books on popular science, of which the latest is *Mathletics: 100 Amazing Things You Didn't Know about the World of Sports*.



Günter Ziegler (Free University of Berlin) was a Gold Medallist in the IMO in 1981 and is known for his research in discrete mathematics and geometry. He has won numerous awards for his work, including the Gottfried Wilhelm Leibniz Prize, Germany's highest research honour, and the Chauvenet Prize for mathematical exposition. Of special interest to Olympiad contestants is his 2003 publication (with Martin Aigner), *Proofs from THE BOOK*. □

GEOMETRY AND REASONING

Geometry represents a high school student's first formal introduction to abstract mathematical thinking. That is, the student is asked to

1. reason about such abstract concepts as points, lines and triangles;
2. understand that certain geometric objects can be defined only in terms of their relations with each other; and
3. prove theorems about the Euclidean structure based on a small set of basic concepts and axioms.

Not only is this type of reasoning fundamental for more advanced courses in mathematics, it is also prevalent in many areas of science and engineering where one reasons about simplified or ideal models.

For example, in the applications of the theory of gravity, one thinks of mass being concentrated in a single point. Similarly, in the kinetics of motion, solid objects are almost always considered as points at their "centre of (inertial) mass", and the most fundamental insight in all of kinetics is that objects proceed in straight line motion unless acted upon by a force. The idea that *point*, *line*, *plane* and space cannot be characterized independently of each other is analogous to situations where a physical quantity can be measured only by observing its interaction with other quantities (e.g., spin can be measured only by observing its interaction with other magnetic fields). Finally, the hypothetico-deductive method is undoubtedly the basis of all science. The practice of proving theorems on the basis of a small set of axioms serves as a cognitive precursor to the vital scientific practice of deriving one law from more fundamental laws; the derivation of Kepler's laws from Newton's laws—one of the greatest triumphs of calculus—is an example. Because high school geometry plays a key role in the development of our students' ability to reason about abstract models in mathematics, science and engineering. we believe that it is absolutely essential that teachers and curriculum developers attend to the question of how to motivate abstract geometric concepts and reasoning to their students.

(Extract of an article by Guershon Harel, Professof of Mathematics at the University of California, San Diego, published in the *Notices of the American Mathematical Society*, January 2014.) □

WAS FERMAT WRONG?

Everybody should know the story of Fermat's Last Theorem (known in the trade as FLT), which was stated by the French mathematician Pierre de Fermat in the 17th Century:

> It is not possible to find positive integers a, b, c and n, with $n > 2$, such that
> $$a^n + b^n = c^n.$$

Fermat's Last Theorem was eventually proved true by Andrew Wiles in 1995, but not without drama. Wiles' first proof, announced with much fanfare in 1993, was found to have a fatal flaw. Back went Andrew Wiles to his drawing board, and after much more work, and a little help from his friends, he produced a flawless proof.

Enter David Cohen, a computer scientist. He decided to prove that Fermat and Wiles were both wrong, by setting his computer to work to find values of a, b, c and n, with $n > 2$, such that $a^n + b^n = c^n$.

To his great surprise, his computer quickly came up with an answer:

$$3987^{12} + 4365^{12} = 4472^{12}.$$

You can check out the numbers with a calculator:

$3987^{12} = 1.613447461 \times 10^{43}$

$4365^{12} = 4.784218173 \times 10^{43}$

$4472^{12} = 6.397665634 \times 10^{43}$

It all adds up. But the above calculations are rounded off to ten significant figures. Perhaps they should be checked with a more powerful calculator. Try it. You will find that the numbers still add up.

In fact, they don't, of course, but ordinary calculators can't tell the difference. But a little bit of numerical thinking can show that

$$3987^{12} + 4365^{12} \neq 4472^{12}.$$

How?

The first thing to check is parity. The two powers on the left hand side are odd, and the power on the right hand side is even. So the numbers pass a simple parity check.

Now let's check last digits.

3587^{12}: Powers of 7 end in 7, 9, 3, 1, 7, 9, 3, 1, ... : cycles of 4. So 3987^{12} ends in 1.

4365^{12}: Powers of 5 obviously end in 5.

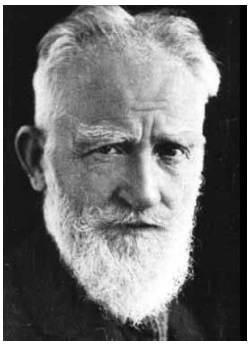
4472^{12}: Powers of 2 end in 2, 4, 8, 6, 2, 4, 8, 6, ... : cycles of 4. So 4472^{12} ends in 6.

Since 1+5 = 6, the last digit check works.

So how can it be proved that $3987^{12} + 4365^{12}$ and 4472^{12} are not equal?

A quick answer can be found on page 31. □

GBS AND MATHEMATICS



George Bernard Shaw was never slow to criticise or attempt to improve any branch of science that took his fancy, with one exception: mathematics. He admitted that he knew very little about maths, blaming his ignorance on the poor teaching he received at school.

> Not a word was said to us about the meaning or futility of mathematics: we were simply asked to explain how an equilateral triangle could be constructed by the intersection of two circles, and to do sums in a, b and x instead of in pence and shillings, leaving me so ignorant that I concluded that a and b must mean eggs and cheese and x nothing, with the result that I rejected algebra as nonsense, and never changed my opinion until in my advanced twenties Graham Wallace and Karl Pearson convinced me that instead of being taught mathematics I had been made a fool of.

The influence of these men (Karl Pearson was an prominent statistician) helped Shaw to appreciate the importance of at least one branch of mathematics: the theory of probability and statistics. That understanding gave him the background for his essay *The Vice of Gambling and the Virtue of Insurance*, which is a chapter in his book *Everybody's Political What's What.* □

AGE AND YOUR SHOE SIZE

Here's a little puzzle going around on the internet. It is apparently a calculation that deduces your age from your shoe size.

Get out your calculator!!
Take your shoe size. (Only whole sizes. If 4 ½ make it 5.)
Multiply it by 5.
Add 50.
Multiply by 20.
Add 1014.
Subtract the year you were born.

The first digit is your shoe size while the last 2 digits are your age.

TRY IT. YOU WILL BE SHOCKED!!

How does the trick work?

Basically, some fancy arithmetic is obscuring what is happening. The first step is to take your shoe size and multiply it by 100. That puts your shoe size at the beginning of the eventual answer.

Then your age is added, disguised as 2014 minus your year of birth, with the figure of 2014 further disguised as 1014 plus 50 times 20. That gives your age as the last two digits of the number.

The reason the trick works is more easily seen with the aid of a little (very little) algebra.

Suppose S is your shoe size and D is your date of birth.

Shoe size: S

Multiply by 5: 5S

Add 50: 5S + 50

Multiply by 20: 100S + 1000

Add 1014: 100S + 2014

Subtract date of birth: 100S + (2014 - D) = 100S + your age.

So there's the answer: the first two digits are your shoe size, and the next two your age. Not really a shock.

Unfortunately the trick sometimes fails. If you are asked to do the calculation before your birthday in 2014, it will give your age plus one.

The algebra also shows that the trick can break down for anybody over 100. Suppose Great Aunt Jemima has a shoe size of 4, and was born in 1912.

Working through the calculation gives

Shoe size: 4

Multiply by 5: 20

Add 50: 70

Multiply by 20: 1400

Add 1014: 2414

Subtract date of birth: 502

Great Aunt Jemima would certainly be shocked to learn that she was 2 years old. And her shoe size is wrong. □

WHO'S SITTING WHERE?

Mr and Mrs Davids, Mr and Mrs Evans and Mr and Mrs Fredericks go to the cinema together. They sit side by side in a row with men and women alternating. No man sits next to his wife.

The men's names are Alan, Barry and Charles, and the women's names are Mary, Nancy and Olive. The men's occupations are shopkeeper, teacher and undertaker, and the women have the same three occupations, but no man has the same occupation as his wife.

The shopkeeper is in one of the two middle seats, next to Charles, who is a teacher. Mrs Davids is at one end of the row, with the undertaker's husband on her right. Nancy is sitting between Alan, on her left, and Barry.

Can you work out everybody's full names and occupations, and who is sitting where?

The answer is on page 31. □

FERMAT'S LITTLE THEOREM

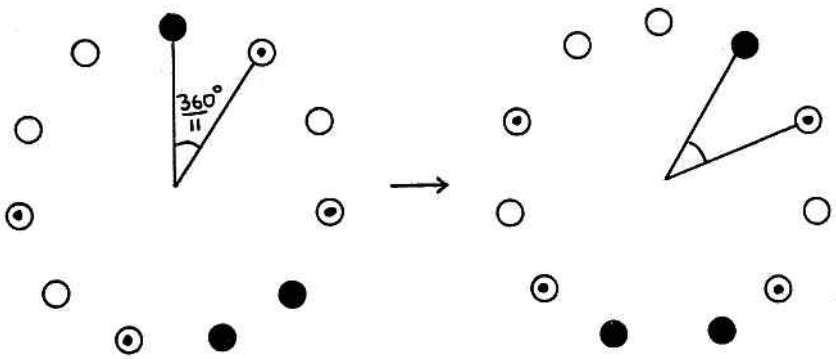
The 17th Century French lawyer Pierre de Fermat is famous for his Last Theorem, which is easy to state and very difficult to prove. Fermat was also responsible for another theorem that today bears his name: Fermat's Little Theorem. It is also easy to state, and fortunately also easy to prove.

> For all natural numbers n and prime numbers p, $n^p - n$ is divisible by p.

We shall prove this result in a special case: $3^{11} - 3$ is divisible by 11.

Suppose you have to arrange coloured beads in a circle, using beads of three different colours. Since in each position you have 3 choices, there are 3^{11} different patterns. Since 3 of these patterns use just one colour, there are $3^{11} - 3$ patterns using more than one colour.

Given one of these multi-coloured patterns, we can rotate it ten times through an angle of $360°/11$, creating ten patterns. Since 11 is a prime number, these patterns are all different. So the patterns can be classified into groups, so that within each group any two patterns are just rotations of each other. No two of these groups contain an element in common, and each group contains 11 patterns. So $3^{11} - 3$ is divisible by 11.



Although this argument is given for the particular numbers 3 and 11, there is nothing special about these numbers except that 11 is prime. It is not difficult to see that the general argument will work: $n^p - n$ is divisible by p, for all n and p, with p prime. $\square$

ONE EQUALS TWO?

Consider the series

$$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\frac{1}{4\times 5}+\cdots$$

Using the identity

$$\frac{1}{k(k+1)}=\frac{1}{k}-\frac{1}{k+1}$$

we split each term into two parts, so that the series becomes

$$\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{4}-\frac{1}{5}\right)+\cdots$$

and now all the terms after the first cancel each other out and we are left with the first term, which is equal to 1, so the sum of the series is 2.

But suppose we split the terms of the series in a different way, using the identity

$$\frac{1}{k(k+1)}=\frac{k+1}{k}-\frac{k+2}{k+1}.$$

The series now becomes

$$\left(\frac{2}{1}-\frac{3}{2}\right)+\left(\frac{3}{2}-\frac{4}{3}\right)+\left(\frac{4}{3}-\frac{5}{4}\right)+\left(\frac{5}{4}-\frac{6}{5}\right)+\cdots$$

and once again all the terms after the first cancel each other out and we are left with the first term, which is equal to 2, and therefore the sum of the series is 2.

So the same series sums to 1 and to 2. So $1 = 2$.

Worse is to follow. Start with yet another simple identity:

$$\frac{1}{k(k+1)}=\frac{2k+1}{k}-\frac{2k+3}{k+1}$$

and use it to split each term of the series in two. The series becomes

$$\left(\frac{3}{1}-\frac{5}{2}\right)+\left(\frac{5}{2}-\frac{7}{3}\right)+\left(\frac{7}{3}-\frac{9}{4}\right)+\left(\frac{9}{4}-\frac{11}{5}\right)+\cdots$$

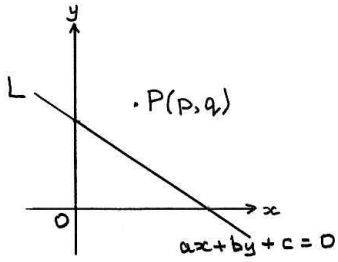
and as before all the terms after the first disappear, and we are left with the first term, which is equal to 3 and is therefore the sum of the series.

There are other ways of splitting the fraction $\dfrac{1}{k(k+1)}$ into two fractions that make the series add up to something else. Can you find them? And what is going wrong?

See page 28 for an explanation. □

A DISTANCE FORMULA

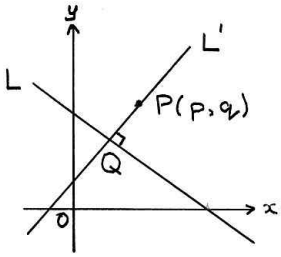
Given a point $P(p, q)$ and a line L with equation $ax + by + c = 0$, how would you find the distance from P to L?



This problem has several quite different solution strategies.

Strategy 1: Solving two equations

The line of shortest distance from P to L (call it L') is perpendicular to L. Since the slope of L is $-b/a$, the slope of the line of shortest distance from P to L is a/b.



So we can set up the equation of L'. By solving the two equations we can find the point of intersection Q of the two lines, and then use the standard distance formula to find the distance PQ.

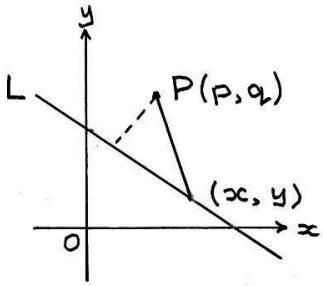
That seems like quite a lot of work.

Since we have to find a *shortest* distance, we could try

Strategy 2: Calculus

Take a general point (x, y) on the line and write down the formula for the distance from (x, y) to (p, q), a function of two variables x and y:

$$\sqrt{(x-p)^2 + (y-q)^2}.$$



Since y depends on x (the point (x, y) lies on the line L), we substitute $y = -(1/b)(ax + c)$ in the distance function, to obtain a function of just one variable, x:

$$\sqrt{(x-p)^2 + (-(1/b)(ax+c) - q)^2}.$$

We want the minimum value of this function, so we differentiate with respect to x, set the derivative equal to zero and solve for x. That gives the x-coordinate of the closest point Q on the line. Using the equation of the line again, we find the y-coordinate of Q, and the distance formula finally gives the required distance PQ.

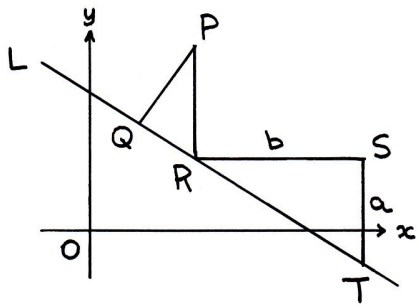
That seems like even more hard work than Strategy 1.

Without carrying out either of these strategies, it is easy to see that both of them are reasonably lengthy, and require some heavy algebra and differentiation. And both of them proceed via finding the closest point Q on the line.

Our third strategy does not require finding Q.

Strategy 3: Geometry

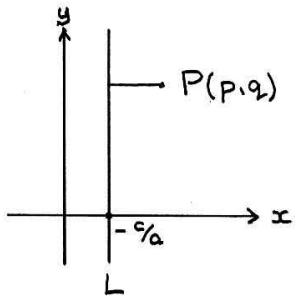
In the figure, the point $R(p, -\frac{1}{b}(ap+c))$ is immediately below P, and the triangle RST has ST parallel to PR, and T on the line L, with $RS = b$, $ST = a$ and $RT = \sqrt{a^2+b^2}$.



Since triangles PQR and RST are similar, $\dfrac{PQ}{PR} = \dfrac{RS}{RT}$, giving

$$PQ = \frac{PR.RS}{RT} = \frac{(q + (1/b)(ap+c))b}{\sqrt{a^2+b^2}} = \frac{ap+bq+c}{\sqrt{a^2+b^2}}.$$

We should finally note that we divided by b in the calculation, which means we were assuming that $b \neq 0$. However, when $b = 0$ the line is vertical (the equation is $ax + c = 0$). In this special case the distance from P to L is $p + c/a$, which is exactly what the formula gives when $b = 0$.



In all of the above figures everything turns out positive. Since a distance is never negative, the general formula for the distance from a point (p, q) to a line with equation $ax + by + c = 0$ is $\dfrac{|ap+bq+c|}{\sqrt{a^2+b^2}}$. □

ERRATUM

In the article "A Third Route to Root Two" (*Mathematical Digest* **174**, January 2014 pp. 2-3) the number 1352 appears twice on page 3. In each case it should be 1351. □

SOLVING A QUADRATIC GRAPHICALLY

How would you solve a quadratic equation $ax^2+bx+c=0$ graphically? The obvious first idea is to sketch the graph of the parabola $y=ax^2+bx+c$ and note where it cuts the x-axis. Since you would have to draw the parabola by hand, that would not be very accurate.

Here's a method that uses only the traditional tools of ruler and compasses, and gives the solutions as accurately as those tools allow.

We may suppose that the quadratic to be solved is $x^2+bx+c=0$ (dividing throughout by a).

Step 1:

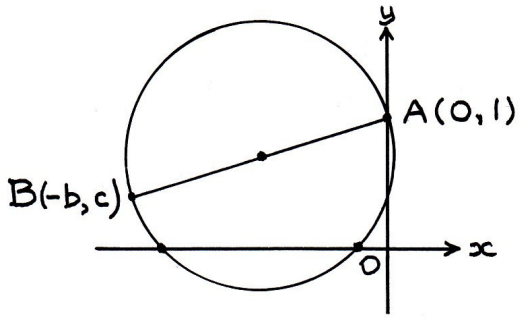
Plot the points $A(0,1)$ and $B(-b,c)$ on a cartesian axis system.

Step 2:

Draw a circle with diameter AB.

Step 3:

The solutions of the quadratic equation are given by the points of intersection of the circle with the x-axis.



Why does the method work?
The centre of the circle is $(-\frac{1}{2}b, \frac{1}{2}(c+1))$, its diameter is $\sqrt{(-b)^2+(c-1)^2}$ and its radius is $\frac{1}{2}\sqrt{(-b)^2+(c-1)^2}$.

So the equation of the circle is

$$\left(x+\frac{1}{2}b\right)^2+\left(y-\frac{1}{2}(c+1)\right)^2=\frac{1}{4}\Big((-b)^2+(c-1)^2\Big).$$

Putting $y=0$ to find where the circle cuts the x-axis gives the equation

$$\left(x+\frac{1}{2}b\right)^2+\left(-\frac{1}{2}(c+1)\right)^2=\frac{1}{4}(-b)^2+\frac{1}{4}(c-1)^2,$$

which simplifies to

$$x^2+bx+c=0,$$

showing that the intercepts on the x-axis are the solutions of the original quadratic equation. $\square$

X+Y = IMO

A movie to be released later this year will be centered around a mythical International Mathematical Olympiad, held in Taiwan. (There was a genuine IMO in Taiwan back in 1998, but that's history.) Here's how the online gossip magazine *Female First* tells the story.

> "Teenage maths prodigy Nathan (Asa Butterfield) struggles with people, not least his mother, Julie (Sally Hawkins), but finds comfort in numbers. Mentored by unconventional and anarchic teacher Mr Humphreys (Rafe Spall), Nathan's talents win him a place representing GB at the International Mathematics Olympiad. When the team go to train in Taiwan, headed up by squad leader Richard (Eddie Marsan), Nathan is faced with unexpected challenges – not least his new and unfamiliar feelings for his Chinese counterpart, the beautiful Zhang Mei (Jo Yang). From suburban England to bustling Taipei and back again, this original and heartwarming film tracks the funny, complex relationship Nathan has with his mentor who ends up learning more from Nathan than he could ever teach him. Ultimately this is Nathan's journey to embrace the irrational nature of love."

To ensure authenticity, a number of real-life IMO personalities were consulted by the producers, and one of them actually appears in the movie — as himself! As we go to press, the movie is nearing completion, but a launch date has not been announced. That allows speculation that it will be premiered in Cape Town during this year's IMO. □

SIMPLE HARMONICS

The wonders of nature, quoth he
Are always a marvel to me:
    That each tick and a tock
    Of a grandfather clock
Is twice pi root l over g.

(From *A Dictionary of Mnemonics*, published by Eyre Methuen) □

VARSITY READINESS TEST

Varsity Readiness Tests are regular features of *MATHEMATICAL DIGEST*. They are designed to highlight parts of school mathematics that are of particular relevance for students planning to study maths at university level.

Try the problems below, without using a calculator or formula sheet. There is no recommended time, but give yourself a couple of hours of concentrated work. Don't give up on a problem just because it looks unfamiliar.

There is no pass mark, but every unsolved problem will highlight a weakness in your mathematical background that needs to be fixed before you land up in Maths 101 at your chosen university.

1. If $10^{2014} - 2014$ is written as an integer, what is the sum of its digits?
   (A) 18138 (B) 18120 (C) 18440 (D) 18669 (E) 18923

2. Two sides of a parallelogram have length 5 and the other two sides have length 7. The length of one of the diagonals is 11. What is the length of the other diagonal?
   (A) 4 (B) $3\sqrt{3}$ (C) 5 (D) $\sqrt{6}$ (E) 6

3. If x, y and z are consecutive positive integers, then
$$\left(1+\frac{1}{x}\right)\left(1+\frac{1}{y}\right)\left(1+\frac{1}{z}\right)$$
   is equal to
   (A) $\dfrac{x^2+y^2+z^2}{xyz}$ (B) $\dfrac{xyz}{x+y+z}$ (C) $\dfrac{(x+y+z)^2}{xyz}$
   (D) $\dfrac{x+3}{x}$ (E) $\dfrac{(x+y+z)^2}{x^2+y^2+z^2}$

4. The fourth power of $\sqrt{1+\sqrt{1+\sqrt{1}}}$ is
   (A) $2+\sqrt{3}$ (B) $\frac{1}{2}(7+3\sqrt{5})$ (C) $1+2\sqrt{3}$ (D) 3 (E) $3+2\sqrt{2}$

5. In triangle ABC, $AB = 11$, $AC = 9$ and the length of the altitude from A to BC is 7. Determine BC.
   (A) $10\sqrt{2}$ (B) $9\sqrt{3}$ (C) $8\sqrt{5}$ (D) $7\sqrt{11}$ (E) $6\sqrt{13}$

6. If $a = \log_8 225$ and $b = \log_2 15$, then
   (A) $a = b/2$ (B) $a = 2b/3$ (C) $a = b$ (D) $b = a/2$ (E) $a = 3b/2$

7. If r is a root of the equation $r^2 + 6r + 3 = 0$, what is the value of $(r+2)(r+4)$?

   (A) 2 (B) 3 (C) 4 (D) 5 (E) 6

8. If $f(x) = \dfrac{2x+1}{3x+2}$ and $f(g(x)) = \dfrac{4x+4}{7x+9}$, then $g(x)$ is equal to

   (A) $\dfrac{x-1}{2x+6}$ (B) $\dfrac{2x+3}{x+2}$ (C) $\dfrac{x+5}{3x-2}$ (D) $\dfrac{3x-2}{x+4}$ (E) $\dfrac{4x-7}{2x+5}$

9. For which values of k does the equation $\dfrac{x^2+x}{x^2+x-2} = k$ have no solution?

   (A) $\frac{1}{9} < k \leq 1$ (B) $1 \leq k < 3$ (C) $0 < k \leq 3$ (D) $-1 \leq k \leq 2$
   (E) $\frac{1}{3} < k \leq 2$

10. The area of a right-angled triangle is 10 cm^2 and its perimeter is 20 cm. What is the length of the hypotenuse of the triangle?

    (A) $7\frac{1}{2}$ (B) 8 (C) $8\frac{1}{2}$ (D) 9 (E) $9\frac{1}{2}$

11. What is the remainder when $2x^3 - x^2 - 6x + 2$ is divided by $2x^2 + 3x + 1$?

    (A) $2x - 1$ (B) $x + 3$ (C) $3x + 5$ (D) $4x - 2$ (E) $-x + 4$

12. The graph with equation $2x + 3y = 8$ is translated and the equation of the new graph is $2x + 3y = 12$. Which of the following could have been the translation used?

    (A) 2 units up and 1 unit to the left
    (B) 2 units down and 3 units to the right
    (C) 3 units up and 2 units to the left
    (D) 3 units down and 2 units to the right
    (E) 2 units down and 3 units to the left

13. The five-digit number 2014A is divisible by 9. What is the remainder when it is divided by 7?

    (A) 3 (B) 4 (C) 5 (D) 6 (E) 0

14. If the quadratic expression $4x^2 + 11x + 3$ is written in the form $a(x+b)^2 + c$, the value of $ab^2 + c$ is

    (A) 3 (B) 4 (C) 5 (D) 6 (E) 7

15. Simplify: $\dfrac{9^n + 8 \cdot 3^n + 12}{3^{n+1} + 6}$.

    (A) $3^n + 10$ (B) $3^{n-1} + 2$ (C) $3^{n-2} + 1$ (D) $3^n + 1$ (E) $3^{n+1} + 6$

16. The inverse of the function $f(x) = 10^{3x-5}$ is the function

(A) $f^{-1}(x) = \log(3x-5)$ (B) $f^{-1}(x) = 5\log x + 3$

(C) $f^{-1}(x) = \dfrac{1}{10^{3x-5}}$ (D) $f^{-1}(x) = \dfrac{1}{3}\log x - \dfrac{5}{3}$

(E) $f^{-1}(x) = \dfrac{1}{3}(\log x + 5)$

17. The roots of the equation $x^2 - 2x - 7 = 0$ are a and b. Which of the following equations has roots $a+1$ and $b+1$?

(A) $x^2 - 3x - 8 = 0$ (B) $x^2 - x - 6 = 0$ (C) $x^2 - 4x - 4 = 0$
(D) $x^2 + 2x - 8 = 0$ (E) $x^2 + 2x - 5 = 0$.

18. What is the last digit of $1 + 3 + 9 + 27 + 81 + \cdots + 3^{100}$?

(A) 2 (B) 4 5 (C) 6 (D) 8 (E) 0

19. The diagonals of a parallelogram have lengths 8 and 12. The two shorter sides have length 6. What is the length of the two longer sides?

(A) $4\sqrt{5}$ (B) 8 (C) $5\sqrt{2}$ (D) 9 (E) $2\sqrt{17}$

20. The graph of $y = (x-3)(1-x)$ is tangent to the graph of $y = kx^2$. Determine k.

(A) $\frac{2}{5}$ (B) $\frac{1}{3}$ (C) -2 (D) $\sqrt{2}$ (E) $-\frac{1}{2}$

21. In triangle ABC, D, E and F lie on sides AB, BC and CA, respectively, so that $BE = DE$ and $CE = CF$. If $\angle A = 40°$, what is the size of $\angle DEF$?

(A) 70° (B) 50° (C) 40° (D) 100° (E) 80°

22. If $\cos 2x = \frac{1}{8}$ and $0° < x < 90°$, then $\cos x$ is equal to

(A) $\frac{1}{4}$ (B) $\frac{1}{2}$ (C) $\frac{1}{16}$ (D) $\frac{2}{3}$ (E) $\frac{3}{4}$

23. What is the remainder when $3x^2 + 2x + 4$ is divided by $x^2 - 2x + 3$?

(A) $8x - 5$ (B) $7x + 3$ (C) $6x - 2$ (D) $5x + 11$ (E) $4x + 9$

24. What is the 2014th digit in the decimal expansion of $\frac{7}{13}$?

(A) 5 (B) 8 (C) 3 (D) 4 (E) 6

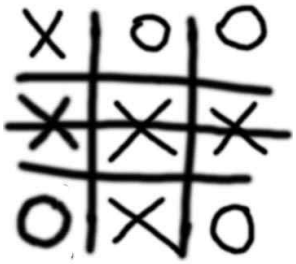
25. If $\dfrac{\sin 3x}{\sin x} = \dfrac{9}{4}$ and $0° < x < 90°$, determine $\dfrac{\cos 3x}{\cos x}$.

(A) $\dfrac{1}{4}$ (B) $\dfrac{3}{4}$ (C) $\dfrac{1}{3}$ (D) $\dfrac{2}{3}$ (E) $\dfrac{1}{2}$

Answers are on page 31.

RECTANGULAR TIC-TAC-TOE

The usual 3×3 game of Tic-Tac-Toe, also known as Noughts and Crosses, is not very interesting. When played between two players with any sort of experience, it will always end in a draw.



Here is an extension of the game that turns out to be much more subtle, and cannot end in a draw.

It is played by two players in just the same way, placing alternate noughts and crosses on a board — but it's a 3×7 board. The aim is to place your symbols so that four of them are at the corners of a rectangle.

Why does this game never end in a draw?

Suppose that the game ends with every one of the 21 squares containing either a nought or a cross.

We shall call two squares that lie in the same column and contain the same symbol a *doublet*. Then each column will contain a doublet. Since there are seven columns, at least four columns will contain doublets of the same symbol. Suppose that the symbol is a cross.

Now look at the rows. Each doublet determines a pair of rows, Since there are four doublets containing crosses, and only three pairs of rows, two doublets of crosses come from the same pair of rows. The crosses in these two doublets form the corners of a rectangle.

Suppose now that you play Tic-Tac-Tow on a 4×6 board. This time a draw is possible. Can you show this? You must find an arrangement of 12 noughts and 12 crosses on a 4×6 board, so that there is no rectangle formed by four of the same symbols.

See page 31 for an answer. □

WHY ONE DOES NOT EQUAL TWO

In the "proof" on page 19 that $1 = 2$, an infinite series was manipulated in what seems like a perfectly reasonable way, using not much more than basic rules of algebra.

Unfortunately, those rules apply to expressions with just a finite number of terms, and things can go wrong with infinite series. Those innocent little triplets of dots ($\cdots$) can be deceptive.

Let's start with the first series, but take it up to the nth term only:

$$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\frac{1}{4\times 5}+\cdots+\frac{1}{n(n+1)}.$$

Using the identity

$$\frac{1}{k(k+1)}=\frac{1}{k}-\frac{1}{k+1}$$

we split each term in two:

$$\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{4}-\frac{1}{5}\right)+\cdots+\left(\frac{1}{n}-\frac{1}{n+1}\right).$$

When the brackets are removed, all but two of the terms disappear, and we are left with $1-\dfrac{1}{n+1}$.

The term "telescoping series" is commonly used to describe what has happened.

Now look at the sum $1-\dfrac{1}{n+1}$, and ask what happens as you take more and more terms (i.e. as $n\to\infty$.) Clearly the second term tends to zero, and the sum of the infinite series can now be seen to be 1.

What happens in the second case? Starting with the same finite sum

$$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\frac{1}{4\times 5}+\cdots+\frac{1}{n(n+1)}$$

we now use the different split:

$$\frac{1}{k(k+1)}=\frac{k+1}{k}-\frac{k+2}{k+1},$$

obtaining the series

$$\left(\frac{2}{1}-\frac{3}{2}\right)+\left(\frac{3}{2}-\frac{4}{3}\right)+\left(\frac{4}{3}-\frac{5}{4}\right)+\left(\frac{5}{4}-\frac{6}{5}\right)+\cdots+\left(\frac{n+1}{n}-\frac{n+2}{n+1}\right)$$

which "telescopes" to

$$2 - \frac{n+2}{n+1}.$$

We now let n tend to infinity. This time the second terms does not tend to zero, but to 1, since

$$\frac{n+2}{n+1} = 1 + \frac{1}{n+1},$$

and the apparent paradox that $1 = 2$ disappears.

Much the same happened when we applied the identity

$$\frac{1}{k(k+1)} = \frac{2k+1}{k} - \frac{2k+3}{k+1}$$

which produced the series

$$\left(\frac{3}{1} - \frac{5}{2}\right) + \left(\frac{5}{2} - \frac{7}{3}\right) + \left(\frac{7}{3} - \frac{9}{4}\right) + \left(\frac{9}{4} - \frac{11}{5}\right) + \cdots + \left(\frac{2n+1}{n} - \frac{2n+3}{n+1}\right)$$

which telescopes to

$$3 - \frac{2n+3}{n+1},$$

which tends to 1 (not 3) as n tends to infinity. We leave it to the reader to fill in the details. □

POETRY QUIZ

A nostalgic poem ends with the couplet

> Stands the church clock at ten to three
> And is there honey still for tea?

What is the name of the poem?

Who wrote it?

What is the angle between the hands of the clock at ten to three?

Answers are on page 31. □

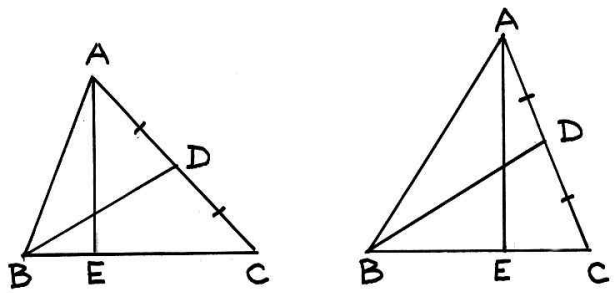
NOT ENOUGH INFORMATION AGAIN

In MATHEMATICAL DIGEST **173** (October 2013) we featured a number of problems in which it appears that not enough information is given. They can nevertheless be solved, because in each case no extra information is needed. Here are two more problems of this type

Problem 1

In triangle ABC, AE is perpendicular to BC (with E on BC), D is the midpoint of AC, and $BD = AE$. Determine $\angle DBC$.

It would seem that $\angle C$ could vary (see the diagrams), so that there is not enough information.



However, from the sine law in triangle BDC we have

$$\frac{CD}{\sin \angle DBC} = \frac{BD}{\sin C} = \frac{AE}{\sin C} = AC = 2CD.$$

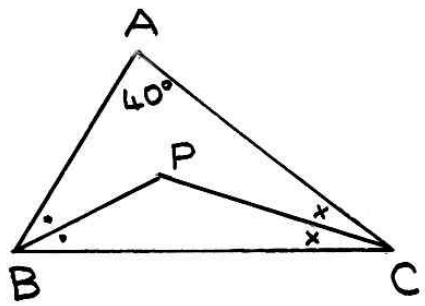
It follows that $\sin DBC = \frac{1}{2}$, and hence $\angle DBC = 30°$.

Problem 2

In triangle ABC, $\angle A = 40°$. The bisectors of $\angle B$ and $\angle C$ meet in P. Calculate $\angle BPC$.

Since only one angle of the triangle is specified, it would seem that there could be many different answers. However, $\angle BPC$ has a unique value, depending only on $\angle A$, calculated as follows:

$$\begin{aligned}
&\angle BPC \\
&= 180° - \angle PBC - \angle PCB \\
&= 180° - \tfrac{1}{2}\angle B - \tfrac{1}{2}\angle C \\
&= 180° - \tfrac{1}{2}(\angle B + \angle C) \\
&= 180° - \tfrac{1}{2}(180° - \angle A) \\
&= 90° + \tfrac{1}{2}\angle A. \qquad \square
\end{aligned}$$



ANSWERS

Sonnet (page 3):
The sonnet is a mnemonic, giving π to 75 places. Just count the letters in each word to get the digits of π. Ten-letter words represent zero. The author of the sonnet is unknown.

Aardvarks and Zebras (page 7):
Connie is an aardvark and all the others are zebras.

Was Fermat Wrong? (page 14):
Why is $3987^{12} + 4365^{12} \neq 4472^{12}$? Look at the two numbers on the left: 3987 and 4365. They are both divisible by 3, and therefore so are their 12th powers. That means that the left hand side is divisible by 3. But 4472 is not divisible by 3, nor is its 12th power. That is enough to show that David Cohen's clever counter-example to FLT doesn't work.

Who's Sitting Where? (page 17):
From the left, the cinema-goers are: Olive Davids (shopkeeper), Charles Fredericks (Teacher), Mary Evans (shopkeeper), Alan Davids (undertaker), Nancy Fredericks (undertaker) and Barry Evans (teacher).

Varsity Readiness Test (page 24):
BBDEA BDAAD EAAAB ECACB AEACA

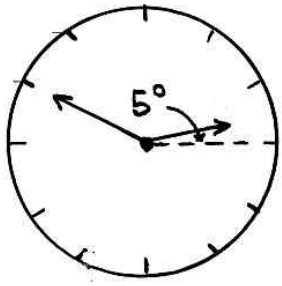
Rectangular Tic-Tac-Toe (page 27):

X	O	O	X	X	O
X	O	X	O	O	X
O	X	O	X	O	X
O	X	X	O	X	O

Poetry Quiz (page 29):
The poem is "The Old Vicarage, Grantchester", by Rupert Brooke.

If you said that the angle between the hands of the church clock at ten to three is 150°, you would be wrong, although not by much.

At ten to three the minute hand is pointing to the 10 on the clock face, and the hour hand is five-sixths of the way between 2 and 3, i.e. one-sixth short. Now the angle between 2 and 3 is 30°, so the hour hand is 5° degrees short of pointing at the 3. So the angle between the two hands is 145°.



□

The University of Cape Town

welcomes the

55th International

Mathematical Olympiad

to the first

IMO in Africa

mathematical digest

No. 176 **July 2014**

IMO 2014

Cape Town - South Africa

mathematical digest

Number 174 **January 2014**

'Tis told by one whom stormy waters threw,
With fellow-sufferers by the shipwreck spared,
Upon a desert coast, that having brought
To land a single volume, saved by chance,
A treatise of Geometry, he wont,
Although of food and clothing destitute,
And beyond common wretchedness depressed,
To part from company and take this book
(Then first a self-taught pupil in its truths)
To spots remote, and draw his diagrams
With a long staff upon the sand, and thus
Did oft beguile his sorrow, and almost
Forget his feeling.

(From *The Prelude*, by William Wordsworth)

Department of Mathematics and Applied Mathematics
University of Cape Town

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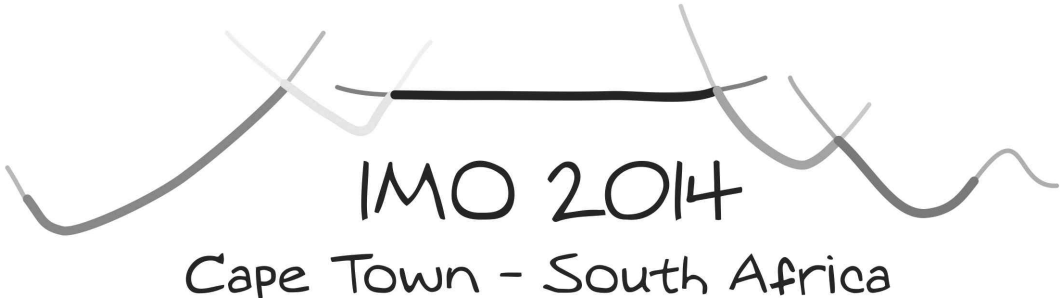
Cover: The cover shows all non-isomorphic homeomorphically irreducible trees of degree 10. See page 16 for a translation!

IMO COUNTDOWN

The 55th International Mathematical Olympiad (IMO) will take place in Cape Town from 3rd to 13th July 2014. Invitations have gone out to over one hundred countries to send teams of their top six high school mathematicians to take part in the biggest, the most prestigious and undeniably the toughest of all the scientific Olympiads for high schools.

First held in 1959 in Romania, the IMO has taken place in every continent except Africa. African participation in the event has been disappointing. Before South Africa's first participation in 1992, only countries on the continent's Mediterranean coast had taken part: Tunisia, Morocco and Algeria. Mozambique joined in 2004, followed by Nigeria (2006), Benin, Zimbabwe and Mauritania (2009), Ivory Coast (2010) and Uganda (2012). Special efforts are being made to encourage participation on IMO2014 by more African countries.



The IMO is the most challenging intellectual contest in the world. Two papers are written, each consisting of just three mathematical problems with what appears to be a generous $4\frac{1}{2}$ hours to crack them. However, the problems are very challenging, even though they may take only a few lines to state and are phrased in the simple language of school mathematics.

Preparations for IMO2014 are well in hand. A Problems Committee of international experts has been appointed to draw up a short list of problems for the contest, with the Jury (the team leaders of the participating countries) making the final selection. Then the papers have to be translated into over 50 languages. After the papers have been written, the team leaders mark them, under the supervision of 50 Coordinators (experienced Olympiad mathematicians, both South African and international), who ensure that the marking is consistent and fair. The Jury then makes the final decisions on the award of Gold, Silver and Bronze medals.

The IMO will take place on the campus of the University of Cape Town. It is expected to be the biggest IMO ever. □

A THIRD ROUTE TO ROOT TWO

A. F. Beardon

University of Cambridge

In *Mathematical Digest* **173** (October 2013) an article appeared with the title "Two Routes to Root Two". Here is a third route.

Suppose that we can solve the equation $X^2 - 2Y^2 = 1$, with X and Y positive integers. Then $X^2 > 2Y^2$ so that

$$0 < \frac{X}{Y} - \sqrt{2} < \left(\frac{X}{Y} - \sqrt{2}\right)\left(\frac{X}{Y} + \sqrt{2}\right) = \frac{X^2}{Y^2} - 2 = \frac{1}{Y^2},$$

and this shows that if Y is large then X/Y is a good rational approximation to $\sqrt{2}$. So how can we solve the equation $X^2 - 2Y^2 = 1$?

A little trial and error gives a solution $(X, Y) = (3, 2)$, but as Y is small, X/Y $(= 3/2)$ is not a good approximation to $\sqrt{2}$. However, there is a way to generate an infinite number of solutions in which the terms Y increase to as much as we like, and where the ratios X/Y converge to $\sqrt{2}$.

Suppose that X, Y, U and V are positive integers with $X^2 - 2Y^2 = 1$, and $U^2 - 2V^2 = 1$. Now define the matrices P and Q by

$$P = \begin{pmatrix} X & 2Y \\ Y & X \end{pmatrix}, \quad Q = \begin{pmatrix} U & 2V \\ V & U \end{pmatrix}.$$

Then

$$PQ = \begin{pmatrix} XU + 2YV & 2[XV + YU] \\ XV + YU & XU + 2YV \end{pmatrix},$$

and the coefficients of this matrix also satisfy the same equation, for

$$\begin{aligned} (XU + 2YV)^2 - 2(XV + YU)^2 &= \det(PQ) \\ &= \det(P)\det(Q) \\ &= (X^2 - 2Y^2)(U^2 - 2V^2) \\ &= 1. \end{aligned}$$

It follows from this that if we write

$$P^n = \begin{pmatrix} X_n & 2Y_n \\ Y_n & X_n \end{pmatrix},$$

then, for all n, $X_n^2 - 2Y_n^2 = 1$.

Now we know that $(X, Y) = (3, 2)$ is a solution so, for example, if we use this solution we obtain

$$P = \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}, \quad P^4 = \begin{pmatrix} 577 & 816 \\ 408 & 577 \end{pmatrix}, \quad P^6 = \begin{pmatrix} 19601 & 27720 \\ 13860 & 19601 \end{pmatrix}.$$

In fact, 577/408 and 19601/13860 are approximations to $\sqrt{2}$ that are correct to five and eight decimal places, respectively.

Of course, we can use the same method for any equation of the form $X^2 - NY^2 = 1$, for if N is not a perfect square then $\sqrt{N}$ is irrational, its decimal expansion is infinite and non-repeating, and the problem is to find a good rational approximation to $\sqrt{N}$.

The earliest evidence we have of good approximations to the irrational square root of an integer is from Greece and India. For example, Baudhâyana (850 BC) gave 577/408 as an approximation to $\sqrt{2}$, and Archimedes (250 BC) gave 1352/780 as an approximation to $\sqrt{3}$; both of these are accurate to five places of decimals.

How were these obtained? Perhaps we shall never know, but surely one possible clue is that

$$577^2 - 2 \times 408^2 = 1, \quad 1352^2 - 3 \times 780^2 = 1.$$

As a second example, consider the equation $X^2 - 7Y^2 = 1$. A little experimentation provides the solution $(X_1, Y_1) = (8, 3)$, and in this case we find that

$$M = \begin{pmatrix} 8 & 21 \\ 3 & 8 \end{pmatrix}, \quad M^6 = \begin{pmatrix} 8193151 & 2167690 \\ 3096720 & 8193153 \end{pmatrix},$$

so that $8193151^2 - 7 \times 3096720^2 = 1$. In fact, 8193151/3096720 is a rational approximation to $\sqrt{7}$ that is correct to fourteen places of decimals. $\square$

LIMERICK

It filled Galileo with mirth
To watch his two rocks fall to earth.
He was heard to proclaim
"Their rates are the same
And quite independent of girth!"

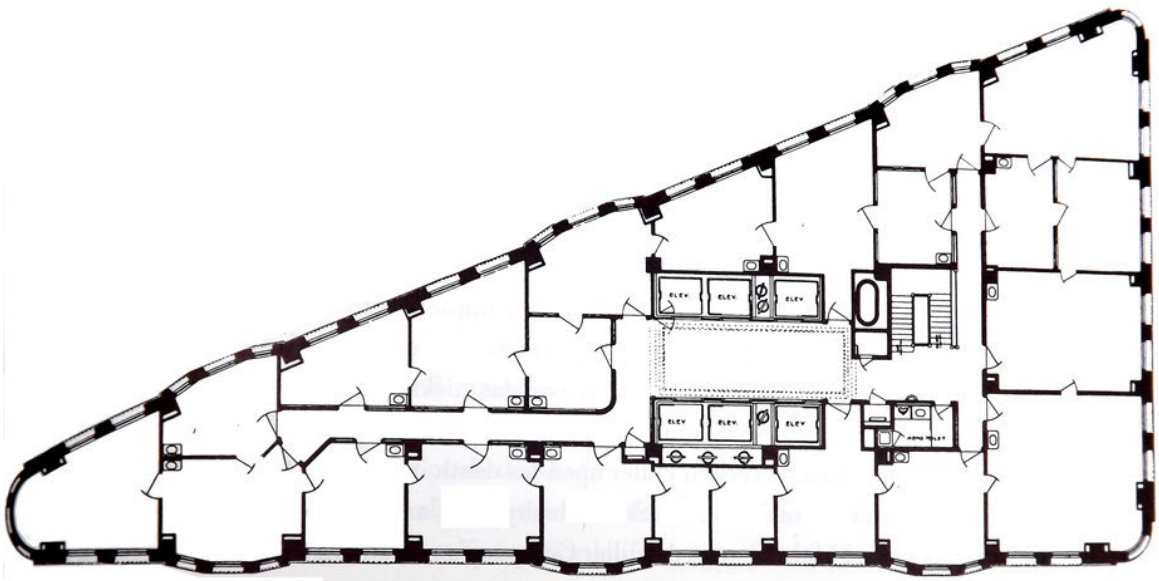
$\square$

PYTHAGORAS AND THE FLATIRON

The Flatiron Building is one of the landmarks of New York. Built in 1902, the Fifth Avenue building is triangular in shape: its footprint is a 5-12-13 triangle.

So it was appropriate that, on the 5th day of the 12th month of the 13th year of the century (5-12-13) the Flatiron Building celebrated its Pythagorean shape when thousands of New Yorkers with glow sticks lit up the building.

The event was organized by MoMath – the new National Museum of Mathematics in New York. It was the biggest mathematics demonstration ever seen in New York, and it will not be for another 92 years (on 13 December 2105 – that's 13-12-05) that the date will again be given by this Pythagorean triple.



□

TRIANGULAR-SQUARE NUMBERS

A. F. Beardon

University of Cambridge

A *square number* is a positive integer of the form m^2, and a triangular number is a positive integer of the from $\frac{1}{2}n(n+1)$, and these, together with hexagonal numbers, appear in the article in "Hexagonal Numbers Again", in *Mathematical Digest* **173** (October 2013).

The great Swiss mathematician Leonhard Euler (1707-1783) listed all positive integers that are simultaneously square and triangular numbers.

To find all such numbers we need to find all integers m and n that satisfy the equation $\frac{1}{2}n(n+1) = m^2$ or, equivalently, $X^2 - 8Y^2 = 1$, where $X = 2n+1$ and $Y = m$. We can find solutions of this equation using the method described in "A Third Route to Root Two" (page 2).

Clearly $(X, Y) = (3, 1)$ is a solution and we have

$$M = \begin{pmatrix} 3 & 8 \\ 1 & 3 \end{pmatrix}, \quad M^2 = \begin{pmatrix} 17 & 48 \\ 6 & 17 \end{pmatrix}, \quad M^8 = \begin{pmatrix} 665857 & 1883328 \\ 235416 & 665857 \end{pmatrix}.$$

Thus, for example, if $2n + 1 = X = 17$ and $m = Y = 6$, then we obtain the square and triangular number 36 with

$$6^2 = 36 = \tfrac{1}{2}(8 \times 9).$$

To obtain a (much) bigger square-triangular number we could use the fact that $665857^2 - 8 \times 235416^2 = 1$.

Interestingly, Euler also proved that while a triangular number can be a square of an integer, no triangular number except 1 is a cube, or a fourth power, of an integer. □

BOTTLE PROBLEM

Place ten milk bottles in a 5×5 crate so that no row, column or diagonal contains more than three bottles.

The answer is on page 29. □

COMPOUND INTEREST IN BABYLON

How long will it take an investment, earning interest at 20% per annum, to double in value?

This is a standard problem in compound interest, which is quickly solved with logarithms. If the number of years is n, then $1.2^n = 2$. Taking logs gives $n = \dfrac{\log 2}{\log 1.2} = 3.80$, to two decimal places.

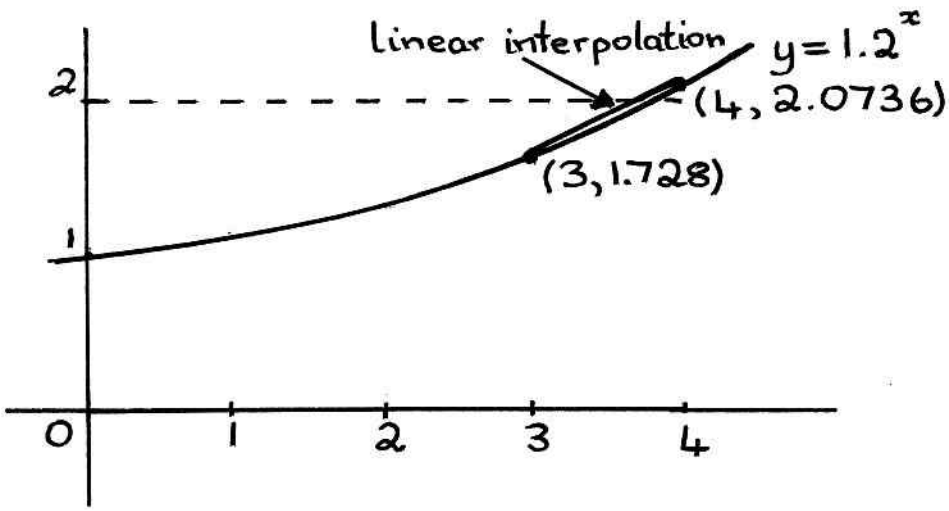
This compound interest problem can be traced back nearly four thousand years, to Larsa, the ancient capital of Babylonia on the Euphrates river in present-day Iraq. This is the biblical Ellasar, mentioned in the Bible (Genesis, chapter 14, verse 1).

The problem was one of five on a clay tablet, which is now in the Louvre in Paris. Archaeologists and experts in ancient languages have been able to read the tablet. The answer given (3.787 years) turns out to be remarkably accurate.

How did the Babylonian mathematicians solve the problem? They did not know about logarithms, which were devised only thousands of years later (in the early 17th Century, by the Scottish mathematician John Napier). Though the clay tablet gives no explanation of the method used, it is possible to make a shrewd guess at how it was done.

The problem is to solve the equation $1.2^x = 2$. Since $1.2^3 = 1.728$, the solution is bigger than 3, and since $1.2^4 = 2.0736$ the solution is smaller than 4. So the solution lies between 3 and 4.

Now look at the graph of $y = 1.2^x$.



The straight line joining the points $(3, 1.728)$ and $(4, 2.0736)$ lies very close to the graph. It has slope $\dfrac{2.0736 - 1.728}{4 - 3} = 0.3456$ and equation

$$y - 1.728 = 0.3456(x - 4).$$

Setting $y = 2$ and solving the equation for x gives $x = 3.787$, which is the solution given on the clay tablet. It is therefore reasonable to assume that the anonymous Babylonian scribed obtained the solution in exactly this way.

This method (of approximating a graph by a straight line segment) is known as *linear interpolation*. The method has been found on many other Babylonian mathematical texts, especially astronomical calculations. □

THE 2014 SOUTH AFRICAN MATHEMATICS OLYMPIAD

The South African Mathematics Olympiad (SAMO) is the biggest Olympiad in the country, with more than 80 000 participants in 2013.

The Olympiad is in three rounds. The first round is at three levels: grade 8, grade 9 and grades 10 to 12. Each paper consists of multiple-choice problems, and those who achieve over 50% progress to round two.

The second-round problems are more challenging, but require answers only. Here there are two levels: grades 8 and 9, and grades 10 to 12. The top hundred at each level qualify for the third round, a paper consisting of real Olympiad problems: tough questions requiring insight, ingenuity, and logically rigorous proofs.

Important dates:
Closing date for entries: 7 February
First round: 13 March
Second round: 13 May
Third round: 10 September

Contact information:
South African Mathematics Foundation
Tel + 27 (0)12 392 9362/24/42/72
E-mail: info@samf.ac.za
Website: www.samf.ac.za □

SAMF
SOUTH AFRICAN MATHEMATICS FOUNDATION

CALCULATING LOGARITHMS

The *logarithm* of a positive real number A to the base a, written $\log_a A$, is the power to which the base must be raised to give A. So

$$a^{\log_a A} = A.$$

In that definition it must be assumed that a is a positive real number, and not 1.

So we have the following:

$$\log_2 8 = 3, \quad \text{since } 2^3 = 8 \qquad\qquad \log_3 81 = 4, \quad \text{since } 3^4 = 81$$

$$\log_4 2 = \frac{1}{2}, \quad \text{since } 4^{\frac{1}{2}} = 2 \qquad\qquad \log_8 4 = \frac{3}{2}, \quad \text{since } 8^{\frac{3}{2}} = 4$$

When logarithms were invented, they were used primarily for calculations. Because our counting system is base 10, logarithms to the base 10 were the main focus. As a result, when the base is omitted, it is assumed that it is 10.

The logs of powers of 10 are easy to calculate. Straight from the definition we have:

$$\log 10 = 1, \qquad \log 100 = 2, \qquad \log \tfrac{1}{10} = -1, \qquad \log 1 = 0, \qquad \log 1000 = 3$$

What about logarithms of other numbers, which are not simple powers of 10? What are the values of $\log 2, \log 3, \log 4$, etc.

With just a little numerical ingenuity we can calculate logs of these numbers (at least to an accuracy of two decimal places).

In what follows, the symbol "$\doteq$" means "approximately equal to" — usually two decimal place accuracy.

Let's start with the calculation of $\log 2$. The key observation is that

$$2^{10} = 1024 \doteq 1000 = 10^3.$$

So $10 \log 2 \doteq 3$, giving $\log 2 \doteq 0.30$ (which turns out to be accurate to two decimal places).

Having calculated $\log 2$, we now tackle $\log 3$. Noting that

$$3^5 = 243 \doteq 250 = \frac{1000}{4}$$

we deduce that

$$5 \log 3 \doteq \log 1000 - \log 4 = 3 - 2 \log 2$$

from which it follows that $\log 3 \doteq 0.48$ (which is again accurate to two decimal places).

The next calculation is easy:

$$\log 4 = 2 \log 2 \doteq 0.60$$

and so is the next:

$$\log 5 = \log(10/2) = \log 10 - \log 2 \doteq 0.70$$

and the next

$$\log 6 = \log 2 + \log 3 \doteq 0.78.$$

There is another way of calculating $\log 6$, based on the observation that

$$6^9 = 10\ 077\ 696 \doteq 10^7$$

from which it follows that

$$\log 6 \doteq \frac{7}{9} \doteq 0.78.$$

To calculate $\log 7$, we note that

$$7^2 = 49 \doteq 50$$

giving

$$2 \log 7 \doteq \log \frac{100}{2} = \log 100 - \log 2 = 2 - 0.03 = 1.70$$

and hence $\log 7 \doteq 0.85$.

The logarithms of $8 = 2^3$ and $9 = 3^2$ should now give no trouble (they are 0.90 and 0.96, respectively). And $\log 10 = 1$, of course.

Since $9 \times 11 = 99 \doteq 100$, $\log 11 \doteq 2 - 2 \log 3 \doteq 1.04$.

With $\log 12 = 2 \log 2 + \log 3 \doteq 1.08$ giving no trouble, our next target is $\log 13$. Here there is an opportunity for nice little bit of numerical ingenuity: the observation that

$$7 \times 11 \times 13 = 1001 \doteq 1000.$$

So $\log 13 \doteq 3 - \log 7 - \log 11 \doteq 1.11$

Passing quickly over the obvious calculations of the logarithms of 14, 15 and 16, we are next confronted by $\log 17$, and once again a numerical coincidence turns out to be useful:

$$17^2 = 289 \doteq 288 = 2^5 \times 3^2$$

from which we get $\log 17 \doteq 1.23$.

We've done enough to leave further calculations left to the reader. $\square$

DOUBLETS

Doublets is a word game invented by Lewis Carroll on Christmas Day in 1877, and has been popular ever since. This is how Carroll described it in the magazine *Vanity Fair*:

> The rules of the Puzzle are simple enough. Two words are proposed, of the same length; and the Puzzle consists in linking these together by interposing other words, each of which shall differ from the next word in one letter only. That is to say, one letter may be changed in one of the given words, then one letter in the word so obtained, and so on, till we arrive at the other given word. The letters must not be interchanged among themselves, but each must keep to its own place. As an example, the word 'head' may be changed into 'tail' by interposing the words 'heal, teal, tell, tall'. I call the given words 'a Doublet', the interposed words 'Links', and the entire series 'a Chain', of which I here append an example:
>
> HEAD → HEAL → TEAL → TELL → TALL → TAIL.
>
> It is, perhaps, needless to state that it is de rigueur that the links should be English words, such as might be used in good society."

Vladimir Nabokov referred to the game as "word golf" in his novel *Pale Fire* and noted that WORD can be changed to GOLF in three links

WORD → WOLD → GOLD → GOLF

and there is a chain from HATE to LOVE requiring only three links:

HATE → LATE → LAVE → LOVE.

It's harder to find a chain from TRIG into GEOM, but there is one of six links:

TRIG → TRIM → TRAM → TEAM → TERM → GERM → GEOM.

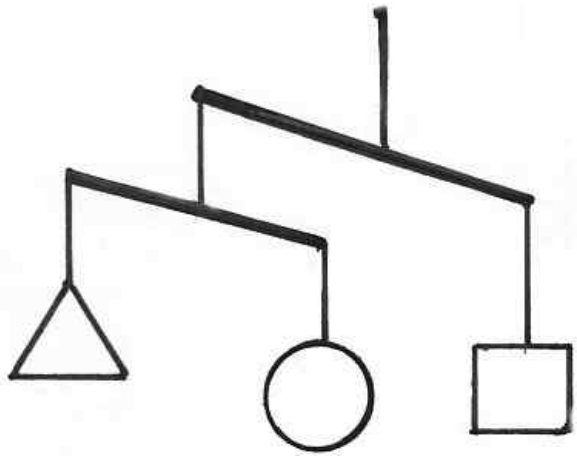
Can you change HAND to FOOT, and navigate SHIP into DOCK?

Six-letter doublets are often impossible, while three-letter doublets can be easy. (Lewis Carroll found that he needed six links to evolve APE into MAN, but it can be done in four. How?)

Answers are on page 29. □

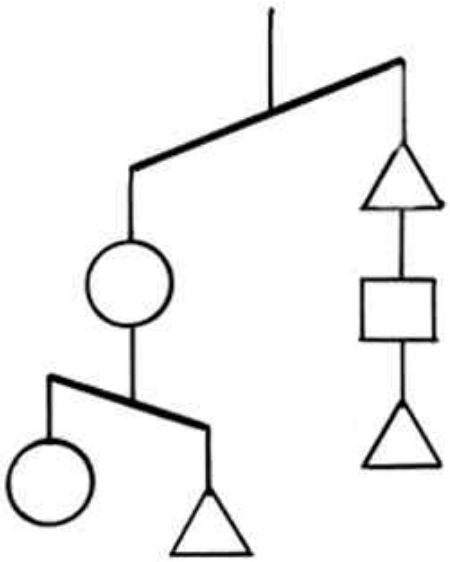
IMBALANCE PROBLEMS

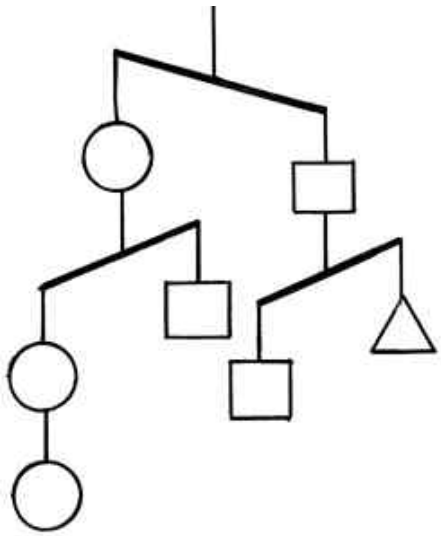
Imbalance Problems were invented by Paul Salomon, who teaches mathematics at the John Burroughs school in St Louis, Missouri, USA.



In the diagram, the slanted lines are thin rods suspended on strings from their centres, and objects are suspended from the ends of the rods. The rods are not horizontal, showing that the objects have different weights, but two objects with the same shape have the same weight. The problem is to work out the relative weights of the objects.

In this simple problem, it is easy to see that the square object is heaviest, followed by the circle, and the triangle is the lightest.

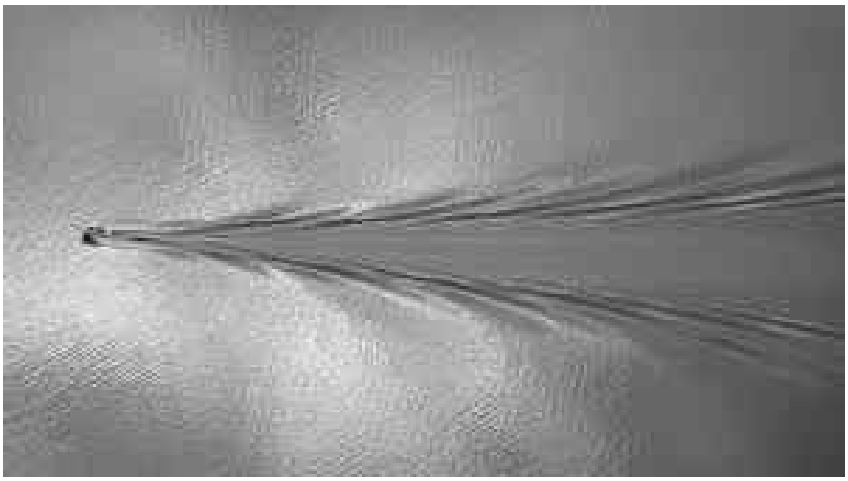




Try to solve the problems (the answers are on page 29) and then invent your own Imbalance Problems and send them to the Editor. The best problems will be published. □

THE KELVIN WAKE

When a ship moves through a calm sea it displaces the water. Since water is incompressible, the ship causes a V-shaped wave which moves outward from its source until it dissipates by friction or dispersion. The study of wakes has important implications for ship design, and for understanding how the banks of navigable rivers can get eroded by the passage of ships.



Wakes were first studied by William Thomson, better known as Lord Kelvin, the famous 19th Century British physicist. He found that the faster the speed of the ship, the faster the wake widens. As a consequence, his calculations showed that the wake angle is a constant 19.47°, independent of the speed of the ship. To be precise, it is the angle whose sine is $\frac{1}{3}$. And it's not just ships that generate this exact wake angle: even a duck paddling across a village pond shows the same wave pattern.

But sometimes narrower wake angles are observed, and these deviations from Kelvin's theory were put down to special situations like shallower water. Recently two French physicists Marc Rambaud and Frédéric Moisy have come up with a more complete analysis.

They went to the Google Earth database and searched for images of ships at sea. They measured the wake angle and length of each ship, and estimated its velocity using the wavelength of the waves of the ship's wake. They then plotted the wake angle against the *Froude number*, a dimensionless physical quantity given by the formula

$$Fr = \frac{v}{\sqrt{gL}} \ ,$$

where v is the velocity of the ship, g is the acceleration due to gravity and L is the waterline length of the ship.

Rambaud and Moisy found that for $Fr < \frac{1}{2}$ the wake angle is constant, but for larger values the wake angle decreases as Fr increases.

The Froude number is analogous to the Mach number for supersonic aircraft, which is the ratio of the velocity of an aircraft to the velocity of sound. The tapering of the Mach cone (the shock wave) for supersonic aircraft resembles the narrowing of the wake of a ship. □

KAPREKAR'S PROCESS

Take any three different digits, arrange them in decreasing and increasing order, and subtract the smaller number from the larger, getting another three digit number. Repeat the process on the result.

For example, starting with 1, 8 and 9, you get $981 - 189 = 792$. Repeat the process: $972 - 279 = 693$. Do it again: $963 - 369 = 594$. But now a repetition gives the same three digits: $954 - 459 = 495$, and the process repeats.

This is known as Kaprekar's Process, discovered by D.R. Kaprekar in 1955.

Can you see why you always end up with the digits 4, 5 and 9? Starting with three digits $a > b > c$, the Kaprekar process will give $abc - cba = d9e$, where $d = a - c - 1$, so that $d - e < a - c$. So the largest and smallest of the digits move closer together, and very quickly you get into the cycle $954 - 459 = 495$.

If you carry out this process with four-digit numbers (not all the same — that's a very uninteresting case), you end up with 6174. For example, starting with 6543, you get in succession:

$6543 - 3456 = 3087 \to 8730 - 0378 = 8352 \to 8532 - 2358 = 6174 \to 7641 - 1467 = 6174$.

Here Kaprekar's process can take a little longer than in the three-digit case, and it is not easy to see why you will always end up with 6174.

When Kaprekar's process is applied to five digit numbers, the situation becomes more complicated, and you have to resort to a computer. It turns out that the process enters in one of three possible cycles:
$\{53955, 59994\}$, $\{71973, 83952, 74943, 62964\}$, $\{75933, 63954, 61974, 82962\}$.
We leave it to the energetic reader to investigate six-digit numbers. □

MATHEMATICS IN OXFORD

In 1868 an Oxford mathematician proposed the development of a School of Mathematics.

The main features of his proposal were:

1. A very large room for calculating Greatest Common Measure. To this a small one might be attached for Least Common Multiple: this, however, might be dispensed with.
2. A piece of open ground for keeping Roots and practising their extraction: it would be advisable to keep Square Roots by themselves, as their corners are apt to damage others.
3. A room for reducing Fractions to their Lowest Terms. This should be provided with a cellar for keeping the Lowest Terms when found, which might also be available to the general body of Undergraduates, for the purpose of "keeping Terms."
4. A large room, which might be darkened, and fitted up with a magic lantern for the purpose of exhibiting Circulating Decimals in the act of circulation. This might also contain cupboards, fitted with glass-doors, for keeping the various Scales of Notation.
5. A narrow strip of ground, railed off and carefully levelled, for investigating the properties of Asymptotes, and testing practically whether Parallel Lines meet or not: for this purpose it should reach, to use the expressive language of Euclid, "ever so far."

Who was the mathematician? The answer is on page 29. □

A NUTTY PERCENTAGE

Reader David Allison alerted us to a consumer column in the *Cape Times* (4 November 2013) in which the deceptive repackaging of a brand of supermarket Pecan Nuts was featured. The former 100g packet was reduced in weight to 75g, and the price was unaltered. The journalist commented that the price had therefore gone up by 25%. Was that right? See page 30 for the answer. □

THE SQUARE ROOT OF THREE

David Feinberg

I fear that I will always be
A lonely number like root three.

A three is all that's good and right,
Why must my three keep out of sight?

Beneath a vicious square root sign,
I wish instead I were a nine.

For nine could thwart this evil trick,
With just some quick arithmetic.

I know Ill never see the sun,
As 1.7321.

Such is my reality,
A sad irrationality.

When hark! What is this I see?
Another square root of a three

Has quietly come waltzing by.
Together now we multiply

To form a number we prefer,
Rejoicing as an integer.

We break free from our mortal bonds
And with the wave of magic wands

Our square root signs become unglued
Your love for me has been renewed.

David Feinberg, professor of computer science at Carnegie Mellon, wrote this poem while at Randolph High School, and published it in the school's literary magazine.

Later, two of his classmates were directing the movie *Harold and Kumar Escape From Guantanamo Bay*, and obtained Feinberg's permission to include it in a pivotal scene. The poem is recited by Kumar (Kal Penn) in a romantic reconciliation at the end of the story. □

HUNTING TREES

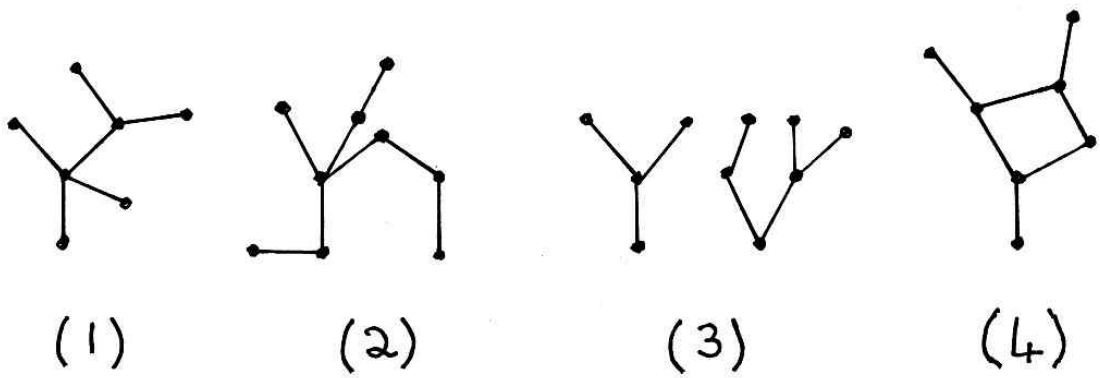
In the 1998 movie *Good Will Hunting*, Matt Damon plays the role of an untutored mathematical genius employed as a janitor in the Massachusetts Institute of Technology. The movie is noted for the amount of genuine mathematics it contains, making the story of the troubled genius seem authentic.

While doing his evening routine of mopping the floors, Will Hunting finds a problem left on a board for students to work on. It is said to be a problem that took MIT professors two years to solve, but Will Hunting cracks it without difficulty. What was the problem?

Succinctly stated, it amounts to finding all non-isomorphic homeomorphically irreducible trees of degree 10. That sounds sufficiently scary, but it can be explained in simple words.

A *tree* is a connected diagram of vertices (dots) joined up by edges (lines), without any circuits. That amounts to saying that for any two vertices, there is exactly on path from one to the other along the edges. The *degree* of a tree is simply the number of vertices.

In the examples below, (1) and (2) are trees, but (3) is not (it is disconnected), nor is (4) (it has a circuit).



Two trees with the same arrangement of vertices and lines, but with different lengths and angles, and differently orientated in space, are not considered different: they are *isomorphic*. We are looking for trees that are *non-isomorphic*: they have an essentially different layout.

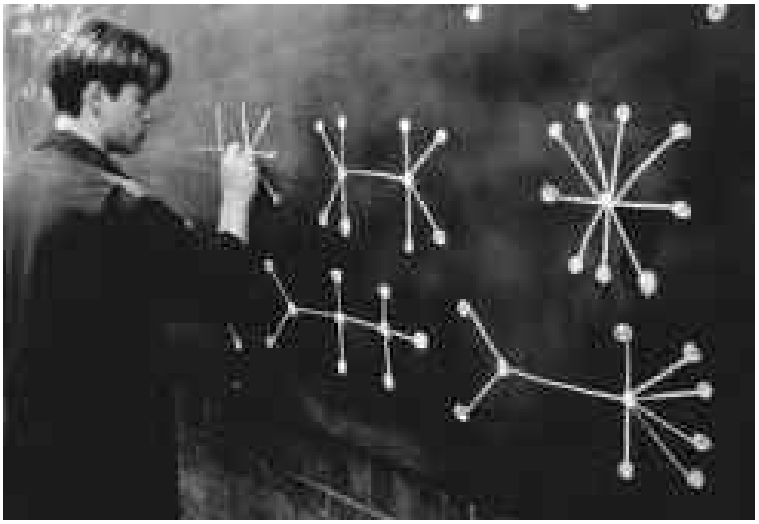
Homeomorphically irreducible means that the tree has no vertex with exactly two lines emanating from it. Tree (1) above is irreducible but (2) is not.

Let us denote by $N(n)$ the number of non-isomorphic homeomorphically irreducible trees of degree n.

It is not difficult to see that $N(1) = N(2) = N(4) = N(5) = 1$ and that $N(3) = 0$.

Then it gets a little harder, but it is still not too difficult to work out that $N(6) = N(7) = 2, N(8) = 4$ and $N(9) = 5$.

The problem in *Good Will Hunting* was to calculate $N(10)$. In the movie, Will Hunting is interrupted when he has drawn eight trees.



What is the value of $N(10)$? Can you find all non-isomorphic homeomorphically irreducible trees of degree 10? The answer is on the cover. □

EQUATIONS

Solving the equations

$$513x + 487y = 8513$$

$$487x + 513y = 8487$$

is, in theory, straightforward: just multiply the first by 513 and the second by 487, then subtract. That step eliminates y and gives an equation from which you can work out the value of x. Now all you have to do is substitute that value of x into either of the two original equations to find y.

That amount of calculation, even with a calculator, is most unattractive. However, it is possible to solve the equations in your head, without writing anything down. Can you see how? The answer is on page 29. □

MARITAL PROBLEMS

Marital Problem 1:
Ahmed is looking at Bongani and Bongani is looking at Carol. Ahmed is married and Carol is unmarried. Prove that a married person is looking at an unmarried person.

Proof
If Bongani is married, then he (a married person) is looking at Carol, who is unmarried.
If Bongani is unmarried, then Ahmed, who is married, is looking at an unmarried person, Bongani.

Marital Problem 2:
Ahmed is looking at Bongani, Bongani is looking at Carol and Carol is looking at Dave. Ahmed is married and Dave is unmarried. Prove that a married person is looking at an unmarried person.

Proof
Suppose Carol is married. Then a married person (Carol) is looking at an unmarried person (Dave).
Suppose Carol is unmarried. Then we may just apply the result in Marital Problem 1.

Marital Problem 3:
Ahmed is looking at Bongani, Bongani is looking at Carol, Carol is looking at Dave and Dave is looking at Ethan. Prove that a married person is looking at an unmarried person.

Proof
Suppose Dave is married. Then a married person (Dave) is looking at an unmarried person (Ethan).
Suppose Dave is married. Then we may just apply the result in Marital Problem 3.

It can now be seen that the Marital Problem can be stated and solved for any line of n people looking at each other as described, with the first person married, and the last unmarried. Somewhere down the line there has to be a married person looking at an unmarried person.

We have seen above that the result is true when $n = 3$, or 4, or 5. (It is, of course, obvious that the result is true when $n = 2$.

We now suppose that the result is true for some natural number k.

If we consider a line of $k + 1$ people looking at each other, with the first person married and the last unmarried, then

EITHER the last but one (in position k) is married, in which case we have a married person (the last but one) looking at an unmarried person (the last),

OR the last but one person is unmarried, in which case we know from the established case of a line of k people that there is a married person looking at an unmarried person.

This method of proof is known as *mathematical induction*.

There is a simple geometric way of solving the marital problem.

If we have a line of people $A, B, C, \cdots, Z$ looking at each other as described, with A married and Z unmarried. Label each person with 0 if married and 1 if unmarried. We then have a string of 0's and 1's, starting with a 0 (A is married) and ending with a 1 (Z is unmarried). Go down the line until you find the first 1. Then the previous number was 0, and that married person is looking at an unmarried person. □

MIXING DRINKS

If you were to mix a litre of pure alcohol (C_2H_6O) with a litre of pure water (H_2O) you would not get two litres of 50% proof alcohol, but 1.92 litres. This may be surprising to barmaids, but is a well-known phenomenon in the chemistry laboratory.

What is more surprising is that the weight of the mixture is *more* than the total weights of the separate litres of water and alcohol. How can that be? Surely the basic law of conservation of matter implies that you can't create extra matter?

Can you explain the paradox? If not, turn to page 30. □

PROGRAMMING OLYMPIAD RESULTS

The results of the 2013 Programming Olympiad were announced at a banquet in Cape Town on Monday 30 September 2013.

Fifteen of the top high school programmers out of an initial entry of 4747 had spent the weekend in a two-day programming competition. Peter Waker, the Manager of the Computer Olympiad, pointed out a remarkable feature: only six of the finalists were in grade 12. He commended the finalists, saying "To be even number 15 out of 4 747 entries is a remarkable achievement. Watch these youngsters; they will create the future."

Gold Medal:
Guy Paterson-Jones (grade 12, Bishops)

Silver Medals
Yaseen Mowzer (grade 9, Fairbairn College)
Shaylan Lalloo (grade 12, Pearson High School)

Bronze Medal
Ulrik de Muelenaere (grade 10, Waterkloof High School)
Darren Roos (grade 11, Pretoria Chinese School)
Robin Visser (grade 11, St George's Grammar School)

Runners-up
Jonathan Alp (grade 11, Parklands College)
Matthew Cherry (grade 12, St Dominic's Priory)
Frederik du Toit (grade 10, Centurion High School)
Pieter Kok (grade 10, Centurion High School)
Demetre Liakos (grade 11, Selly Park Secondary School)
Jan de Wet Linde (grade 12, Jim Fouché High School)
Kiuran Naidoo (grade 12, Clifton School)
Thomas Orton (grade 11, Bishops)
Ryno Swart (grade 12, Durbanville High School)

Computer Olympiad
South African Computer Olympiad: a project of the Computer Society of South Africa.

The 2014 Computer Olympiad starts with a Talent Search, held in schools on Tuesday 11 March.

The first round of the Applications Olympiad will be held in schools on Friday 9 May, with the finals in Cape Town on Tuesday 1 July and the Awards Ceremony on Wednesday 2 July.

The first round of the Programming Olympiad will be held in schools on Friday 1 August, with the finals in Cape Town on Saturday 11 and Sunday 12 October and the Awards Ceremony on Monday 13 October.

For full details, contact:

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THIRTY DAYS HATH SEPTEMBER

Have you ever been caught by a smartypants who asks you: "How many months of the year have 30 days?"

The natural response is to recite quickly to yourself the little nursery rhyme

> Thirty days hath September
> April, June and November.
> All the rest have thirty-one,
> Excepting February alone,
> Which hath but twenty-eight days clear
> And twenty-nine in each Leap Year.

You then give the answer: four.

Your smartypants friend cries triumphantly: "Wrong! Eleven months of the year have thirty days in them."

Here's a way of getting your own back if caught out by this catch question. Just say: "Well, if you are going to interpret 'thirty' as 'thirty or more' in your question, then you must accept my answer of 'four' as 'four or more', which is correct. □

SOLAR POWER MATHS

The Sun is the source of life on the Earth, pouring energy onto our planet at the rate of 1.4 kilowatts per square metre. If all that energy could be harnessed, our power problems would be no more.

The solution is, however, not that simple. For a start, that figure of 1.4kW/m^2 applies to the top of the Earth's atmosphere. Much of the Sun's power is lost in passing through the atmosphere, blocked by dust particles or reflected by clouds.

In higher latitudes, the Sun's rays pass through a thicker layer of air than at the equator, so more energy is lost. Then there is the Earth's rotation, which means that for half the time the Sun does not shine, and some countries are less sunny than others. In South Africa we have an average of about 2500 hours of sunshine per year.

Every country has an "availability factor" k_a, which describes what proportion of the Sun's energy is available. In South Africa $k_a = 0.16$, a relatively high figure. That means that the average solar radiation per square metre in South Africa is $1400 \times 0.16 = 224$ watts.

How can solar energy be captured and used? One way is to use the Sun's rays to heat water. Solar panels on the roofs of houses are now a common sight and can bring down the cost of hot water in the home by a significant amount. However, a constant and cheap hot water supply will not run domestic appliances such as the fridge, TV or microwave oven. Nor is it of any use in charging a cellphone or electric toothbrush. For all those purposes solar radiation must be converted to electricity.

This can be done by using panels of photovoltaic cells, which convert sunlight to electicity directly. This process is neither efficient nor cheap.

The currently level of efficiency of photovoltaic cells is about 15%. A solar panel of 1 m^2 in South Africa will generate $224 \times 0.15 = 34.6$ watts. That means that if you want a constant power supply to your home of 1 kW you will need need $1000 \div 34.6 = 29.8$ square metres of solar panels. That will be enough for low-power appliances such as a fan, radio or cellphone charger, but not nearly enough for cooking.

How much will it cost to set up a house for full solar-powered electricity? Apart from the solar panels, you will need an inverter (to convert the solar-generated DC current to AC) and a bank of batteries to store energy so

you can watch TV after sunset. Inverters and batteries cause further losses of efficiency, and any engineer will recommend a safety margin of 20% on the installation specifications. Overall, the bill may run up to R250 000. Bearing in mind that the equipment will need maintenance and has a finite life-span, the cost of running a normal household on solar power may be around ten times the cost of power from the Eskom grid. □

APPLICATIONS OF MATHEMATICS

In universities, mathematics is taught mainly to men who are going to teach mathematics to men who are going to teach mathematics to ... . Sometimes, it is true, there is an escape from this treadmill. Archimedes used mathematics to kill Romans, Galileo to improve the Grand Duke of Tuscany's artillery, modern physicists (grown more ambitious) to exterminate the human race. It is usually on this account that the study of mathematics is commended to the general public as worthy of State support.

(Bertrand Russell, in *Human Society in Ethics and Politics*, 1954) □

BAR JOKES

A neutrino walks into a bar and asks how much the drinks cost. The barman replies "For you, no charge."

A chemist walks into a bar and asks for a glass of hijklmno. The barman thinks "H to O" and gives him a glass of water.

A physicist walks into a bar, and calls out: "Ten drinks for everybody in the house!" The barman comments: "Now that's an order of magnitude."

An infinite number of mathematicians walk into a bar. The first orders one pint of beer, the second orders half a pint, the third orders a quarter of a pint, and so on. The barman waits until all the orders are in, then puts two pints on the counter, saying "You mathematicians clearly don't know your limits!" □

VARSITY TEST

Varsity readiness tests are now a regular feature of MATHEMATICAL DIGEST. They are designed to highlight those parts of school maths that are of particular importance for students planning to study mathematics at university level. These include:

- arithmetical ability (without needing a calculator)
- algebraic skills
- thorough knowledge of trig formulas, and how to derive them when you don't have a formula sheet
- geometrical and three-dimensional perception
- understanding of logic (proofs, counter-example and contradiction)

The present Senior Certificate examination is, unfortunately, not very challenging when it comes to these areas, and experience has shown that even an A Symbol for SC Mathematics in these inflationary times is no guarantee of success when it comes to tackling varsity maths. There is, of course, rather more to university maths courses than multiple-choice problems, but any difficulty you might have in tackling the problems below will highlight a weakness in your mathematical background that may cause you trouble in a first-year university maths course.

1. $\left(1+\dfrac{1}{a-1}\right)\left(1+\dfrac{1}{a}\right)\left(1+\dfrac{1}{a+1}\right)$ is equal to

   (A) $1+\dfrac{1}{a+2}$ (B) $1+\dfrac{3}{a-1}$ (C) $1+\dfrac{1}{3a}$ (D) $1+\dfrac{1}{a^3-a^2}$ (E) $1+\dfrac{2}{a}$

2. $7^4 - 2\times 42^2 + 6^4$ is equal to

   (A) 169 (B) 179 (C) 189 (D) 199 (E) 209

3. For which values of k does the equation $\dfrac{x^2+x}{x^2+x-2} = k$ have no solution?

   (A) $\frac{1}{9} < k \leq 1$ (B) $1 \leq k < 3$ (C) $0 < k \leq 3$ (D) $-1 \leq k \leq 2$
   (E) $\frac{1}{3} < k \leq 2$

4. The area of a right-angled triangle is 10 cm^2 and its perimeter is 20 cm. What is the length of the hypotenuse of the triangle?

   (A) $7\frac{1}{2}$ (B) 8 (C) $8\frac{1}{2}$ (D) 9 (E) $9\frac{1}{2}$

5. The fourth power of $\sqrt{1+\sqrt{1+\sqrt{1}}}$ is

(A) $2+\sqrt{3}$ (B) $\frac{1}{2}(7+3\sqrt{5})$ (C) $1+2\sqrt{3}$ (D) 3 (E) $3+2\sqrt{2}$

6. If $a=\log_8 225$ and $b=\log_2 15$, then

(A) $a=b/2$ (B) $a=2b/3$ (C) $a=b$ (D) $b=a/2$ (E) $a=3b/2$

7. If r is a root of the equation $r^2+6r+3=0$, what is the value of $(r+2)(r+4)$?

(A) 2 (B) 3 (C) 4 (D) 5 (E) 6

8. In triangle ABC, $\angle A=50^\circ$. The bisectors of $\angle B$ and $\angle C$ meet in O, inside the triangle. Then $\angle BOC$ is equal to

(A) 100° (B) 105° (C) 110° (D) 115° (E)120°

9. $(a^2+b^2)(c^2+d^2)$ is equal to

(A) $(ac-bd)^2+(bc+ad)^2$ (B) $(ad-bc)^2+(ab+cd)^2$

(C) $(ab-cd)^2+(ac+bd)^2$ (D) $(ac-bd)^2+(ab+cd)^2$

(E) $(ab-cd)^2+(ab+cd)^2$

10. If $\cos 2x=\frac{1}{8}$ and $0^\circ<x<90^\circ$, then $\cos x$ is equal to

(A) $\frac{1}{4}$ (B) $\frac{1}{2}$ (C) $\frac{1}{16}$ (D) $\frac{2}{3}$ (E) $\frac{3}{4}$

11. The solution of the inequality $3^{2x+1}+3<3^{x+2}+3^x$ is the interval

(A) $(-1,1)$ (B) $(1,\frac{3}{2})$ (C) $(-\frac{1}{2},3)$ (D) $(\frac{1}{2},3)$ (E) $(1,2)$

12. In triangle ABC, D, E and F lie on sides AB, BC and CA, respectively, so that $BD=DE$ and $CE=CF$. If $\angle A=80^\circ$, what is the size of $\angle DEF$?

(A) 70° (B) 50° (C) 40° (D) 100° (E) 80°

13. Find all values of x such that $-90^\circ\le x\le 270^\circ$ and $2\sin x\cos x=\sin x$.

(A) $-60^\circ, 0^\circ, 60^\circ, 180^\circ$
(B) $-60^\circ, -30^\circ, 0^\circ, 60^\circ, 180^\circ$
(C) $0^\circ, 60^\circ, 150^\circ, 180^\circ$
(D) $-60^\circ, 0^\circ, 60^\circ, 120^\circ, 180^\circ$
(E) $-30^\circ, 0^\circ, 30^\circ, 120^\circ, 180^\circ$

14. A boy runs from his home to school at 9 km/h, then walks back home at 5 km/h. What was his average speed, to one decimal place, for the trip to school and back?

(A) 6.3 km/h (B) 6.4 km/h (C) 6.6 km/h (D) 6.9 km/h (E) 7.1 km/h

15. If $(3x+1)(4-x)(2+7x)$ is multiplied out to the form ax^3+bx^2+cx+d, then $a+b+c+d$ is equal to

(A) 108 (B) 144 (C) 96 (D) 104 (E) 98

16. Which of the following is a factor of the quadratic $6x^2-7x-24$?

(A) $x+12$ (B) $x-3$ (C) $6x+1$ (D) $x-24$ (E) $3x-8$

17. The set of all real numbers x such that $x^2<5x+24$ is the interval

(A) $(-3,8)$ (B) $(1,7)$ (C) $(3,8)$ (D) $(-8,-3)$ (E) $(-2,4)$

18. How many negative integers satisfy the inequality $6x^2-29x-65<0$?

(A) none (B) 1 (C) 5 (D) 8 (E) infinitely many

19. A right-angled isosceles triangle has an area of 8. The perimeter of the triangle is

(A) $8+4\sqrt{2}$ (B) $4+4\sqrt{2}$ (C) 16 (D) 12 (E) $8\sqrt{2}$

20. Triangle ABC is right-angled at B. If $AB=5n$ and $BC=12n$, where n is a positive integer, which of the following is not a possible length for AC?

(A) 39 (B) 69 (C) 91 (D) 104 (E) 143

21. $\sqrt[3]{27^{27x^{27}}}$ is equal to

(A) 27^{3x^9} (B) 27^{3x^3} (C) 27^{27x^9} (D) $27^{9x^{27}}$ (E) $27^{3x^{27}}$

22. $\log_2 500$ is equal to

(A) $4+2\log_2 5$ (B) $1+3\log_2 5$ (C) $3+2\log_2 10$
(D) $3+2\log_2 5$ (E) $2+3\log_2 5$

23. If $f(x)=\dfrac{2x+1}{3x+2}$ and $g(x)=\dfrac{x-2}{2x-3}$ then $g(f(x))$ is equal to

(A) $\dfrac{5x-7}{11x-12}$ (B) $\dfrac{4x-5}{7x-11}$ (C) $\dfrac{3x-7}{7x-11}$ (D) $\dfrac{3x-8}{5x-11}$ (E) $\dfrac{4x+3}{5x+4}$

24. The parabola with equation $y=x^2-3x+2$ is translated 2 units to the right and 3 units up. The x-coordinate of the point of intersection of the two parabolas is

(A) $\frac{5}{2}$ (B) -3 (C) $\frac{13}{4}$ (D) $\frac{7}{4}$ (E) -2

25. When $(3x-2)(2x+5)(x+3)$ is multiplied out, the coefficient of x is

(A) 29 (B) 33 (C) 17 (D) 23 (E) 31

Answers are on page 30. □

MAGIC RECTANGLES

A *magic rectangle* is an $m \times n$ array of the positive integers from 1 to $m \times n$ such that the numbers in each row have a constant sum and the numbers in each column have a constant sum. These two sums will be different if the rectangle is not a square.

An example of a 3×5 magic rectangle is shown below, with row sum 40 and column sum 24.

6	7	8	9	10
13	3	1	11	12
5	14	15	4	2

When can the integers from 1 to 24 be arranged to make a magic rectangle? The grid to be filled must be either 2×12, 3×8 or 4×6. Can you put the numbers from 1 to 24 into each of these three grids to make magic rectangles?

See page 30 for answers. □

PYTHAGOREAN DATES

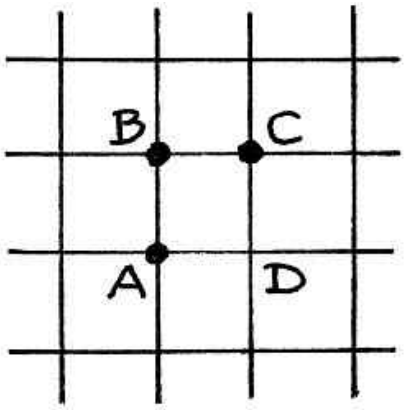
A *Pythagorean date* is one formed of the three numbers of a Pythagorean triple. The most recent example was 5-12-13 (see *Pythagoras and the Flat-iron*, page sss).

When will the next Pythagorean date occur? The numbers in a Pythagorean triple can be used in any order.

The answer is on page 30. □

FROG JUMPS

Frog Jumps is a game played on a grid of squares. It starts with three frogs (counters) placed at the vertices of one of the squares, with the fourth vertex vacant.



Frogs may jump over each other according to a simple rule, "reflection in another frog". What that means is that a frog at vertex P may jump over another frog at vertex Q to a vacant vertex R, where PQR is a straight line of vertices and $PQ = QR$. A frog that is jumped over is not removed.

The object of the game is to jump the three frogs over each other from one grid point vertex to another in such a way that eventually one of them lands on the vacant vertex D, and to do this in the smallest number of jumps. Can you do it? The answer is on page 30. □

GUILTY PARTY

Beane, Gorne and Dunnit is a firm of attorneys well-known for their dishonest ways.

When a client made indignant enquiries about the disappearance of money in a trust fund administered by the partnership, and accused them of theft, he got three responses:

Beane said: "I didn't do it."
Gorne said: "Oh yes you did!"
Dunnit said: "Beane is innocent."

The client learnt that two of these statements were lies, and the third true.

Who stole the money?

The answer is on page 31.

ERRATUM

In *Mathematical Digest* **173** (October 2013), question 1 of the Varsity Test (page 18): $\sin 15^\circ \cos 75^\circ$ should be $\sin 15^\circ \cos 15^\circ$. □

ANSWERS

Bottle Problem (page 5):

o	x	x	o	x
x	x	o	x	o
o	x	o	x	x
x	o	x	x	o
x	o	x	o	x

Doublets: (page 11):
HAND → BAND → BOND → FOND → FONT → FOOT
SHIP → SLIP → SLAP → SOAP → SOAK → SOCK → DOCK
APE → OPT → OAT → MAT → MAN

Imbalance Problems (page 11):
In each problem, the weights of the circle, square and triangle are denoted by a, b and c, respectively.
1. $b > c > a$
2. $c > b > a$
3. $c > a > b$

A Nutty Percentage (page 14):
The journalist was wrong. While the weight reduction of the packet of nuts was indeed 25%, the price increase was more than that.

The price of the packet of nuts was not mentioned in the report, so we suppose it was P. Then the price per gram of the nuts went from $P/100$ to $P/75$. So the price increase was $P/75 - P/100$ and the percentage increase was $\dfrac{P/75 - P/100}{P/100} \times 100$ which comes out at $33\frac{1}{3}\%$.

Mathematics in Oxford (page 14):
The mathematician was the Christ Church don, Charles Lutwidge Dodgson, better known as Lewis Carroll.

Hunting Trees (page 16):
The answers are on the cover.

Equations (page 17):
The key observation is that $487 + 513 = 1000$. So if you add the two equations you get $x + y = 17$ and if you subtract them you get $x - y = 1$. So $x = 9$ and $y = 8$.

Mixing Drinks (page 19):
Mixing 1 litre of alcohol and 1 litre of water to give 1.92 litres of mixture means than the mixture is now displacing 80 millilitres of air less than the two separate litres, and 80 millitres of air weighs about a tenth of a gram. So the denser mixture experiences less buoyancy from the air in which it is weighted, and will weigh 0.1g more. (Of course, if the mixture was weighed in a vacuum, there would be no increase of weight.)

Varsity Test (page 24):
BAABE BDDAE ABABA EABAB DEECD

Magic Rectangles (page 27):
Making a 2×12 Magic rectangle is not difficult. Since the sum of the numbers from 1 to 24 is $\frac{1}{2}(24)(24+1) = 12 \times 25 = 300$, the column sum would have to be 25. One possibility is shown below.

1	23	3	21	5	19	7	17	9	15	11	13
24	2	22	4	20	6	18	8	16	10	14	12

In fact, all the other solutions can be obtained by permuting the columns of this rectangle. There are therefore 12! solutions, or more than 4.79 million.

Cutting the rectangle above into two 2×6 pieces and putting one on top of the other quickly gives a 4×6 Magic Rectangle.

1	23	3	21	5	19
24	2	22	4	20	6
7	17	9	15	11	13
18	8	16	10	14	12

Given that there are millions of solutions in the 2×12 and 4×6 cases, it may come as a surprise that it is not possible to make a 3×8 Magic Rectangle. Why not? Just think of the column sums. Since the sum of the numbers from 1 to 24 is 300 (as noted above), each of the 8 column sums would have to be $300 \div 8$, which is not an integer.

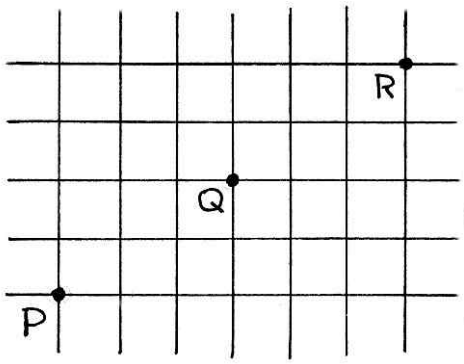
Pythagorean Dates (page 27):
There are no Pythagorean dates in 2014. The next Pythagorean date will be 17 August 2015, or 17-8-15 (since $8^2 + 15^2 = 17^2$). Noting that $9^2 + 12^2 = 15^2$, we can find two other Pythagorean dates next year: 12-9-15 and 9-12-15.

Guilty Party (page 28):
Beane stole the money and lied, Gorne told the truth and Dunnit lied.

Frog Jumps (page 28):
The task is impossible, no matter how many jumps are made.

Suppose that the grid points are labelled in the normal way, and that the three frogs are initially at $(0,0)$, $(1,0)$ and $(0,1)$. The object of the game is to land a frog at $(1,1)$.

Suppose that in general a frog at $P(a,b)$ jumps over a frog at $Q(c,d)$. It will then land at the grid point $R(a+2(c-a), b+2(d-b))$.



The important point to note is that the coordinates of the landing point have the same parity as the coordinates of the initial point, since the horizontal and vertical components of the jump are both even.

The three frogs start of at points $(0,0), (0,1)$ and $(1,0)$, with parity patterns (even, even), (even, odd) and (odd, even), respectively, and however they jump they will always land at points with the same parity pattern. However, the target point has parity pattern (odd, odd). So none of the three frogs can ever land there. □

THE TWIN PRIME CONJECTURE

Do there exist infinitely many pairs of prime numbers differing by two? Examples are easy to find: (3, 5), (5, 7), (11, 13), (17, 19), etc. There are infinitely many primes, but are there infinitely many prime pairs? This is the famous *twin prime conjecture*, which has been around for many years.

In April 2013 Yitang Zhang announced that he had a proof that there is an integer N, such that there are infinitely many pairs of primes that differ by N, and that N is at most 70 million. Terence Tao subsequently proposed a Polymath project collaborative effort to reduce Zhang's bound of 70 million, and in October the figure was reduced to 4680. □

mathematical digest

No. 177 October 2014

IMO 2014 Champions

mathematical digest

Number 177 **October 2014**

Many arts there are which beautify the mind of man; but of all none do more garnish and beautify it than those arts which are called mathematical, unto the knowledge of which no man can attain, without perfect knowledge and instruction of the principles, grounds, and Elements of Geometry.

(John Dee, in *The Mathematical Preface*, published in 1570 as an introduction to Henry Billingsley's translation into English of Euclid's *Elements of Geometry*)

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Cover: IMO champions: Alexander Gunning (Australia), Jiyang Gao (China) and Po-Sheng Wu (Taiwan) tied for the top place in the 2014 IMO, with perfect scores.

MD QED

This is the last edition of *Mathematical Digest*.

The magazine has been published without break since 1971. Many of South Africa's top maths professors, computer scientists and actuaries first discovered through *Mathematical Digest* that there was a world of fascinating mathematics beyond the narrow confines of the school syllabus. In its heyday over two thousand schools received free copies of every edition, four times a year. However, with no corporate support for the last three years, free distribution of the magazine had to be cut back. At the same time, the cost of printing and postage have spiralled, while paid subscriptions and bulk orders from schools have dwindled.

We are ending this year on a high note with the biggest edition ever, including a full account of the 55th International Mathematical Olympiad. The first IMO in Africa was hosted by UCT in July. It was almost (but not quite) the biggest IMO of all time.

Apart from the IMO story (pages 2 to 14), we have reports on the South African Mathematics Olympiad (page 26), the Computer Programming Olympiad (page 30) and the Interprovincial Mathematics Olympiad (page 32). Mathematics Olympiads are an important factor in promoting mathematics and identifying and developing talent at schools. It is noteworthy that two of the four Fields Medals this year went to young mathematicians who first showed their prowess at the IMO (see pages 17 and 18).

The magazine has always been an admirer of Martin Gardner, whose centenary was celebrated in October. You can read a tribute to Martin Gardner, the most successful populariser of mathematics of the 20th Century, on page 22.

With challenging articles on geometry and probability, and puzzles, jokes and mathematical verse scattered throughout the magazine, there is much to extend the horizons of promising young mathematicians, while the "Varsity Readiness Test" (page 36) is a self-assessment challenge for those who plan to continue maths at university level.

While the magazine will no longer be printed, the *Mathematical Digest* website `www.mth.uct.ac.za/digest` will be maintained, and many back numbers and other publications are still available. □

THE IMO2014 STORY

John Webb
Director, IMO2014

In *Mathematical Digest* **176** (July 2014) the story was told of how South Africa was selected to host the 2014 International Mathematical Olympiad. The story went up to July 2011, when the IMO2011 Jury accepted South Africa's bid to host the IMO in 2014. In March 2012 the University of Cape Town (UCT) was selected as the host venue. That gave us two years and four months to get ready.

The IMO2014 Logo

One of the first achievements of the local organizing committee was to come to a quick agreement on a logo. Professional designers came up with some ideas, but they were roundly rejected in favour of Dirk Laurie's inspired design: five sweeping curves outlining Cape Town's iconic Table Mountain in the five colours of the IMO logo, which are also the colours of the five Olympic rings and (if you include the white background), the colours of the South African flag.

IMO Programme

In order to make use of university accommodation and other facilities, the dates of the IMO were determined so as to fit into the UCT July vacation. With the further requirement that contest papers should not be set on a Friday, Saturday or Sunday (to accommodate religious sensitivities), the programme was soon settled.

Day 0: Wed 2 July Arrival of IMO Advisory Board.
Early arrival of Team Leaders (optional).
Day 1: Thu 3 July Arrival of Team Leaders
Day 2: Fri 4 July Jury meeting (setting papers)
Day 3: Sat 5 July Jury meeting (setting papers)
Day 4: Sun 6 July Jury meeting (setting papers). Arrival of teams
Day 5: Mon 7 July Opening Ceremony
Day 6: Tue 8 July Contest paper 1
Day 7: Wed 9 July Contest paper 2
Day 8: Thu 10 July Coordination Excursion for teams
Day 9: Fri 11 July Coordination Entertainment programme teams
Day 10: Sat 12 July Closing Ceremony and Medal presentation

Farewell Banquet
Day 11: Sun 13 July Departure

Deadlines

Working back from those dates, a sequence of deadlines were set:
28 January 2014: Invitations sent out
15 February 2014: Online confirmation of participation by countries
31 March 2014: Deadline for receipt of problem proposals
19 April 2014: Online registration of Leaders, Deputy Leaders, Observers and the number of Contestants
31 May 2014: Online registration of Contestants
15 June 2014: Online registration of travel details

Organizers of past IMOs had warned us that IMO countries are notoriously casual about deadlines, so we were prepared to be tolerant and flexible.

Finding Sponsors

The main task facing the organizers was financial. An IMO host country undertakes to cover all expenses of the visiting teams, including accommodation, meal and excursions. Visiting countries cover their own international travel.

The most expensive item was accommodation for over 1000 people. Hotels are expensive, but with university accommodation available we were able to keep the budget below R12m. The Netherlands-based IMO Foundation, set up with a Google donation of one million Euros in 2011, provided a kick start of two hundred thousand Euros (about R3m), but there was still a long way to go. A fundraising campaign was launched, offering supporters sponsorship levels following the traditional IMO award structure: Gold, Silver, Bronze, Honourable Mention and Participation.

The first contributions came from ex-UCT students, now successful entrepreneurs in the United States. A personal campaign led by Roelof Botha brought in over R1m, including well-known entrepreneurs such as Elon Musk and Mark Shuttleworth.

Meanwhile the local fundraising campaign was making heavy weather. It was disappointing that none of the big banks and insurance companies showed interest in supporting the IMO, even though skilled financial mathematicians and actuaries are at the core of their businesses.

Then came the big one: six million rands from the Department of Basic Education was pledged in March 2013 (the end of their fiscal year), and

government agencies such as Telkom, Sanral and Sasol soon followed. The funding was secure.

Conference Management

An IMO is rather more complicated to organize than a standard academic conference. Not only does it have no income from its participants (half of whom are students and therefore legally minors—which has significant legal implications), but it starts off in two separate streams at different times and places, with the Leaders arriving early to set the papers, and only joining the students after the papers have been written. The overall administration (accommodation, meals, excursions, airport transfers etc) was taken on by the UCT Conference Management Centre.

Running an IMO is supported by a comprehensive software suite covering all aspects from online registration to marks processing and printing certificates. The program was developed by the IMO Webmaster Matjaž Željko, who visited UCT in January 2014 to acquaint everybody with his system. Local IT support was organized by Julian Gordon, the UCT Mathematics Department's IT Manager. He was ably assisted by Andrew Cruise, a member of South Africa's first IMO team back in 1992.

Invitations

Using the IMO address list, official invitations to IMO2014 were sent out by the South African Minister of Basic Education, Angie Motshekga, in January 2014. The response from 106 countries promised to make 2014 a record IMO year, overtaking the 2009 record of 104 countries and 565 contestants in Germany. However, a number of countries subsequently withdrew (some quite late), and the final numbers (101 countries and 560 contestants) were just short of the record.

Visas

Participating countries were pleased to be informed that IMO teams would be given visas free of charge, and visa applications proceeded smoothly, with just one most unfortunate exception. Kosovo, an independent country since 2008, has taken part in the IMO every year since 2011. Though recognized by 108 countries worldwide, Kosovo is not recognized by South Africa, which refuses to accept Kosovo passports. A personal plea for a special concession for Kosovo was sent by the Minister of Basic Education to her counterpart in Home Affairs, but fell on deaf ears. To our huge embarrassment, Kosovo's participation in IMO2014 was blocked.

Africa and the IMO

African participation in the IMO has always been low. Until South Africa joined in 1992, only the Maghreb countries of North Africa (Algeria, Morocco and Tunisia) had taken part. South Africa, through participation in the Pan African Mathematics Olympiad, has been instrumental in bringing other African countries into the IMO, but African participation has been irregular. In 2013 only five African countries took part in the IMO in Colombia: Morocco, Nigeria, South Africa, Tunisia and Uganda.

As this was the first IMO under the African sky, special efforts were made to increase African participation. The usual procedure for a new country is to send an Observer to an IMO a year before becoming a full participant. The IMO Advisory Board agreed to allow South Africa to relax this requirement: we were permitted to invite all African countries that had taken part in the Pan African Olympiad to send teams. As a result, we were able to invite Burkina Faso, Gambia, Ghana and Tanzania to send teams to Cape Town. With participation by Benin, Ivory Coast, Morocco, Nigeria, South Africa, Tunisia, Uganda and Zimbabwe, African IMO participation in 2014 went up to a record number of twelve. In addition, Botswana and Madagascar sent Observers, and have undertaken to take part in the Pan African Mathematics Olympiad and the IMO in future.

All African countries were invited to arrive a week early or a special week-long training programme, hosted by the African Institute for Mathematical Sciences in Muizenberg. There was no charge for attending the camp. A team of experienced IMO trainers ran the programme: Nic Heideman (South Africa), Paul Vaderlind (Sweden), Mark Saul (USA), David Hatton (South Africa) and Charles Leytem (Luxemburg).

Problem Selection

However good the administrative structures, however comfortable the budget and however efficient the technology, the key to the success of an IMO is the quality of the contest itself. The contest papers must be top-class, the examinations scrupulously run and the marking accurate.

IMO problems have to be of a high standard of originality and mathematical elegance. To ensure that this happened, a Problem Selection Committee (PSC) was appointed, headed by Johan Meyer (University of the Free State) and Stephan Wagner (University of Stellenbosch). In preparation for this task, Johan Meyer attended the IMOs in Argentina in 2013 and Colombia in 2014, giving him an opportunity to gain IMO experience. He

was thus able to meet and invite a strong team of international experts to join his committee: Ilya Bogdanov (Russia), Christian Meyer (Germany), Waldemar Pompe (Poland) and Géza Kós (Hungary).

The first task of the PSC was to solicit problems from all participating countries and screen them for quality, suitability for an IMO and, above all, originality.

When in January 2014 the IMO invitations were sent out, all countries were invited to propose problems for the contest. The IMO regulations specify that the host country does not propose problems. Mathematicians consider it a significant achievement to have a problem accepted by the IMO—perhaps even equivalent to publishing a research paper in a top academic journal—so there was no shortage of submissions. Problems started arriving from all over the world early in 2014, and it was the unenviable task of the Problem Selection Committee to work through every one, without first looking at the solution. Long discussions on email and skype ensued. In June the PSC met for the first time at the University of the Free State for face-to-face discussions, moving on to the University of Stellenbosch before finally submitting their IMO2014 Short List to the Jury. It consisted of 30 problems, classified in the IMO traditional categories as Algebra (6), Combinatorics (9), Geometry (7) and Number Theory (8). From these the Jury would have to choose six.

The Short List is a valuable document, even after six of its problems have been chosen for an IMO, as it provides a wonderful source of original problems. Copies of the Short list are strictly controlled, and countries are allowed to use them only in their in selection tests for IMO teams for the next year. Only after that is the Short List published for general use.

The IMO Jury

The Team Leaders of the participating countries comprise the IMO Jury. The task of the Jury is to to select three problems for each of the two days: one easy, one medium and one difficult, and to represent the four categories fairly. A system of selection, trialled in 2013, was again followed in 2014. The Jury first chose two easy and two medium problems, one from each of the four categories. The two difficult problems were then chosen from two of the four categories.

The role of the Jury Chairman is crucial, and Sizwe Mabizela (Deputy Vice-chancellor of Rhodes University) filled this position very well. The Jury enjoyed his style: he would begin a session with a homily on the most

recent FIFA World Cup results, and was careful to keep the Jury focussed. While English is the agreed language of Jury discussions, all votes are translated when needed into the other official languages: French, German, Russian and Spanish. The Chairman's skilful summaries of arguments before crucial votes were vital in moving matters smoothly forward, so that the papers were set and ready in good time.

With the problems selected, a small committee of the Jury set about finalising the official English versions, which were then translated into the other four official languages, and then into all other 55 languages. All translations were checked by the Jury before approval.

Arrival of the Teams

The arrival date for the Deputy Leaders and their teams was 6 July, but a number of teams had arrived before then, partly to allow their teams time to overcome jet lag and travel fatigue, partly to do some sight-seeing, but mainly to do some last-minute training. African teams had arrived early, as described. The Australian and United Kingdom teams came a week early and set themselves up in a comfortable guest house quite close to UCT. A programme of intense problem-solving followed, culminating in a special contest paper as a counterpart to their traditional cricket rivalry: the Mathematics Ashes. The results of this paper determined that the Ashes (a sombre funeral urn containing the ashes of the papers written in the first contest in 2008) will remain in the UK for another year.

Team Guides

Each team was assigned a Team Guide. Mainly senior students at the local universities, the Team Guides were selected and trained by a group of Senior Guides, maths teachers Neil Eddy, Justin Biggs, Robyn Biggs, Charles Bradshaw, Charles Smith, Julian Taylor and Geraldine van der Westhuizen. Team Guides had the responsibility of looking after their team for their whole stay. They met them at the airport, stayed with them in their UCT residence, accompanied them on shopping trips, outings and excursions, ensured that they got up and got to the contest venue on time, organized the seating in the Opening and Closing Ceremonies and generally insured that everybody had a great time.

The Opening Ceremony

The Opening Ceremony took place in UCT's Jameson Memorial Hall. Arriving teams and guests were greeted on the plaza by stilt walkers and a vibrant marimba band. With the teams, deputies and guides filling most

of the seats in the body of the hall, the Leaders were kept strictly separate in the gallery, from which they could only wave to their teams.

The proceedings opened with a stirring rendition of the National Anthem by an up-and-coming star of the UCT College of Music, Makudu Senaoana. Compere Mike Strohman smoothly led the proceedings through Welcome Speeches by the UCT Vice-Chancellor Max Price (an interesting thumbnail sketch of the history and prehistory of mathematics in Africa) and the Minister of Education of the Western Cape, Debbie Schäfer. A short Welcome video from Sanral, the sponsor of the ceremony, was screened.

The Parade of Teams is a time-honoured feature of every IMO Opening Ceremony. Teams, with their Deputy Leaders and Guides, come onto the stage, waving flags, posing for photographs and sometimes throwing small gifts into the audience. Usually the teams are called up in alphabetical order, but this year they were called up in historical order of attendance at the IMO. First in the parade was Romania, in 1959 the first country to host an IMO. Bulgaria, Hungary, Poland followed (they were also there in 1959). There were other countries at that first IMO (Czechoslovakia, East Germany and the Soviet Union), but they no longer exist, and their heirs had to wait their turn.

With this order, South Africa (whose first IMO was in 1992 in Moscow) came halfway down the list, and the team was given a special welcome, being escorted onto the stage by the award-winning Limited Edition Drum Corps, with a spectacular drumming and stick-juggling performance.

The second half of the parade began with all the new countries which resulted from the breakup of the Soviet Union, Yugoslavia and Czechoslovakia, and ended with the four new African countries, Burkina Faso, Gambia, Ghana and Tanzania getting special recognition.

Entertainment by Cape Town's ZipZap Circus followed: spectacular "gravity" juggling, acrobatics in hoops and a brilliant mime artist with "volunteers" from the audience who played along with great spirit.

The Opening Ceremony ended with the IMO Oath, read on behalf of all the teams by South African team member Sanjiv Ranchod. IMO Advisory Board Chairman Nazar Agakhanov (Russia) then declared the IMO open, and the IMO flag was raised.

After the ceremony the teams went back to their residences for an early night in preparation for the contest days ahead, while the Leaders, with everything ready, enjoyed a reception in Smuts Hall.

The IMO Contest

The UCT Sports Centre was an ideal venue for writing the contest papers in all but one respect: in Cape Town's winter its cavernous interior is draughty and cold, as UCT students who write their mid-year exams there know well. They come warmly dressed, some even with knee rugs, and the IMO students were advised to wrap up well in sweaters and the warm IMO jackets and beanies provided.

The task of running the examinations was taken over by a strong team of Cape Town maths teachers led by Alison Kitto. Administering IMO contest papers is far more complex than running a conventional examination. Although 560 students is not an unusual size for a university examination paper, the IMO papers have to be distributed in 60 languages. Seating had to be arranged in six blocks, so that any two students from the same team were seated some distance apart. Students received personalised contest folders, with the question papers in their own language. All this complication was neatly handled by Matjaž Željko's software, which arranged seating and printed out labels for desks and folders and the question papers in the requisite languages.

An IMO paper is a marathon of $4\frac{1}{2}$ hours. During the first 30 minutes students may submit questions in writing, which are transmitted to the Jury (back in their hotel). The Jury then decides on an appropriate response, which may take the form of an explanation ("The orthocentre of a triangle is the point of intersection of the altitudes") or may be less helpful ("No answer" or "Read the question again").

With the questions answered and the contestants toiling away at the problems, the Jury went off to Kirstenbosch for lunch at Moyo's and a stroll in the sunshine around the gardens, including the new Canopy Walkway, also known as the *Boomslang*. On the way back, a stop at Rhodes Memorial was another photo opportunity.

Back in the Sports Centre, the exam process proceeded smoothly and ended on time. For the invigilators it did not end with the collection of the papers, as every sheet had to scanned before the folders were returned to their Team Leaders for marking. This process had to be meticulously done, and took several hours. The papers were ready for the Leaders that evening.

The next day saw the teams tackling the second paper, and everything went as smoothly as on the first day. When the question-and-answer session was over, the Leaders were moved to the UCT campus where they settled into

Graca Machel Hall, just across from the student residences. They then were able to be at the doors of the Sport Centre as their students, whom they had not been able to talk to for a week, came out at the end of paper 2. Much excited discussion ensued on how they had done.

Coordination

Marking of the papers was coordinated by a team of experienced IMO veterans, appropriately called Coordinators, and sometimes (humorously) Marking Police. The Chief Coordinator, Dirk Laurie, had assembled a team of 54 Coordinators, led by six Problem Captains, all with substantial IMO experience: Johan Meyer, David Hatton, Jay van Zyl, Nic Heideman, Maciek Stankiewicz and Stephan Wagner were responsible for problems 1 to 6, respectively, and each headed a team of eight Coordinators.

Half of the coordinators were South African (they included many IMO veterans), while the rest were from overseas. They formed a strong team. Among them were three of the top four of the IMO "Hall of Fame": Lisa Sauermann (Germany), Nipun Pitimanaaree (Thailand) and Christian Reiher (Germany), all winners of a remarkable four Gold Medals in their IMO careers.

As soon as the papers were set, each coordination team tackled their problem, aiming to produce as many different solutions as possible, with corresponding marks schemes. Alternative solutions had meanwhile also been produced by members of the Jury while the problems were being debated and selected. After a fair amount of midnight oil, comprehensive sets of solutions for each problem, with appropriate marks schemes, were produced. With the Jury's approval of all mark schemes, the two-day process of Coordination began.

Here the Coordinator's role of "Marking Police" took over. Having access to all the scanned solutions, a pair of Coordinators (usually one South African and one from overseas) would meet the Team Leader and Deputy of a country to agree on the marking of a question. Team Leaders always fight hard for points for their teams, but often have to concede to the Coordinators' firm "That solution is worth two points, not three, and that is what we have given in all other cases." Discussion is robust, but usually ends in agreement. If not, a dispute is passed to the Problem Captain, and if there was still no satisfaction, the Chief Coordinator is brought in. If the Chief Coordinator's ruling is not acceptable, the matter goes to the final Court of Appeal (the Jury). In the only case of an appeal this year, the Jury upheld the judgements of the lower courts.

To an IMO Team Leader, good coordination means a good IMO, and the general opinion of the 2014 coordination was a unanimous "excellent".

Excursion to Cape Point

With the exams over and Leaders and Deputies hard at work, the teams could relax and enjoy three days of excursions. While Table Mountain and Robben Island are Cape Town's top attractions, neither could be included in the programme. Both provided logistical difficulties: the Robben Island ferry takes only 220 passengers (and is notoriously unreliable), while a Table Mountain cable car takes just 65. Worse, being outdoor attractions, they could have been wiped out by a Cape winter storm when the cableway and the ferry don't run.

A trip to Cape Point offered an alternative and workable excursion, with the thinking that it would be worth seeing even in a good winter storm, when perhaps one could catch a glimpse of the ghostly Flying Dutchman battling to round the Cape. The weather, however, continued to smile on the IMO, and the day was sunny and windless as a convoy of IMO coaches headed south down the Indian Ocean coast of the Cape Peninsula through the old Simonstown naval base to see the famous Boulders Penguins. Crossing to the Atlantic side, the party enjoyed lunch and entertainment laid on by the community of Ocean View, before continuing down to Cape Point to see the meeting point of the Atlantic and Indian Oceans.

Celebrity Lectures

With Coordination in full swing, a special programme of lectures was offered for the students, presented by three world-renowned mathematicians.

First on the bill was Günther Ziegler, the winner of a Gold Medal at the IMO in 1981. Today he is Professor of Mathematics at the Free University of Berlin and is known for his work in discrete mathematics and geometry, both of which are important IMO themes. Of special interest to Olympiad contestants is his book *Proofs from the Book*, written with Martin Aigner. Now in its fifth edition, it has been translated into a dozen languages. His lecture title was *Firing Cannons at Sparrows: Cutting Polygons via Topology of Configuration Spaces.*

John Barrow is Professor of Mathematics at the University of Cambridge and a mathematician, theoretical physicist and cosmologist. He is the Director of the Mathematics Millennium Project and is the author of some twenty books on mathematics for the general reader, his most recent being

Mathletics: 100 Amazing Things You Didn't Know about the World of Sports. His lecture title was *The Maths of Whole Universes.*

South African Peter Sarnak is Professor of Mathematics at Princeton University. A champion chess player in his youth, he came to see mathematics as a source of far more interesting intellectual challenges, and is today a leader in the worldwide attack on the major unsolved problem in mathematics, the Riemann Hypothesis. Just before coming to Cape Town for the IMO he received an Honorary Doctorate from his alma mater, the University of the Witwatersrand. His lecture title was *Number Theory and the Circle Packings of Apollonius.*

African Games

The Celebrity Lectures packed a lot of maths into the morning. Though the afternoon was more relaxed, it was not without some challenging maths, this time of an indigenous African variety. On the two previous evenings the Department of Cultural Affairs and Sport of the Western Cape Provincial Government had laid on a special programme of African music, drumming and dance for the students. The students returned to the Sports Centre, now stripped of its rows of desks and set up for a programme of African games, music and dance. The strategic board game *morabaraba* is very popular in South Africa, while the "African stone game", variously known as *mancala*, *bao* and *wari* in different African countries, is gaining support. Experts in these games explained the rules and set up tables for playing them. For the more energetic there was coaching in skipping and space for indoor soccer. The afternoon culminated with music and dance. The three celebrity lecturers were also there, joining in the fun.

In choosing IMO Problem 6 the Jury is not just interested in finding a really difficult problem, but one that represents attractive and interesting mathematics. This year's problem 6 had an extra feature: it encouraged extensions. The USA Team Leader, Po-Shen Loh, was particularly interested in this problem and its generalizations, and organized a late-night discussion which was packed out.

The Final Jury Meeting

The Final Jury Meeting includes reports from the IMO Advisory Board on various issues, the most important being the approval of recommendations of the IMO Advisory Board on future hosts. The list now extends for five years: 2015 (Thailand), 2016 (Hong Kong), 2017 (Brazil), 2018 (Romania) and 2019 (United Kingdom). Then came the serious business of finalising

the results of IMO2014. There was just one appeal against the decision of the Coordinators, and the Jury upheld the recommendation of the Chief Coordinator.

Then the scores were published in numerical order (without names attached) and the Jury had to decide on where the cuts for Gold, Silver and Bronze Medals were to be placed. The regulation stipulates that the total number of medals should not exceed half the number of contestants, and that the ratio of the numbers of Gold, Silver and Bronze medals should be approximately $1 : 2 : 3$. With 560 contestants, the medal numbers came out as 47 Gold, 93 Silver and 140 Bronze (280 altogether). We had minted 10% more than that figure (55 Gold, 103 Silver and 150 Bronze Medals), giving what we thought was a comfortable margin in each category. However, the Jury controversially voted to award 295 medals: 49 Gold Medals, 113 Silver Medals and 133 Bronze Medals, far in excess of the numbers specified in the Regulations. As a result we were short of 10 Silver Medals.

With 24 hours to go before the Award Ceremony, and not enough time to mint more Silver Medals, we had to resort to spraying ten Bronze Medals with silver paint. The recipients of the counterfeit medals were sent genuine Silver Medals afterwards.

Waterfront Excursion

The next day was a sunny Saturday, and for the first time the Leaders could join their teams for an excursion. The Cape Town Waterfront, ranked as the most popular tourist venue in Africa, was the destination. Apart from walking around the quaysides and watching yachts and small craft come and go with the spectacular backdrop of Table Mountain, there were many souvenir and African curio shops to explore and harbour cruises or restaurants to enjoy. Free entry to the popular Two Oceans Aquarium and the new Rugby Museum had been organized for everybody wearing their IMO badge.

The Closing Ceremony

The Closing Ceremony was co-sponsored by the Sasol Foundation and the Shuttleworth Foundation. The proceedings began with a rousing performance of *Shosholoza* by the Joyful Harmonies Choir led by Mondi Mdingi. Welcome addresses were were given by UCT Deputy Vice-Chancellor Professor Sandra Klopper, the Vice President of the Sasol Foundation MS Mpho Letlhape. A video by the Shuttleworth Foundation was shown, and then it was time for the medals.

The medal presenters for the 133 Bronze Medals were drawn from IMO staff: Problems Selection Committee, Coordinators, Senior Guides and Invigilators.

The Bronze Medals were followed by a dance performance choreographed especially for the 2014 IMO by Celeste Botha and performed by the Cape Towns' *New World Dance Company*.

Silver Medals for 113 contestants were presented by members of the IMO Advisory Board and distinguished academics from the University of Cape Town, the University of the Western Cape and the University of Stellenbosch.

It was now time for the 49 Gold Medals, and they were given a special welcome in true African tradition by praise singer Zama Batyi.

The Gold Medals were presented by the IMO Celebrity Lecturers and representatives of the sponsors, with the top three Gold Medallists Alexander Gunning (Australia), Jiyang Gao (China) and Po-Sheng Wu (Taiwan)—who had also achieved perfect scores—receiving their medals from Minister Angie Motshekga.

With the IMO drawing to a close, it was time to look ahead to the 2015 IMO. After a short video about next year's IMO venue (Chiang Mai, Thailand), the IMO flag was lowered and the South African team came on stage to hand it over with due ceremony to the team from Thailand. Professor Nazar Agakhanov, the Chairman of the IMO Advisory Board, then declared the IMO ended, and the guests were invited to the Farewell Party, a short walk from Jameson Hall.

Arriving in the spacious foyer of UCT's Robert Leslie Building the guests were welcomed by a "cellist in a bubble" and the Joyful Harmonies Choir. An important feature of any Closing Ceremony is the award of the Golden Microphone for the most frequent speaker in the Jury sessions. This year the microphone was replaced by a vuvuzela. The Golden Vuvuzela was presented with appropriate ceremony, to the voluble United Kingdom Leader, Geoff Smith. Towards the end of the evening some of the party allowed themselves to be distracted by football: the World Cup match between the two losing semifinalists.

Departure

Sunday 13 July started very early for those with early flights, and all day buses were shuttling back and forth between UCT and the Cape Town International Airport for homeward flights. All the Team Guides were

there to see off their teams off, whom they had got to know so well in the last week. Farewells were emotional, sometimes even tearful.

Some teams had different plans, either to stay on for a few days in Cape Town, or to take the opportunity of visiting other parts of South Africa and even further, such as the Kruger Park, Okavango and the Victoria Falls.

By the end of the day the UCT campus was quiet. The first IMO in Africa was over. □

THE POWER OF ALGEBRA

My earliest encounter with algebra came about at an early age, when, having long been intrigued by the identity $2 + 2 = 2 \times 2$, I had hit upon $1\frac{1}{2} + 3 = 1\frac{1}{2} \times 3$. Wondering whether there might be other examples, and using some geometrical consideration concerning squares and rectangles, or something—I had never done any algebra—I hit upon some rather too-elaborate formula for which I had guessed might be a general expression for the solution to this problem.

Upon my showing this to my older brother Oliver, he immediately showed my how my formula could be reduced to $\frac{1}{a} + \frac{1}{b} = 1$, and he explained to me how this formula indeed provided the general solutions to $a + b = a \times b$. I was amazed by this power of simple algebra to transform and simplify expressions, and this basic demonstration opened my eyes to the wonders of the world of algebra.

(Roger Penrose, in *The Best Writing on Mathematics 2013*) □

FERMAT LIMERICK

With an integer greater than 2
It's something one simply can't do.
If this margin were fat,
I'd show you all that,
But it's not, so the proof is on you!

(Ted Munger) □

IMO 2014 PROBLEMS

Problem 1. Let $a_0 < a_1 < a_2 < \cdots$ be an infinite sequence of positive integers. Prove that there exists a unique integer $n \geq 1$ such that

$$a_n < \frac{a_o + a_1 + \cdots + a_n}{n} \leq a_{n+1}.$$

Problem 2. Let $n \geq 2$ be an integer. Consider an $n \times n$ chessboard consisting of n^2 unit squares. A configuration of n rooks on this board is *peaceful* if every row and every column contains exactly one rook. Find the greatest positive integer k such that, for each peaceful configuration of n rooks, there is a $k \times k$ square which does not contain a rook on any of its k^2 unit squares.

Problem 3. Convex quadrilateral $ABCD$ has $\angle ABC = \angle CDA = 90°$. Point H is the foot of the perpendicular from A to BD. Points S and T lie on sides AB and AD, respectively, such that H lies inside triangle SCT and

$$\angle CHS - \angle CSB = 90°, \quad \angle THC - \angle DTC = 90°.$$

Prove that line BD is tangent to the circumcircle of triangle TSH.

Problem 4. Points P and Q lie on side BC of acute-angled triangle ABC so that $\angle PAB = \angle BCA$ and $\angle CAQ = \angle ABC$. Points M and N lie on sides AP and AQ, respectively, such that P is the midpoint of AM, and Q is the midpoint of AN. Prove that lines BM and CN intersect on the circumcircle of triangle ABC.

Problem 5. For each positive integer n the Bank of Cape Town issues coins of denomination $\frac{1}{n}$. Given a finite collection of such coins (of not necessarily different denominations) with total value at most $99 + \frac{1}{2}$, prove that it is possible to split this collection into 100 or fewer groups, such that each group has total value at most 1.

Problem. A set of lines in the plane is in *general position* if no two are parallel and no three pass through the same point. A set of lines in general position cuts the plane into regions, some of which have finite area. Prove that for all sufficiently large n it is possible to colour at least $\sqrt{n}$ of the lines blue in such a way that none of the finite regions has a completely blue boundary.

Note: Results with $\sqrt{n}$ replaced by $c\sqrt{n}$ will be awarded points depending on the value of the constant c. □

THE 2014 FIELDS MEDALS

The International Congress of mathematicians (ICM) is a four-yearly get-together of the world's top mathematicians. This year it was held in August in Seoul, South Korea. The highlight of the ICM Opening Ceremony is the announcement of the latest batch of Fields medals.

Fields Medals, officially the "International Medal for Outstanding Discoveries in Mathematics", are awarded at each ICM to up to four mathematicians under the age of 40. The medals were established by the Canadian mathematician John Charles Fields. Fields motivated and established the award, designed the medal and funded the monetary component: $15 000 (Canadian). That is about one percent of the value of a Nobel Prize, and also substantially less than other prizes for mathematical achievement, such as the Abel Prize (see *Mathematical Digest* **176**, (July 2014)). Nevertheless, the Fields Medals are still regarded as the highest accolade that a mathematician can win.

The 2014 Fields medals went to

- **Artur Avila** (Paris Diderot University), for his work on dynamical systems and spectral theory;
- **Manjul Bhargava** (Princeton University), for developing powerful new methods in the geometry of numbers, including elliptic curves used in cryptography;
- **Martin Hairer** (Warwick University), a world leader in the field of stochastic partial differential equations;
- **Maryam Mirzakhani** (Stanford University) for her work on hyperbolic geometry and Riemann surfaces. She is the first woman to win a Fields Medal.

None of the winners currently works in their county of birth — a not uncommon feature in the fluent world of mathematical research. Avila hails from Brazil, Bhargava was born in Canada of Indian immigrants, Hairer comes from Austria and Mirzakhani from Iran.

Avila and Mirzhakani share another distinction: both were Gold Medallists in the International Mathematical Olympiad: Avila in 1995 and Mirzhakani in 1994 and 1995 (with a perfect score in 1995). □

BREAKTHROUGH PRIZES

The Breakthrough Prize Foundation, baseed in San Francisco, has announced five winners of the inaugural Breakthrough Prize in Mathematics:

- **Simon Donaldson**, Stony Brook University and Imperial College London, for the new revolutionary invariants of 4-dimensional manifolds and for the study of the relation between stability in algebraic geometry and in global differential geometry, both for bundles and for Fano varieties.
- **Maxim Kontsevich**, Institut des Hautes tudes Scientifiques, for work making a deep impact in a vast variety of mathematical disciplines, including algebraic geometry, deformation theory, symplectic topology, homological algebra and dynamical systems.
- **Jacob Lurie**, Harvard University, for his work on the foundations of higher category theory and derived algebraic geometry; for the classification of fully extended topological quantum field theories; and for providing a moduli-theoretic interpretation of elliptic cohomology.
- **Terence Tao**, University of California, Los Angeles, for numerous breakthrough contributions to harmonic analysis, combinatorics, partial differential equations and analytic number theory.
- **Richard Taylor**, Institute for Advanced Study, for numerous breakthrough results in the theory of automorphic forms, including the Taniyama-Weil conjecture, the local Langlands conjecture for general linear groups, and the Sato-Tate conjecture.

The Breakthrough Prize in Mathematics was launched by Facebook founder Mark Zuckerberg and Russian tech billionaire Yuri Milner last December. Its aim is to recognize major advances in the field, honour the worlds best mathematicians, support their future endeavours and communicate the excitement of mathematics to the general public.

The laureates will be presented with their trophies and $3 million each at the Breakthrough Prize ceremony in November.

All five recipients of the Prize have agreed to serve on the selection committee responsible for choosing subsequent winners of the prize from the pool of contenders nominated by the mathematics community. From 2015

onwards, one Breakthrough Prize in Mathematics will be awarded every year.

Mark Zuckerberg said: "Mathematics is essential for driving human progress and innovation in this century. This year's Breakthrough Prize winners have made huge contributions to the field and we're excited to celebrate their efforts."

Yuri Milner commented: "Mathematics is the most fundamental of the sciences – the language they are all written in. The best mathematical minds benefit us all by expanding the sphere of human knowledge." □

THE POTATO PROGRESSION

This exercise is recommended by the United States Marine Corps.

Begin by standing on a comfortable surface, where you have plenty of room on each side.

With a 5-lb potato bag in each hand, extend your arms straight out from your sides. Hold them there as long as you can. Try to reach a full minute, and then relax.

Each day you'll find that you can hold this position for just a bit longer.

After a couple of weeks, move up to 10-lb potato bags. Then try 50-lb potato bags, and then eventually, try to get to where you can lift a 100-lb potato bag in each hand and hold your arms straight for more than a full minute.

After you feel confident at that level, put a potato in each bag. □

COIN PROBLEM

You have nine coins which look identical, but one is slightly heavier than the others. You have a balance scale, but no weights. What is the smallest number of weighings needed to determine which is the heavy coin?
The solution is on page 44. □

CUBES AS THE SUM OF TWO SQUARES

Duncan Samson

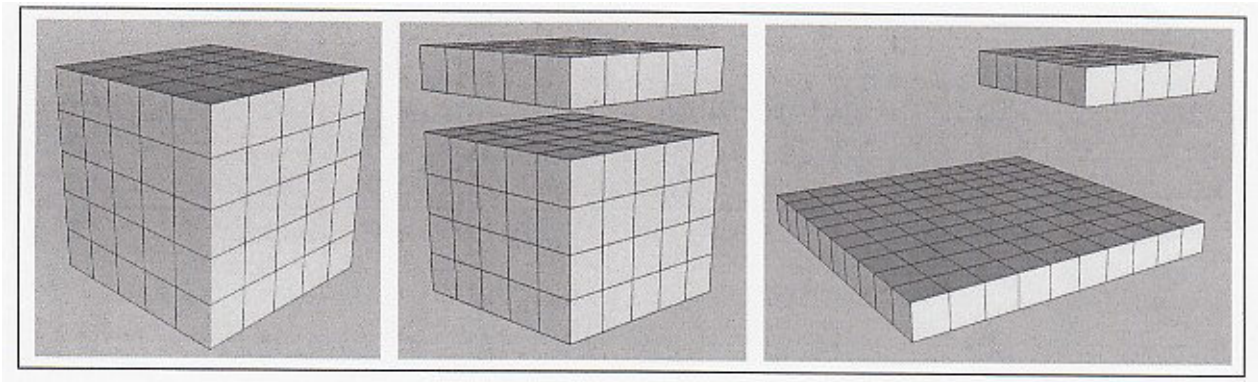
Rhodes University

On a recent roadtrip to Namibia I noticed a road sign that indicated we were 125 km from the Namibian border. The next distance indicator that I saw showed that we were 100 km from the border. It immediately struck me that the first number, i.e. 125, was a perfect cube while the second number, i.e. 100, was a perfect square. Not only that, but the difference between them, i.e. 25, was also a perfect square! This got me wondering - how often is the difference between a perfect cube and a perfect square also a perfect square? Formulated slightly differently, how many perfect cubes can be expressed as the sum of two perfect squares? A little bit of playing around quickly revealed the following initial cases:

$$\begin{array}{ccc} 8 = 4 + 4 & \texttt{i.e.} & 2^3 = 2^2 + 2^2 \\ 125 = 100 + 25 & \texttt{i.e.} & 5^3 = 10^2 + 5^2 \\ 1000 = 900 + 100 & \texttt{i.e.} & 10^3 = 30^2 + 10^2 \end{array}$$

From here on the numbers get rather large and cumbersome to explore. Nonetheless, one is immediately struck by the observation that the general structure seems to be $b^3 = a^2 + b^2$, although it is not immediately obvious whether there is a relationship between a and b, nor what such a relationship might be.

However, the general structure of $b^3 = a^2+b^2$ does suggest a visual approach towards establishing a generalisation. If we visualise a large cubic structure containing b^3 individual unit cubes, then clearly one layer of the large cube would be a square array of b^2 unit cubes. There would thus be $(b-1)$ remaining layers each containing b^2 unit cubes. The scenario is illustrated for a 5 by 5 by 5 cube containing 125 individual unit cubes which can be thought of as 5 stacked square arrays of 25 unit cubes each. If the top layer is removed there are 4 remaining layers each containing 25 unit cubes. And since 4 is itself a perfect square, these 4 remaining layers can be rearranged into a large 10 by 10 square array of 100 unit cubes.



If we consider the case of $10^3 = 30^2 + 10^2$ then we could visualise it as a 10 by 10 by 10 cube containing 1000 individual unit cubes. Removing the top layer leaves 9 remaining layers each containing 100 unit cubes. And since 9 is itself a perfect square, these 9 remaining layers can be rearranged into a large 30 by 30 square array of 900 unit cubes.

We can now generalise the scenario. In order for a perfect cube b^3 to be expressible as the sum of two perfect squares, we require that $b^3 - b^2$ is also a perfect square. This will happen when the number of *layers* in the perfect cube is *one more than a perfect square.* In other words, the perfect cube must be of the form $(n^2+1)^3$ where $n\epsilon\mathbb{N}$. Each layer of such a cube will have $(n^2+1)^2$ unit cubes. When the top layer is removed there will be n^2 remaining layers which can be rearranged into a large square array with side length $n(n^2+1)$, the square array containing a total of $(n(n^2+1))^2$ individual unit cubes.

In addition to showing that for a perfect cube to be expressible as the sum of two perfect squares it needs to be of the form $(n^2+1)^3$ where $n\epsilon\mathbb{N}$, this visualisation can also be seen as a "proof without words" of the following algebraic identity:

$$(n^2+1)^3 = (n^2+1)^2 + (n(n^2+1))^2$$

It is left to the reader to check the truth of this identity algebraically. □

POLYNOMIAL PROBLEM

What is the coefficient of x^{13} in the expansion of the 26th-degree polynomial

$$(a-x)(b-x)(c-x)\cdots(z-x)?$$

The answer is one page 44. □

THE MARTIN GARDNER CENTENARY

Martin Gardner, though not a professional mathematician, was the most successful populariser of mathematics of all time. During his long life he wrote hundreds of articles and books on mathematics and science, as well as works of fiction and philosophy.

To celebrate the hundredth anniversary of Martin Gardner's birth (21 October, 1914), a special centennial memorial lecture was held in the UCT Department of Mathematics on his birthday.

Martin Gardner was born during the opening months of the First World War. He grew up in Tulsa, Oklahoma, and attended the University of Chicago, where he majored in philosophy. After time spent in the US Navy during the Second World War he took to journalism, writing for a children's magazine, *Humpty Dumpty*.

A crucial event in his life was writing an article for *Scientific American* on hexaflexagons. The magazine liked it so much that they invited him to write a regular column, called *Mathematical Games*. At this time Gardner had very little background in mathematics, and scoured secondhand bookshops in New York for whatever he could find on recreational mathematics.

Mathematical Games ran for 25 years, and was an astonishing success. Gardner built up a relationship with many top mathematicians, and through his columns the general public was introduced to many new developments in mathematics, including Penrose tilings, Mandelbrot sets, Conway's *Game of Life*, Ramsey theory, the RSA "unbreakable" codes and many more. Thousands of high school and university students reacted to his columns with delight. These and many other essays, articles and reviews were put together in over one hundred books, fifteen of which are compilations of his Scientific American columns.

Martin Gardner had more interests than just mathematics. One was the works of Lewis Carroll, and his *Annotated Alice* became a bestseller, explaining the mathematical and logical riddles hidden in the *Alice in Wonderland* and *Alice through the Looking-Glass*. It was his most successful book. He was also the scourge of pseudoscientists, and *Fads and Falla-*

cies in the Name of Science pilloried cranks and charlatans (flying saucers, Atlantis, parapsychology, medical quacks, dianetics, spoon-bending and ESP). With similar-minded friends he founded a magazine, *the Skeptical Inquirer*, which criticised with devastating effect the writings of astrologers, numerologists, parapsychologists and other devotees of crank science.

He loved practical jokes and hoaxes, and a famous April Fool column in 1975, entitled "Six Sensational Discoveries that Somehow or Another have Escaped Public Attention" caught out many readers. It included the surprising fact that $e^{\pi\sqrt{163}}$ is an integer, a counter-example to the Four-Colour Theorem and the discovery of a drawing by Leonardo da Vinci proving that he had invented the flush toilet. Another column on Pyramid Power featured a range of wonderful spoof stories describing how the shape of the Great Pyramid of Cheops could be exploited to preserve food, cure diseases and keep razor blades sharp forever. He included a story that when Uri Geller stroked a key while sitting under a large pyramid, the key burst into flames.

After the column appeared, he was contacted by a publisher of pseudoscience, offering him an advance of $15 000 for a quick book to be called *Pyramid Power*, which he felt sure he could promote into a best-seller. Gardner explained that all of the stories in the column were untrue. The publisher, who had believed every one of them, was taken aback, but only briefly. He said that Gardner should still write the book, under a pseudonym, because there was a huge market of gullible readers for such sensational stuff. Moreover, the publisher said, after a year's sales Gardner could reveal that the book had been a hoax.

After retiring as a regular columnist for *Scientific American* (causing the magazine's subscriptions to drop significantly), he returned to his philosophical roots and produced an autobiography, *The Whys of a Philosophical Scrivener*. He then executed one of his finest hoaxes: he wrote, under a pseudonym, a scathing review of the book in the *New York Review of Books* (8 December 1983) which had the effect of reducing the sales of his book until the joke was revealed.

Martin Gardner died at the fine old age of 95 in 2010. His influence in promoting interest and enthusiasm for mathematics in a wide audience is his lasting legacy. In an introduction to one of his books, one of his protégés, Princeton mathematician Persi Diaconis, wrote "Warning: Martin Gardner has turned dozens of innocent youngsters into mathematicians and thousand of mathematicians into innocent youngsters." □

THE OLYMPIAD SYNDROME

University mathematicians are noticing that an increasing number of students arriving at university appear to be suffering from what has been identified as an "Olympiad Syndrome". They have given this name to the syndrome because it seems to be particularly prevalent among those who were exposed to mathematical competitions, camps and Olympiads during their schooldays. Even when sufferers come from different schools, they appear to be good friends with others exhibiting the same syndrome, so there is undoubtedly an infectious virus involved. The virus appears to have originated in Eastern Europe during the 1960s and in recent years has spread rapidly in China and other Eastern countries, where many severe cases have been identified. It peaks in July every year.

No cure has been found for sufferers, although prolonged exposure to television while young appears to be effective in slowing the development of the disease.

The Olympiad Syndrome is characterised by the following symptoms:

- an ability to tackle problems that are not routine variations of textbook exercises and routine school exam questions
- a sound understanding of mathematical logic, the concept of proof, and counter-example
- a readiness to explore ideas beyond the confines of the school syllabus
- never asking "What's the rule to solve this problem?"
- knowing the standard formulas – without needing a formula sheet
- having good numerical skills – without needing a calculator
- willing to spend more than five minutes thinking about a problem whose solution is not immediately obvious
- having good geometrical insight, in both two and three dimensions
- being good at algebraic manipulation
- knowing that geometry can be used to think of algebraic problems, and algebra can solve geometric problems
- able to construct logical proofs
- connecting apparently different areas of mathematics

- thinking flexibly and creatively
- when faced with a difficult question that appears insoluble, to think of a simpler version of the problem, solve it, and then upgrade the solution to a solution of the original problem
- after solving a problem, reflecting on its solution, thinking about finding a different solution, relating the solution approach to other problems and asking how it might be used to solve different (and harder) problems
- developing good style in writing mathematics
- appreciating generalization and overall mathematical structures
- being constantly aware of applications of mathematics to science, engineering, commerce and the real world
- always being on the lookout for fallacious logic in arguments – and not just mathematical arguments
- being interested in the history of mathematics
- working with their peers who have the same passion for mathematics, with a willingness to match their minds against others in friendly rivalry
- reading popularisations of mathematics by authors such as Martin Gardner, Ian Stewart and John Barrow
- looking for opportunities to take part in further Olympiads at university level. □

EUCLID AND HOBBES

He was forty years old before he looked on Geometry; which happened accidentally. Being in a Gentleman's library, Euclid's *Elements* lay open, and 'twas the 47 El. libri 1 [the Theorem of Pythagoras]. He read the Proposition. "By G..." sayd he (he would now and then sware an emphatical oath by way of emphasis) "this is impossible!". So he reads the Proposition of it, which referred him back to a Proposition, which Proposition he read. That referred him back to another, which he also read. *Et sic deinceps* that at last he was demonstratively convinced of that trueth. That made him in love with Geometry.

(John Aubrey, writing about Thomas Hobbes in *Brief Lives*) □

SA MATHEMATICS OLYMPIAD RESULTS

The Western Cape high school students once again took most of the top places in the annual South African Mathematics Olympiad, winning fifteen of the twenty medals (including both Gold Medals) in the 2014 event. The results of the Olympiad were announced at the SAMO award ceremony in Cape Town on Saturday 11 October.

Junior Olympiad

The Dawie du Toit Gold medal was won by Nicholas Lamprecht (grade 9, Westerford High School).

The runners up (in alphabetical order) were

Christopher Aubin (grade 9, Bishops)
Matthys Carstens (grade 9, Hoërskool Durbanville)
Charl du Toit (grade 9, Hoërskool Parel Vallei)
Joseph Forman (grade 9, Herzlia Middle School)
Tae Beom Kim (grade 8, Rondebosch Boys' High School)
Michael Mariano (grade 9, Hoërskool Bellville)
Aaron Naidu (grade 8, St. Henry's Marist Brothers College)
Stehann van Jaarsveld (grade 9, St. Andrew's School, Bloemfontein)
Jialiang Yu (grade 9, Crawford College, Pretoria)

Senior Olympiad

The Dirk van Rooy Gold was won by Robin Visser (St. George's Grammar School). He scored a full 42 points in the final round—only the second contestant in the history of the Olympiad to achieve a perfect score. Robin won the same Gold Medal in 2013. He was also a Senior Silver Medallist in 2012 and a Junior Silver Medallist in 2011.

The runners-up (in alphabetical order) were

Chris Kim (grade 12, Reddam House College, Constantia)
Nicholas Kroon (grade 10, St Andrew's College)
Andrew McGregor (grade 10, Rondebosch Boys' High School)
Mohammed Yaseen Mowzer (grade 10, Fairbairn College)
Tae Jun Park (grade 11, Rondebosch Boys' High School)
Sanjiv Ranchod (grade 10, Westerford High School)
Bronson Rudner (grade 10, South African College High School)

Roger Song (grade 11, Reddam House College, Bedfordview)
Christoff van Zyl (grade 11, Paarl Gymnasium)

The South African Mathematics Olympiad is conducted by the South African Mathematics Foundation, and sponsored by Harmony Gold and the South African Institute of Chartered Accountants.

Dates for 2015
Closing date for entries: 6 February
First round: 12 March
Second round: 13 May
Third round: 9 September

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THE QUEEN'S PROOF

Geoff Smith
University of Bath

This came out of the United Kingdom Mathematics Trust Oxford maths camp at The Queen's College, and all Euclidean geometers who don't know this will want to know it.

Let K be the area of triangle ABC with sides a, b and c in the natural order. As Hero(n) knew, the square of K is $s(s-a)(s-b)(s-c)$ where $s = (a+b+c)/2$, but most proofs are sordid.

A student demanded a better proof. We found it.

Let the inradius be r, and the exradii be r_a, r_b, r_c. As is well known, $K = rs = r_a(s-a)$. So the square of K is $rs.r_a(s-a)$.

Now for the good bit. The circle BCI (Gamma) through the incentre has the excentre I_a at the other end of the diameter through I.

Now consider the contact point of the incircle with BC. The intersecting chords theorem (power of a point) applied to this point and Gamma tells us that $(s-b)(s-c) = r.r_a$ (do it!). Enjoy! □

IOI IN TAIWAN

Ulrik de Muelenaere, who is in grade 11 at Waterkloof High School in Pretoria, won a Bronze Medal for South Africa at the 26th International Olympiad in Informatics (IOI), which took place in Taipei, the capital of Taiwan, from 13th to 20th July.

The other members of the South African team were Shaylan Lalloo (at Pearson High, Port Elizabeth when selected) and Thomas Orton (grade 12, Bishops). The Team Leaders were past IOI medallists Sean Wentzel and Robert Spencer, both of the University of Cape Town.

The International Olympiad in Informatics is a programming competition held in a different country every year. IOI2014 attracted participation from 82 countries.

Top spots were taken by three participants, all with full marks: Ishraq Huda from Australia, Scott Wu from the USA and Yinzhou Xu of China. The top twelve positions included the entire Chinese team of four, three participants from the USA, two from Australia, and one each from Russia, Bulgaria and Georgia.

"The competition gets more difficult every year," said South African Delegation Leader Sean Wentzel. "Many countries have potential participants selected years in advance and put them through a rigorous programme."

Medal winner Ulrik de Muelenaere attributes his success to participating in many online programming competitions to sharpen his skills. "I enjoy programming and problem-solving, so I really enjoyed the IOI."

The contest itself only took up two days. The rest of the time was spent sightseeing, soaking up Taiwanese culture, visiting an amusement park and making new friends.

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MATHS WITH A PUNCH

A US college dropout has become an expert at maths and physics - after being punched in the back of the head.

Jason Padgett was a party-loving 31-year-old who wore his hair in a mullet and made a living working in his father's furniture shop when he was mugged in a karaoke bar near his home in Tacoma, Washington, in 2002. He was knocked out but released from hospital the same day, unaware that he had suffered a profound brain injury. The next day he woke up and found that his vision had changed to include details he never spotted before. Turning on the bathroom tap, he noticed "lines emanating out perpendicularly from the flow".

"It was so beautiful that I just stood in my slippers and stared," he said. Suddenly he could see elaborate geometric shapes in everyday objects. These endlessly repeating patterns are known as fractals and are regarded by mathematicians as the building blocks of everything in the known universe. That led to an obsession with maths and physics. Not only that, he had the ability to recreate the fractals by hand in extreme detail, despite having no previous talent for art. Some took him months to create.

Padgett, now 43, has since been diagnosed as one of only 40 people in the world with acquired savant syndrome, in which once-ordinary people become skilled in maths, art or music after a brain injury. At first, he was frightened by his new ability. He stopped going to work and spent all of his time at home, refusing visitors. He became obsessed with germs and would wash his hands until they were red, and wouldn't even hug his own daughter until she washed her hands as well.

But after seeing a documentary on savantism he contacted leading expert Dr Darold Treffert, who diagnosed him with "acquired savant syndrome". Further research found that the left side of his brain was more activated, especially in the area we use for mathematics. It seems that after his injury, neurotransmitters flooded the left side of his brain and ultimately changed its structure, making him hyper-specialised.

Padgett has now returned to college and written a book, *Struck By Genius*. In it, he writes: "I believe I am living proof that these powers lie dormant in all of us. If it could happen to me, it could happen to anyone.

(From the *Daily Mail*, 22 April 2014) □

COMPUTER OLYMPIAD RESULTS

The results of the 2014 Computer Programming Olympiad were announced at an Awards Dinner in Cape Town on Monday 13 October. The final round papers had been written at UCT over the weekend 11-12 October.

Of the 17 finalists taking part, no fewer than 13 were from the Western Cape.

Gold Medal
Thomas Orton (grade 12, Bishops)

Silver Medals
Ulrik de Muelenaere (grade 11, Waterkloof High School)
Robin Visser (grade 12, St George's Grammar School)

Bronze Medals
Jonathan Alp (grade 12, Parklands College)
Yawseen Mowzer (grade 10, Fairbairn College)
Laurens Weyn (grade 11, Abbott's College Milnerton)

Runners-up
Ashim Asharaph (grade 12, Sol Plaatje Secondary School)
Frikkie du Toit (grade 11, Centurion School)
Scott Hallauer (grade 11, Reddam College, Constantia)
Pieter Kok (grade 11, Centurion High School)
Ralph McDougall (grade 8, Curro High School, Durbanville)
Tae Jun Park (grade 11, Rondebosch Boys' High School)
Matthew Poulter (grade 12, Rondebosch Boys' High School)
Sanjiv Ranchod (grade 10, Westerford High School)
Bronson Rudner (grade 10, South African College High School)
Reuben Steenekamp (grade 12, Reddam College Constantia)
Jeremy Wilkinson (grade 11, Bishops)

The Computer Programming Olympiad is a project of the Institute of Information Technology Professionals South Africa (IITPSA), formerly the Computer Society of South Africa, and is sponsored by Standard Bank, with the support of the Oracle Academy and the University of Cape Town.

Standard Bank ORACLE®

Dates for 2015

Talent Search:
Any time from 16 to 20 March

Applications Olympiad:
First Round: 8 May
Finals: 30 June
Awards function: 1 July (evening)

Programming Olympiad:
First Round: 7 July 2015
Final Round: 10 and 11 October 2015
Awards function: 12 October 2015 (evening)

International Olympiad of Informatics
South Africa will be sending a team of four to the 2015 IOI, which will be held in Astana, Kazakhstan, from 26 July to 2 August.

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ALGEBRA ANTICS

Mr Mercaptan laughed. "I know the sort of thing. 'Look after the past and the future will look after itself.' 'God squared minus man squared equals Art-plus-life times Art-minus-life.' "

(From *Antic Hay*, by Aldous Huxley) □

THE 2014 INTERPROVINCIAL MATHEMATICS OLYMPIAD

The South African Mathematics Team Competition, better known as the Interprovincial Mathematics Olympiad, took place on Saturday 13 September. Over one hundred teams from twelve "provinces" took part (the definition is informal, being more in line with sporting provinces than the nine official ones). Provinces enter teams at two levels: Junior (up to grade 9) and Senior (grades 10 to 12). Many provinces entered A, B and C teams, and with entries from Botswana, Uganda and Swaziland more than 100 teams took part.

The format of the Olympiad has remained the same since it was first held back in 1980. Two papers are written. The first paper is for individuals: a multiple-choice test, with a maximum score of 100. The second paper consists of ten difficult problems, to be tackled by the team working together, with each question worth 100 points. Thus the maximum team score is 2000.

Western Province teams repeated their 2013 performance, winning both divisions comfortably.

JUNIOR	**Individual**	**Team**	**Total**
Western Province	760	1000	1760
Boland	649	1000	1649
Free State	636	900	1536
KwaZulu-Natal	634	900	1534
Gauteng South	569	900	1469
SENIOR			
Western Province	722	900	1622
KwaZulu-Natal	535	600	1135
Gauteng South	525	600	1125
Free State	513	600	1113
Eastern Cape	399	700	1099

The Western Province Junior A Team was captained by Timothy Schlesinger (grade 8, Rondebosch Boys' High School), with team members Chris Aubin (grade 9, Bishops), Michael Botha (grade 9, Melkbosstrand High School), Daniël Hugo (grade 9, Hoërskool DF Malan), Harry Kim (grade 8, El Shaddai Christian School), Tae Beom Kim (grade 8, Rondebosch Boys'

High School), Taariq Mowzer (grade 7, Panorama Primary School), Razeen Parker (grade 8, Rondebosch Boys' High School), Brandon Snider (grade 8, Bishops) and Emile Tredoux (grade 9, Parklands College).

The Western Province Senior A Team was captained by Robin Visser (grade 12, St George's Grammar School), with team members David Broodryk (grade 10, Westerford High School), Jandré du Toit (grade 12, Hoërskool De Kuilen), Soo-Min Lee (grade 11, Bishops), Sang Eun Lee (grade 10, St. George's Grammar School), Andrew McGregor (grade 10, Rondebosch Boys' High School), Yaseen Mowzer (grade 10, Fairbairn College), Tae Jun Park (grade 11, Rondebosch Boys' High School), Sanjiv Ranchod (grade 10, Westerford High School) and Bronson Rudner (grade 10, South African College High School).

The 2014 IPMO is organized by Dr Sudan Hansraj, of the School of Mathematical Sciences, University of KwaZulu-Natal, on behalf of the South African Mathematics Foundation. Financial support provided by the Actuarial Society of South Africa means that teams can be kitted out with provincial T shirts, while Casio watches are won by the members of the top teams. □

THE 2015 UCT MATHEMATICS COMPETITION

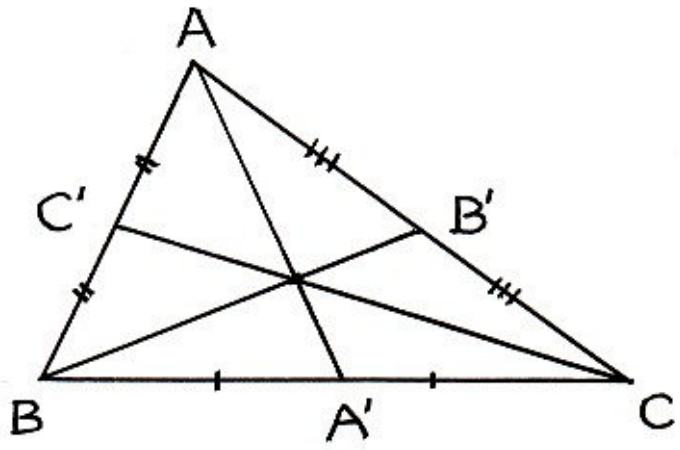
Next year's University of Cape Town Mathematics Competition will take place on Thursday 16th April on the UCT campus. The event is open to all high schools in the Western Cape. It is the biggest maths competition in the world in which everybody writes in the same place at the same time.

Every year groups of schools outside the Western Cape—and even outside South Africa—are sent the papers after the event so that they can take part after the official event at UCT

Full details are on the website `www.mth.uct.ac.za/competition`. □

THE MEDIAN TRIANGLE

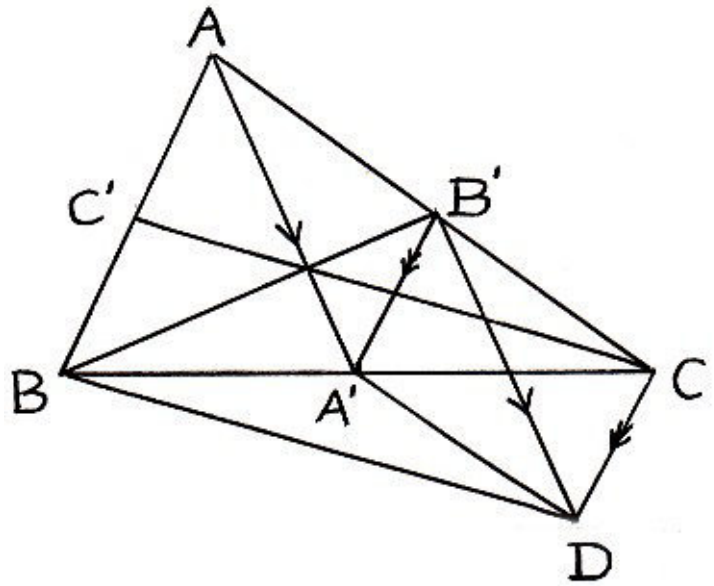
The *medians* of a triangle are the lines joining the vertices to the midpoints of the opposite sides.



It is a basic result in school geometry that the medians of a triangle are *concurrent*, i.e. they pass through a common point, called the *centroid*. Moreover, the line joining the midpoints of two sides is parallel to the third side and equal to half its length.

What is less well-known is that the three medians can form a triangle, called the *median triangle*, and this triangle has area exactly three-quarters of the original triangle.

These two facts can be proved without writing anything down. Just stare at the diagram below until everything is obvious.



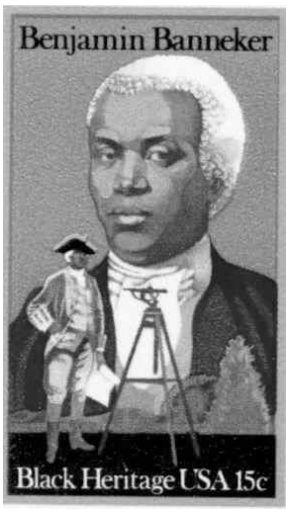
For those who don't trust a Proof Without Words, an explanation is given on page 44. □

A MATHEMATICAL PROBLEM

Benjamin Banneker

A Cooper and Vinter sat down for a talk,
Both being so groggy, that neither could walk,
Says Cooper to Vinter, "I'm the first of my trade,
There's no kind of vessel, but what I have made,
And of any shape, Sir—just what you will
And of any size, Sir—from a ton to a gill!"
"Then," says the Vinter, "you're the man for me,
Make me a vessel, if we can agree.
The top and the bottom diameter define,
To bear that proportion as fifteen to nine,
Thirty-five inches are just what I crave,
No more and no less, in the depth, will I have;
Just thirty-nine gallons this vessel must hold,
Then I will reward you with silver or gold.
Give me your promise, my honest old friend?"
"I'll make it tomorrow, that you may depend!"
So the next day the Cooper his work to discharge,
Soon made the new vessel, but made it too small,
And then cursed the vessel, the Vinter and all.
He beat on his breast, "By the Powers" he swore,
He never would work at this trade any more.
Now my worthy friend, find out, if you can,
The vessel's dimensions and comfort the man!

Benjamin Banneker, the son of a former slave, was born in 1731 on a farm near Baltimore, USA. He was self-taught in surveying, astronomy, and mathematics and published several astronomical almanacs. He composed many mathematical problems, setting them in verse. Above is one of the few that have survived.
He died in 1806 and in 1980 was honoured on a United States postage stamp.



□

VARSITY READINESS TEST

Are your ready for varsity maths? The problems below will probe for weaknesses in your maths background, which you should try to fix before starting a first-year maths course at university. Answers are on page 45.

1. From 9.45 am to 10.30 am, Thabo drove a distance of 84 km. What was his average speed, in km/h?

   (A) 60 (B) 80 (C) 112 (D) 63 (E) 48

2. Which number is the largest?

   (A) $8^3 \times 2^8$ (B) $2^7 \times 4^6$ (C) $\dfrac{16^8}{8^3}$ (D) $16^3 \times 8^2$ (E) $32^3 \times 8^2$

3. If $\sqrt{\sqrt{y-1}\sqrt{y+1}} = x$, where $y > 1$, then y is equal to

   (A) $x^4 + 1$ (B) $\sqrt{x^2+1}$ (C) $\sqrt{x^4+1}$ (D) $\sqrt{x^4-1}$ (E) $x^2 + 1$

4. If $4^n + 4^n = 2^{2013}$, then n is equal to

   (A) 1006 (B) 2012 (C) 671 (D) 503 (E) 2005

5. In the grade 12 maths class there are three boys for every two girls. The average age of the boys is 14 years and 2 months and the average age of the girls is 13 years and 4 months. What is the average age of the whole class?

   (A) 13 years and 6 months (B) 13 years and 8 months
   (C) 13 years and 10 months (D) 13 years and 9 months
   (E) Not enough information has been given.

6. A square has sides of length 2. What is the sum of the distances from one vertex of the square to the midpoints of each of the four sides?

   (A) $2 + \sqrt{3}$ (B) $2 + 2\sqrt{3}$ (C) $2 + \sqrt{5}$ (D) $2 + 2\sqrt{5}$ (E) $2\sqrt{5}$

7. Determine $\sqrt{4^{2013} + 2^{2014} + 1}$.

   (A) $4^{2013} + 1$ (B) $2^{2013} + 1$ (C) $2^{2014} + 1$ (D) $2^{2014} - 1$ (E) $4^{2012} + 1$

8. If $\dfrac{3(x+y) - 2(x-y)}{3(x-y) + 2(x+y)} = 1$, then $\dfrac{x}{y}$ is equal to

   (A) 2 (B) $\dfrac{3}{2}$ (C) 5 (D) $\dfrac{7}{9}$ (E) $-\dfrac{9}{7}$

9. $(3 - \sqrt{2})^3$ is equal to

   (A) $27 - 2\sqrt{2}$ (B) $38 - 15\sqrt{2}$ (C) $55 - 31\sqrt{2}$
   (D) $35 - 14\sqrt{2}$ (E) $45 - 29\sqrt{2}$

10. If $2x^2 - 7xy - 9y^2 = 0$, then x/y is equal to
(A) $-9/2$ or -1 (B) $9/2$ or 1 (C) $-2/9$ or 1 (D) $2/9$ or 1
(E) $9/2$ or -1

11. If $2^x = 3,\ 3^y = 5$ and $5^z = 8$, then xyz is equal to
(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

12. If $y^2 = y + 4$, then $y^4 - y^3 - 16$ is equal to
(A) 0 (B) $3y$ (C) 16 (D) $4y$ (E) -16

13. The diagonals AC and BE of a regular hexagon $ABCDEF$ intersect in G. If $AB = 1$, what is the area of triangle ABG?
(A) $\dfrac{\sqrt{3}-1}{4}$ (B) $\dfrac{\sqrt{3}}{8}$ (C) $\dfrac{9}{40}$ (D) $\dfrac{1+\sqrt{3}}{12}$ (E) $\sqrt{34}$

14. If $0 < x < 1$, which of the following has the biggest value?
(A) x^3 (B) $x^2 + x$ (C) $x^3 + x^2$ (D) x^4 (E) $x^3 + x$

15. Solve for x: $1 + \dfrac{10}{1 + \dfrac{1}{1 + \dfrac{1}{x}}} = 7.$
(A) 1 (B) -2 (C) $-\dfrac{1}{3}$ (4) -3 (E) 2

16. Straight lines PRT and QRS intersect in R. If $PQ = PR$, $RS = RT$ and $\angle RST = 55°$, find $\angle QPR$.
(A) $40°$ (B) $45°$ (C) $50°$ (D) $55°$ (E) $60°$

17. If $3x^2 < 3x$ then
(A) $x < \dfrac{1}{3}$ (B) $x = 0$ (C) $0 < x < 1$ (D) $x = 1$ (E) $x > 1$

18. Let X be the area of a triangle with sides of length 25, 25 and 30. Let Y be the area of a triangle with sides of length 25, 25 and 40. What is the relation between X and Y?
(A) $X = \dfrac{3}{4}Y$ (B) $X = \dfrac{9}{16}Y$ (C) $X = Y$ (D) $X = \dfrac{4}{3}Y$ (E) $X = \dfrac{16}{9}Y$.

19. The radius of the inscribed circle of a right-angled triangle is 6 cm, and the length of one of the shorter sides is 24 cm. What is the area of the triangle, in cm^2?
(A) 240 (B) 300 (C) 210 (D) 216 (E) 192

20. The difference between the squares of two consecutive integers is 199. What is the sum of the squares of these two consecutive integers?
(A) 19 801 (B) 39 601 (C) 19 602 (D) 20 201 (E) 19 405 □

THE CORNFLAKES PROBLEM

The makers of a well-known brand of cornflakes plan to promote their sales by putting cards with pictures of famous sportsmen and women in each box of cornflakes they produce. They hope that this will entice sports fans to buy more cornflakes. The cards will be produced in equal numbers and distributed at random in the boxes of cornflakes, one in each box.

Suppose there are n cards in the series. How many boxes of cornflakes would one expect to have to buy in order to get a complete set?

It turns out that the number of boxes you can be expected to buy is

$$nH(n)$$

where $H(n)$ is the *harmonic series*

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}.$$

This is how it works.

The first box you buy has a card which starts your collection. After that, you may get duplicates, and the more cards you have in your collection, the more likely you are to get a duplicate in the next box of cornflakes.

Suppose that after some time you have collected k different cards. How many boxes can you expect to buy before getting a card different from any you have?

The chance of getting a duplicate in the next box you buy is k/n, while the chance of getting a different card from those you already hold is $1 - k/n$.

The chance of buying r boxes containing duplicates before getting a box with a new card is

$$\left(\frac{k}{n}\right)^r \left(1 - \frac{k}{n}\right).$$

So the expected number of boxes you will have to buy to get a new card (given that you already hold k different cards) is

$$\left(1 - \frac{k}{n}\right) + \left(\frac{k}{n}\right)\left(1 - \frac{k}{n}\right) \times 2 + \left(\frac{k}{n}\right)^2 \left(1 - \frac{k}{n}\right) \times 3 + \left(\frac{k}{n}\right)^3 \left(1 - \frac{k}{n}\right) \times 4 + \cdots .$$

We need to sum this series.

Putting $\dfrac{k}{n} = a$, the series is

$$(1 - a) + 2a(1 - a) + 3a^2(1 - a) + 4a^3(1 - a) + 5a^4(1 - a) \cdots$$

$$
\begin{aligned}
&= 1 + 2a + 3a^2 + 4a^3 + 5a^4 + \cdots \\
&\qquad - a - 2a^2 - 3a^3 - 4a^4 - \cdots \\
&= 1 + a + a^2 + a^3 + \cdots \\
&= \frac{1}{1-a} \qquad \text{(summing the geometric series)} \\
&= \frac{k/n}{1-k/n} \qquad \text{(substituting back for } a\text{)} \\
&= \frac{n}{n-k}.
\end{aligned}
$$

Adding up all these expected numbers of boxes (from $k = 0$ to $n-1$) gives the total number of boxes of cornflakes you can expect to buy to get the full set of n different cards:

$$
\begin{aligned}
&\frac{n}{n} + \frac{n}{n-1} + \frac{n}{n-2} + \cdots + \frac{n}{1} \\
&= n\Big(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}\Big) \qquad \text{(changing the order of the terms)} \\
&= nH(n).
\end{aligned}
$$

□

GOD'S ALPHABET

Eric Hoffer (1898–1983) was an American moral and social philosopher whose working-class background as a migrant worker and dock labourer earned him the nickname "the longshoreman philosopher".

Hoffer was an atheist, but was sympathetic to religion. In his article "God and the Machine Age" (reprinted in *The Ordeal of Change*) he made the point that early scientists such as Galileo and Kepler "really and truly believed in a God who had planned and designed the whole of creation—a God who was a master mathematician and technician."

"It sounds odd in modern ears that it was a particular concept of God that prompted and guided the men who were at the birth of modern science. They felt in touch with God in every discovery they made. Their search for the mathematical laws of nature was to some extent a religious quest. Nature was God's text, and mathematical notations were His alphabet."

One of the best of Hoffer's many thought-provoking and quotable observations was: "The hardest arithmetic to master is that which enables us to count our blessings." □

QUITE INTERESTING NUMBERS

Are there any uninteresting numbers?

The Indian mathematical genius Srinivasa Ramanujan claimed that 1729 is interesting, as related in the famous anecdote by his mathematical friend and mentor G.H. Hardy:

> "I remember once going to see him when he was ill at Putney. I had ridden in taxi cab number 1729 and remarked that the number seemed to me rather a dull one, and that I hoped it was not an unfavorable omen. 'No,' he replied, 'it is a very interesting number; it is the smallest number expressible as the sum of two cubes in two different ways.' "

The two different ways are $1729 = 1^3 + 12^3 = 9^3 + 10^3$.

One must assume that Ramanujan was thinking of positive cubes, since otherwise 91 is a smaller candidate for the sum of two cubes in two different ways: $91 = 3^3 + 4^3 = (-5)^3 + 6^3$.

We can prove in just a few lines that ALL natural numbers (positive integers) are interesting. The proof is by contradiction.

Assume that there is at least one uninteresting number. That means that the set of uninteresting numbers is non-empty. Now it a fundamental property of the set of natural numbers that every non-empty subset has a smallest member.

Consider now that member: it is the smallest uninteresting number.

But that is enough to make it interesting! We have a contradiction, and must conclude that all numbers are interesting.

On the BBC show *QI*, which stands for *Quite Interesting*, Quizmaster Stephen Fry offered a good candidate for a not-very-interesting number: 12407. He asserted that 12407 is the smallest number that does not appear in any of the the sequences listed in Neil Sloane's *Encyclopaedia of Integer Sequences*.

We had to check this, and unfortunately discovered that Stephen Fry is wrong. The number 12407 does appear in Sloane's *Encyclopaedia*, but only once, and in a rather esoteric and uninteresting sequence.

So all numbers are interesting, but some are less interesting than others. It would therefore seem that 12407 is a good candidate for the least interesting of all numbers. Is there any number less interesting than 12407? □

SMUTS HALL IN WINTER

The IMO Coordinators were housed in Smuts Hall, the oldest of UCT's student residences. With lawned quads and stone staircases, Smuts Hall reflects its Oxbridge ancestry. The chilly, if never freezing, Cape winter, was another reminder of its English heritage. Dirk Laurie, the IMO Chief Coordinator, was moved to pen these lines.

'Twixt veins of leafless ivy hung
On dank and dismal walls,
The lofty cisterns shed their load
Like raging waterfalls.

'Tis feared Smuts Hall that we must brave,
Whose charms are stark and subtle,
For, mark my words, they bred a man
Whose wealth was worth a shuttle.

The Leaders of the IMO
Descend en masse by bus
On modern Graça Machel Hall,
But that is not for us.

Coordinator! A grand name:
But neither young nor old,
Find that the title also grants
Imperviousness to cold.

When from these green and icy lawns
In numbness we depart,
The spartan spirit learnt of yore
Will live in every heart.

The quaint and chilly squalor still
Shall captivate and charm,
Like ancient hewn-stone cottages
Upon a backveldt farm. □

NUMBERS AND FACES

W.H Auden

The Kingdom of Number is all boundaries
Which may be beautiful and must be true;
To ask if it is big or small proclaims one
The sort of lover who should stick to faces.

Lovers of small numbers go benignly potty,
Believe all tales are thirteen chapters long,
Have animal doubles, carry pentagrams,
Are Millerites, Baconians, Flat-Earth-Men.

Lovers of big numbers go horridly mad,
Would have the Swiss abolished, all of us
Well purged, somatotyped, baptised, taught baseball:
They empty bars, spoil parties, run for Congress.

True, between faces almost any number
Might come in handy, and One is always real;
But which could any face call good, for calling
Infinity a number does not make it one.

(W.H. Auden: *Collected Poems*) □

SIMPLE HARMONICS

The wonders of nature, quoth he
Are always a marvel to me:
That each tick and a tock
Of a grandfather clock
Is twice pi root l over g.

(From *A Dictionary of Mnemonics*, published by Eyre Methuen) □

CALCULATOR TIP

To handle imaginary numbers on a calculator you just have to turn it through 90°. □

HAPPY FAMILIES

In our street there are fifteen children from five different families, and each family has a different number of children. The five wive have five different careers and the five husbands play five different sports.

From the ten facts below, work out the details of each family: surnames, number of children, wife's career and husband's sport.

- There is only one child in the Collins family.
- The doctor has one more child than the tennis player.
- The school teacher's husband does not like ball games.
- The swimmer had two children.
- The Adams family has three more children than the engineer.
- The cricketer has five children.
- The Barnes family does not have three children.
- The physiotherapist has twice as many children as the soccer player.
- The computer programmer has fewer children than the Davids family.
- Mr Ellis is a marathon runner.

Answers are on page 45. □

FOLDING A CUBE

If the net of six squares shown in figure 1 is cut out, it can be folded into a cube. However, the net shown in figure 2 cannot be folded into a cube.

How many different nets are there which can be folded to make a cube?

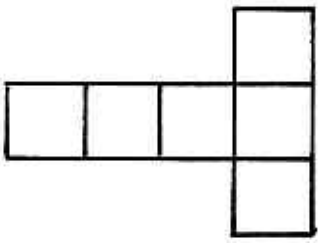


Fig. 1

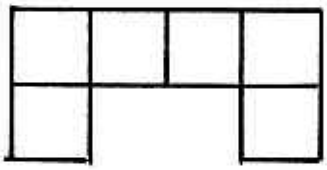


Fig. 2

The solution is on page 45. □

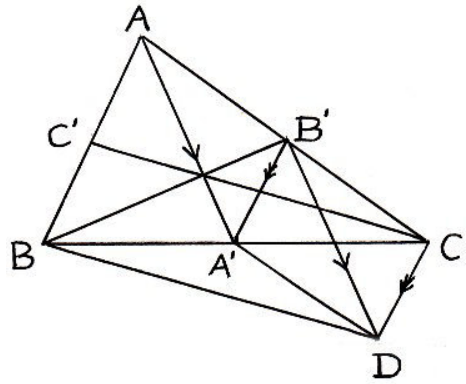
ANSWERS

Coin Problem (page 19):
Divide the nine coins into three groups of three, and weigh two of the groups against each other.

If they balance, you know that the heavier coin is in the third group.
If they do not balance, the the heavier coin is in the heavier group.
Either way, you have identified a group of three that contains the heaver coin.
Weigh two of the coins in this group against each other.
If they balance, the third coin is the heavy one.
If they do not balance, the heavier coin is in the pan that goes down.

Polynomial Problem (page 21):
The polynomial contains the factor $(x - x)$, so is identically zero.

Median Triangle (page 34):
In triangle ABC the medians AA', BB' and CC' are drawn (with A', B' and C' the midpoints of BC, CA and AB.
Draw $B'D$ equal and parallel to AA'. (1)
Then $AA'DB'$ is a parallelogram.
Now $B'C$ is half of AC', which is equal and parallel to $A'D$.



So $B'CDA'$ is a parallelogram, with CD equal and parallel to $B'A'$, which is equal and parallel to $C'B$. It follows that $CEDC'$ is a parallelogram.

Hence $CC' = DB$. (2)

Now look at the triangle $BB'D$. From (1) and (2) we see that the three medians AA', CC' and BB' and are equal in length to $B'D$, DB and BB', respectively, which form a triangle, and therefore the three medians can form a triangle.

To see that the median triangle $BB'D$ is three-quarters of the area of triangle ABC, we note that the median triangle can be divided into three triangles: $DA'B$, $B'A'B$ and $BA'D$, each of which is equal in area to $A'BC$, which in turn is one-quarter of the area of triangle ABC.

So the median triangle has an area equal to three-quarters of the area of the original triangle.

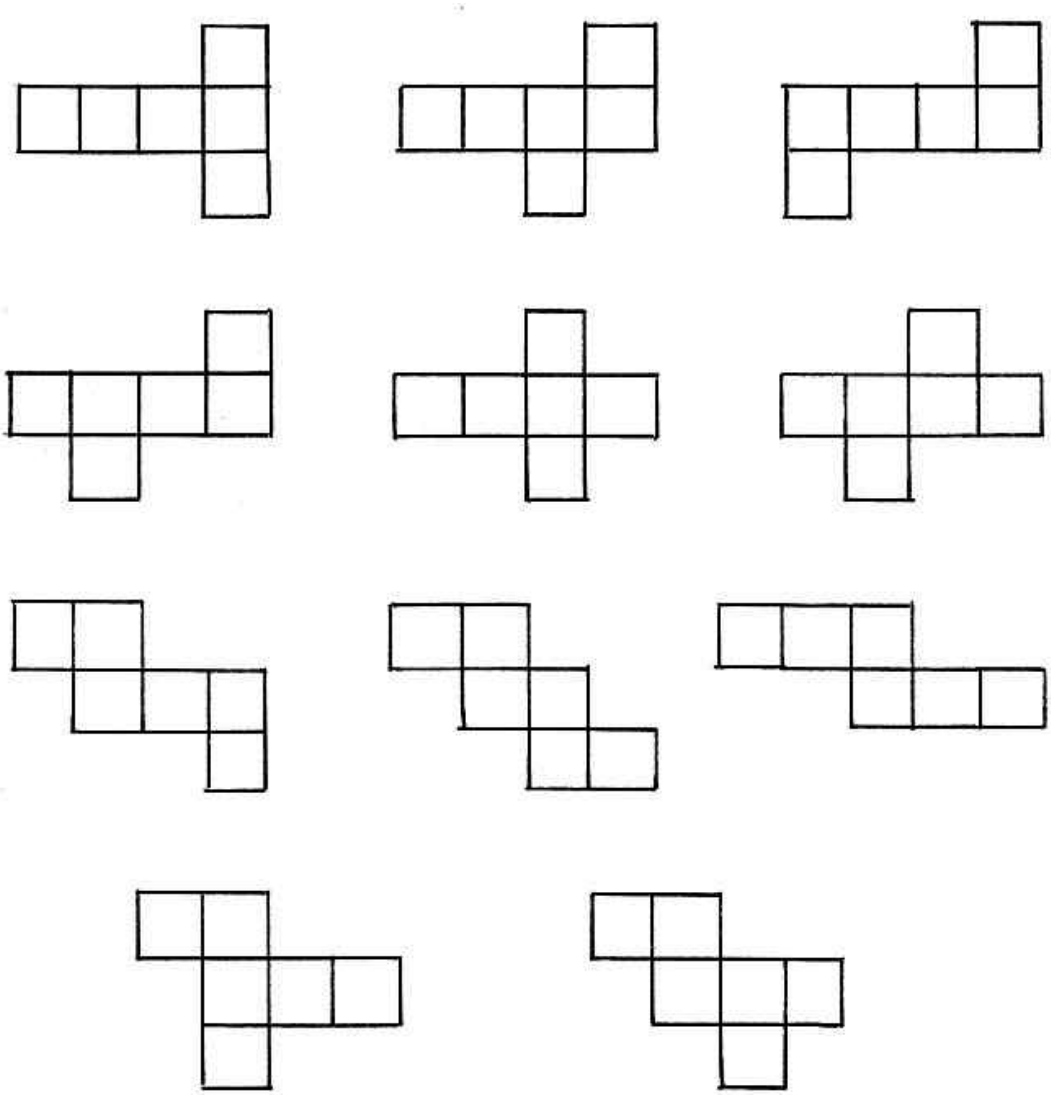
Varsity Test (page 36):
CCCAC DBBEE CDBBE ACCDA

Happy Families (page 43):

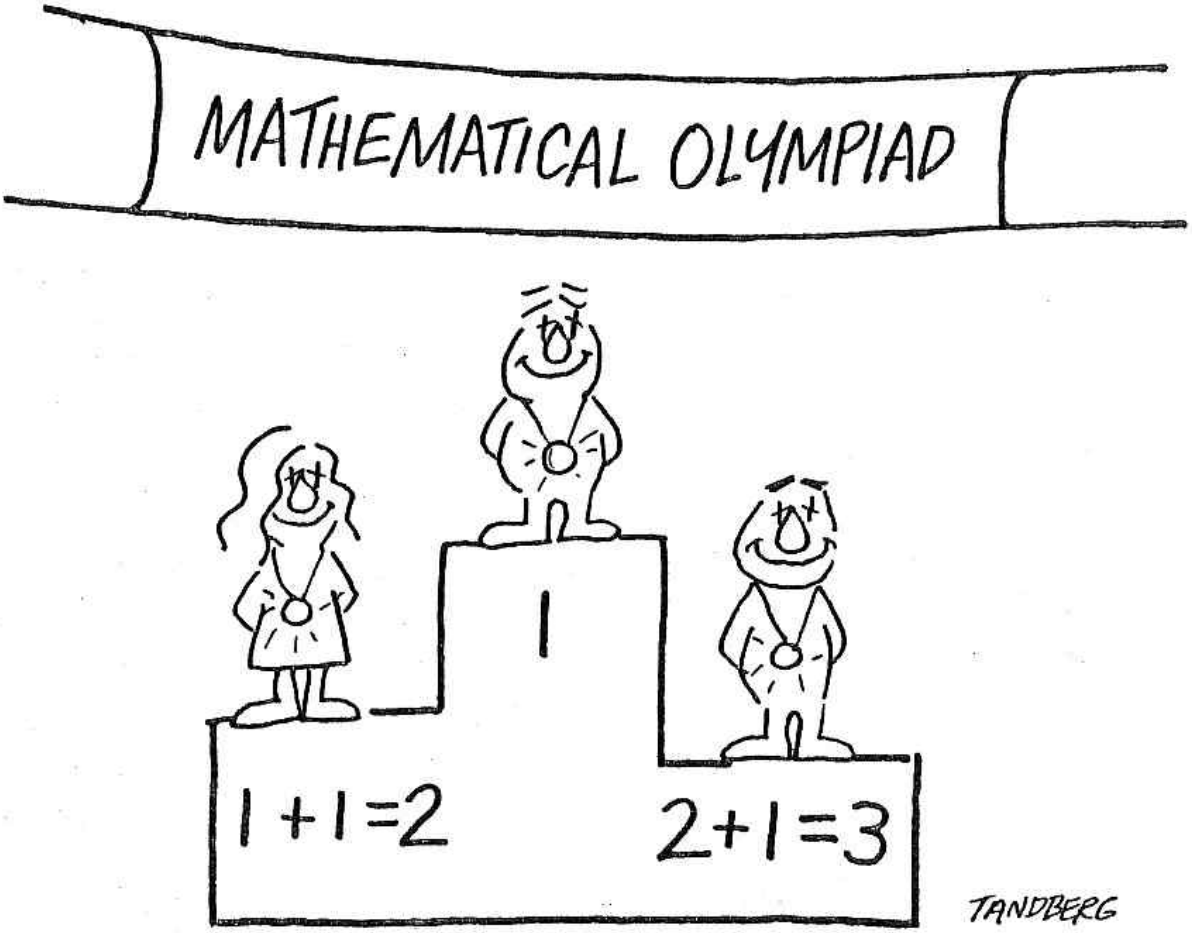
Names	Husband's sport	Wife's career	Number of children
Adams	tennis	computer programmer	4
Barnes	swimming	physiotherapist	2
Collins	soccer	engineer	1
Davids	cricket	doctor	5
Ellis	marathon	school teacher	3

Folding a Cube (page 43):

There are 12 different nets that can be folded to make a cube.



□


MATHEMATICAL OLYMPIAD
1
1+1=2
2+1=3
TANDBERG

mathematical digest

No. 178 January 2015

Mathematical Typewriter

mathematical digest

Number 178 **January 2015**

I spend most of my time nowadays working in one way or another in connection with a computing machine. It is a rather niggly business in a way, like publication. If you have a single hole in the wrong place on the tapes you punch everything goes wrong, and whereas a reader will forgive two or three misprints the machine forgives nothing.

(Alan Turing, in a letter to Stanley Skewes, 9th April 1953)

Department of Mathematics and Applied Mathematics
University of Cape Town

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EDITORIAL

The last edition of *Mathematical Digest* was number 177, which appeared in October 2014. A number of readers and supporters were saddened, but given the lack of sponsorship and dwindling subscriptions, the decision was inevitable.

Prompted by readers, we are planning to continue with publication online. While there is no longer the cost of printing and postage, the question remains whether an online magazine would be read. But since we had decided to continue the magazine's website, and indeed revamp it and post previous editions online, it was decided that number 177 would be the last *print* edition.

So here we are online with number 178, January 2015. □

COVER

For many years *Mathematical Digest* was typed on the twin-keyboard Imperial typewriter shown. On the left of the typewriter is a conventional keyboard QWERTY keyboard, while the right-hand keyboard has a wide range of mathematical symbols. It could be withdrawn and replaced with a third keyboard with a further range of symbols. This typewriter was used in the UCT Department of Mathematics from the 1960s until superseded by "golf ball" typewriters and, eventually, computer typesetting.

WHAT IS A WHOLE NUMBER?

The staff in the Mathematics Department of a Cape Town school started the new school year by arguing about what "whole numbers" are.

Some said they are the numbers $1, 2, 3, 4, \ldots$.

Others insisted that the set must include zero: $0, 1, 2, 3, \ldots$.

Yet another group said the negative numbers must be included in the definition: $\ldots, -3, -2, -1, 0, 1, 2, 3 \ldots$.

They appealed for a ruling by the Editor, who had never thought about this important issue until then, and consulted his bookshelf. What do the dictionaries say about the terms used in school mathematics: integer, whole number, natural number, counting number?

A general English dictionary may not be helpful in giving precise definitions of mathematical terms. The *Concise Oxford Dictionary* entry for *integer* gives the meaning as *whole number*, and the entry for *whole number* gives *integer.* Neatly circular.

A selection of dedicated mathematics dictionaries produced confusion.

The *Mathematics Dictionary*, by C B Dreyer and S J R Vorster, published by De Jager-Haum, says the whole numbers are $0, 1, 2, 3, \ldots$.

The Cambridge *Mathematical Dictionary for Schools*, by Brian Bolt and David Hobbs, says the whole numbers are $1, 2, 3, \ldots$.

The Oxford *Concise Dictionary of Mathematics*, by Christopher Clapham, does not list *whole number*, but under the entry *integer* we read "One of the 'whole' numbers $\ldots - 3, -2, -1, 0, 1, 2, 3, \ldots$."

The Oxford *Mathematics Study Dictionary* says that integers are the numbers $\ldots -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5 \ldots$ and then says "Whole numbers is a term used rather loosely to mean either the natural numbers or the integers; it depends upon the context".

That about sums it up. Mathematicians usually agree that the natural numbers are $1, 2, 3, \ldots$, and denote the set by $\mathbb{N}$, except for French mathematicians who insist that 0 is a natural number. There is also consensus that the integers are the numbers $\ldots - 3, -2, -1, 0, 1, 2, 3, \ldots$, denoted by $\mathbb{Z}$. The numbers $0, 1, 2, 3, \ldots$ are usually referred to as *non-negative integers.* The terms *whole numbers* or *counting numbers* seem to be confined to school use with no agreement on their definitions. $\square$

ALAN TURING AND THE UCT CONNECTION

The computer pioneer and codebreaker Alan Turing is today very much in the news, with the screening of the movie *The Imitation Game*, in which Benedict Cumberbatch plays Turing.

Before the Second World War, Alan Turing had established himself at Cambridge as a pioneer of theoretical computer science and artificial intelligence. At the outbreak of the war he led a team of mathematicians at the secret codebreaking centre at Bletchley Park, tasked with the challenge of cracking the Nazi's "unbreakable" Enigma code. In a massive and brilliantly planned attack, Turing and his team broke the code, furnishing such valuable information on the German war machine that it is today generally agreed that they shortened the war by four years. This is the story that is generally known, and recounted in the movie.

What is not generally known is that, both before and after the war Turing tackled a completely different mathematical problem, in collaboration with the South African mathematician Stanley Skewes. The problem concerned the mysteries of prime numbers and the Riemann zeta function. To this day the greatest unsolved problem in mathematics is the Riemann Hypothesis.

Skewes and Turing met in Cambridge in the 1930s, where they were students in King's College. Apart from their interest in mathematics, they were keen oarsmen, and rowed in the King's first boat. With Turing at bow and Skewes at number 2, they were able to talk about the zeta function during training sessions on the Cam.

Stanley Skewes, known to everybody at "Sammy", was a research student of the famous mathematician Jack Littlewood (who, incidentally, attended Wynberg Boys' High School). Inspired and led by Littlewood, Skewes achieved fame in 1933 for a piece of highly technical work on the Riemann

Hypothesis in which an enormous number was crucial:

$$10^{10^{10^{34}}}.$$

For over 40 years the "Skewes Number" featured in the Guinness Book of Records as the biggest number ever to have had a significant application in mathematics. It lost its record high number status in 1977.

Returning to South Africa after getting his PhD at Cambridge, Skewes was on the staff of the Department of Mathematics and established himself as an excellent lecturer, a popular Warden of Driekoppen Residence (now Kopano), and a founder of the UCT Rowing Club. He retired in 1965, and died in 1988.

Skewes and Turing met again in Cambridge after the war, when Skewes was on a sabbatical from UCT. Shortly after that meeting Turing left for Manchester University, where one of the earliest stored-program computers was being developed, the Manchester Mark 1. Turing was the first to tackle the Riemann Hypothesis with a computer.

The Skewes-Turing friendship and collaboration continued, and they wrote a joint paper which, however, was not published until many years later.

A letter from Turing to Skewes, dated 9 April 1953, was found in Skewes' papers after his death. In the letter, Turing reminisces about rowing, speculates on the nature of existence and the philosophy of mathematics, and complains about the niggling details of computer programming. There is no reference or hint in the letter about the personal tragedy that had beset Turing earlier that year: his conviction for homosexuality and subjection to hormonal treatment imposed by the court, which drove him to suicide in June 1954. □

MANURE MATHS

> This week the Cape Town International Convention Centre is stabling 200 horses that are being sold on the annual Cape Premier Yearling Sale. ... Event logistics manager Ann Dalton said that between 800 and 900 cubic metres of horse manure were moved out of the stables twice a day, as well as 2000 cubic metres of straw.
>
> *The Times*, 23 January 2015

Eight cubic metres of manure per horse per day? — Editor □

PROOFS, PLEASURES AND PUDDINGS

John Webb
UCT Summer School
January 2012

Deductions

Proofs are the glue that holds mathematics together. This is the legacy we have inherited from the Greeks. Other civilizations such as Egypt, Babylon, India and China developed significant mathematical ideas, but did not have the concept of proof. Greek mathematics with its insistence on rigorous proof is still today the benchmark of success.

A proof is a connected sequence of logical deductions, starting with certain assumptions (axioms) and ending with the final result, the theorem.

Let's start with the deduction. Aristotle gave us classical deductive logic. If we take it for granted that all men are mortal, he said, and then observe that Socrates is a man, then Socrates is necessarily mortal.

That seems obvious, but the true nature of deduction is not always understood. Just consider the most famous fictional expositor of deductive logic: Sherlock Holmes. When they first meet (in *A Study in Scarlet*) Holmes staggers Watson (not for the last time) by remarking that the good doctor has evidently recently returned from Afghanistan. Watson presses Holmes to explain, which he does, as follows: "Here is a gentleman of a medical type, but with the air of a military man. Clearly an Army doctor, then. He has just come from the tropics, for his skin is dark, and that is not the natural tint of his skin, for his writs are fair. He has undergone hardship and sickness, as his haggard face shows clearly. His left arm has been injured. He holds it in a stiff and unnatural manner. Where in the tropics could an English army doctor have seen much hardship and got his arm wounded. Clearly in Afghanistan."

Sherlock Holmes became famous for this type of reasoning, which is very often referred to as "deductive". Aristotle would not agree. While Afghanistan is indeed a possible answer to Holmes' question, an equally plausible guess would have been Natal. The date of the first meeting of Holmes and Watson was, after all, the early 1880s, and the British Army had been quite heavily involved in affairs such as Isandhlwana, Rorke's Drift and Majuba just before then. Watson could have got his suntan, and his injury, in Natal.

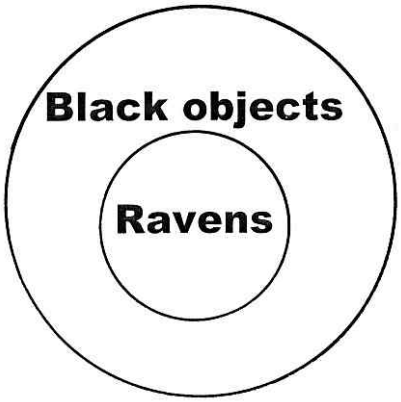
The type of reasoning used by Sherlock Holmes, Hercule Poirot, Lord Peter Wimsey, Father Brown, Nero Wolfe, Miss Marple and all fictional detectives from then on, is seldom deductive, more often inductive. Induction is the process of going from some body of empirical information to a hypothesis which implies that information, but is not logically implied by it. Deduction, as in Aristotle's syllogism, goes from the general "All men are mortal" to the particular "Socrates is mortal".

Deductive reasoning was wrapped up by Aristotle, and is sterile. It validates arguments, but does not create them. Inductive reasoning is how we discover new facts about the world. We note that the Sun rises every day, and predict that it will rise tomorrow, and nobody will query our reasoning. But consider Halley's comet, which is subject to the same laws of astronomy as the Sun. Even though Halley's comet has reappeared every 75-76 years since 240 B.C., its last appearance in 1986 was decidedly unimpressive, and whether it will be observed in any decent form in 2061 is doubtful.

Induction

Deduction reasons from the general to the particular. Induction, reasoning from the particular to the general, is a thorny problem for philosophers. Let us consider Hempel's Paradox.

After a series of observations of birds, we notice that every raven we have seen has been black. From this empirical observation we construct a hypothesis: "All ravens are black." This is a perfectly reasonable and scientifically respectable hypothesis which no ornithologist would question, but it is not a logical deduction from the evidence. After all, ornithologists agreed that all swans were white, until they discovered *cygnus atratus*, the Australian black swan.



Back to the blackness of ravens. Even if we examine all the living ravens in the world and find them all to be black, our hypothesis embraces all dead

and yet unborn ravens. Nor, says Carl Hempel, can we point to a bird that is both black and a raven and claim it as an instance confirming our hypothesis. For the statement "All ravens are black" is logically equivalent to the statement "All non-black things are non-ravens." A confirming instance of the latter statement is a green caterpillar: it is not black, and not a raven, but nobody would accept a green caterpillar as support for the statement that all ravens are black. Indeed, a green caterpillar is a confirming instance of the hypotheses "All ravens are white." Worse, the statement "All ravens are black" is logically equivalent to "Anything is either black or not a raven" and this latter statement is confirmed by any black object (raven or not) as well as any non-raven (black or not).

There is another problem about confirming instances. From the observations of the heights of millions of people, we construct the hypothesis "All humans are less than four metres tall." Every person under four metres is then a confirming instance of our hypothesis. But suppose we come across a man who is 3 meters and 99 centimetres in height. Far from confirming our hypothesis, this instance shakes it severely.

Mathematical Induction

So much for the philosopher's induction. In mathematics, there is an important method of proof known as *mathematical induction*. It was used implicitly by Euclid, as we shall see, but the first explicit exposition of the idea can be found in a 1573 textbook by Franciscus Maurolycus. He sets out to prove that the sum of the first n odd numbers is n^2:

$$1 + 2 + 3 + \ldots + (2n - 1) = n^2.$$

Maurolycus begins his argument by noting that (in our modern notation)

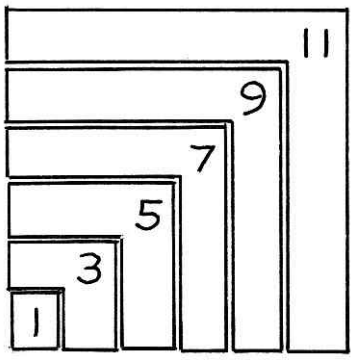
$$n^2 + (2n + 1) = (n + 1)^2.$$

This is his Proposition 13.

Freely translated, his argument now runs as follows:

> "By Proposition 13 the first square number (unity) added to the following odd number (3) makes the following square number (4); and this second square number (4) added to the third odd number (5) makes the fourth square number (9); and likewise the third square number (9) added to the next odd number (7) gives the next square number (16). And so successively to infinity the proposition is demonstrated by repeated application of Proposition 13."

The Greeks had demonstrated this result by geometry: at each stage the extra odd number, which they called a *gnomon* (because it looks like the L-shaped centre-piece of a sundial) to the previous square, to get the next square.



In his classic essay "A Mathematician's Apology" G.H. Hardy cites two proofs which to him epitomise mathematics. One of them is the irrationality of the square root of 2, discussed below. The other is the proof that there are infinitely many primes. Both date back to Euclid, and both use mathematical induction.

Let's look at Euclid's proof that there are infinitely many primes. Euclid did not put it that way, because the careful Greek mind avoided references to infinity. What Euclid actually states is "Prime numbers are more than any assigned multitude of numbers." In other words, for each natural number n we can find n distinct primes.

This statement is obviously true for $n = 1$: just note that 2 is a prime number. Now suppose we can find n primes $p_1, p_2, \ldots, p_n$. We are going to show that there is another one. Here comes Euclid's inspired observation: consider the number

$$N = p_1 p_2 p_3 \ldots p_n + 1.$$

There are now two options:

EITHER N is prime, in which case we have found our $(n+1)$th prime.

OR N is composite, in which case it has a prime factor q. This prime number cannot be any of the primes $p_1, p_2, p_3 \ldots, p_n$, for when they are divided into N there is a remainder of 1. So this prime number q is our $(n+1)$th prime.

Mathematical induction turns out to be an essential tool in the mathematicians's toolbox.

Reductio ad absurdum

Another method is proof by contradiction, or *reductio ad absurdum*. Suppose that the result **P** you are required to prove is NOT true. Then, by a sequence of logical steps, you arrive at something that is clearly wrong. The original assumption (not-**P**) is therefore seen to be false, and so **P** is true.

The second of Hardy's examples of a quintessential proof uses this method. It is the proof of the irrationality of the square root of 2. Let me illustrate this by proving that $\sqrt{2}$ is irrational. We assume the contrary, namely that $\sqrt{2} = \frac{p}{q}$. It then follows that $p^2 = 2q^2$. We then note that the number of factors of p^2 is even, while the number of factors of $2q^2$ is odd. Since a number cannot be simultaneously even and odd, we have a contradiction. So $\sqrt{2}$ is not rational.

Reductio ad absurdum relies on the Law of the Excluded Middle. If something is not false, i.e. not "not true", then it is true.

Why Proof?

Since the time of Euclid, proofs have been the nuts and bolts that hold mathematics together.

The words "proof" and "prove" are used today in a variety of contexts, some of which are not always clearly understood. For example:

> The proof of the pudding is in the eating.
>
> The exception proves the rule.
>
> Come live with me, and be my love
> And we shall all the pleasures prove.

In all these cases the word is being used in the sense of a test. The pudding must be proved (i.e. tested) by eating it. By the way, this metaphor is often misused as "The proof is in the pudding."

My second example is often misunderstood, and incorrectly applied. The exception does not establish the logical validity of the rule, but puts it to the test and shows its inadequacy. An exception is an example (what the mathematician refers to as a *counter-example* that shows that a certain conjecture is incorrect. And in my last example, Christopher Marlowe's Passionate Shepherd is not extending an invitation to his sweetheart to establish the validity of all the pleasures through a system of axioms, propositions and corollaries. Let's try them out, he says.

Actually, he does it rather more poetically. After listing the pleasures, he concludes:

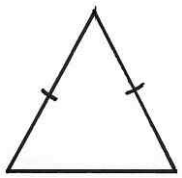
> And if these pleasures may thee move,
> Come live with me and be my Love.

"Proof" comes from the same root as "probe", an investigation or test. We use "proof" in the sense of a test when we *proofread* a document, or when we "put something to the proof."

A watch is "waterproof" or "shockproof" if can stand up to the test of being submerged in water, or knocked around.

This interpretation of "proof" as a test is relevant in mathematics. Let us start with the famous *Pons Asinorum* of Euclid (Proposition 5 of Book I of *The Elements*.

> The angles at the base of an isosceles triangle are equal.



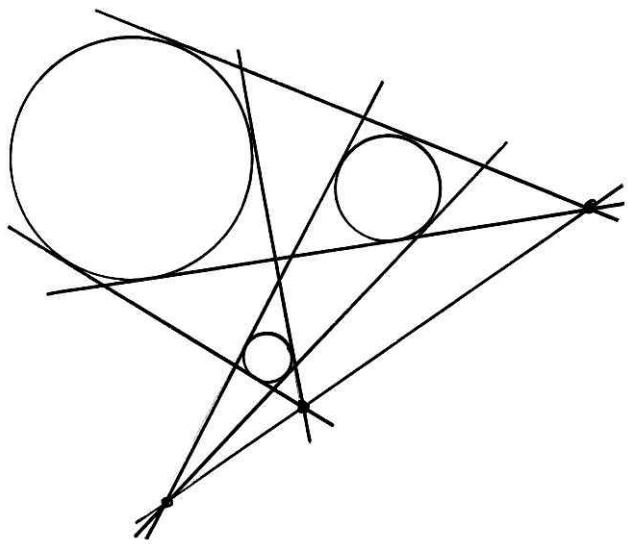
(It was called "the Bridge of Asses" because it was a benchmark of mathematical ability. If you could not get beyond Proposition 5, you were (mathematically) an ass.)

It's really a pretty obvious result. Just think of picking the triangle up, turning it over and putting it down on itself. Everything fits, with the angle on the right covering the angle on the left exactly, so the angles are equal. So why does Euclid make such a meal of it?

The point is that Euclid sets out with just a handful of basic assumptions, and his goal is to establish all geometrical knowledge on these axioms. The *Pons Asinorum* is then a test: can the theorem be established on the basis of the axioms? If it could not, then the axioms are insufficient.

Later proofs in geometry are different. Here's a theorem, discovered some two thousand years after Euclid, by Gaspard Monge.

Three non-overlapping circles of different radii are drawn in the plane. The three pairs of tangents are drawn, intersecting in three points. It turns out that the three points of intersection always lie on the same straight line. This is by no means obvious, and clearly needs a proof.



I want to give another example of a proof as a test. In the early 20th century Bertrand Russell and Alfred North Whitehead wrote their monumental *Principia Mathematica*, in which they set out to show that all of mathematics could be reduced to logic. It's a fearsome three-volume affair. After a 674-page Volume I, on page 83 of Volume II, Russell and Whitehead eventually manage to prove the result

$$1 + 1 = 2$$

and remark, with rare humour, "This result is occasionally useful."

Now if ever there was an opportunity to hold mathematicians up to ridicule for demonstrating something that is blindingly obvious as one plus one is two, it is here.

The ridicule, however, would miss the point. The proof that $1 + 1 = 2$ is not there to convince the reader of the truth of the result, but to test the logical structure on which the whole work is based. Proving that $1 + 1 = 2$ was a crucial test of the massive enterprise of proving that mathematics could be based on logic. Had Russell and Whitehead been unable prove the result, they would have had to go back to the basics and rebuild their axiomatic system.

If and only if

The two-letter word "if" is crucial to mathematics. Bertrand Russell wrote

> Pure mathematics consists entirely of such asseverances that, if such and such a proposition is true of anything, then such and such another proposition is true of that thing. It is essential not to discuss whether the first proposition is really true, and not to

> mention what the anything is of which it is supposed to be true. ... Thus mathematics may be defined as the subject in which we never know what we are talking about, nor whether what we are saying is true.

That little word "if" is often misused. Consider the following little domestic scene.

Child: "May I watch television?"
Mother: "If you have done your maths homework, you may watch television."

Later the mother finds the child watching television, with the maths homework not done. Mum is just about to blow her top, when the child responds: "You told me what I was allowed to do if my homework was done. But you did not put any restriction on my activities in the event that my homework was not done."

What the mother should have said was "If you have **not** done your homework, you may **not** watch television", or "You may watch television *only if* you have not done your homework."

There's a world of difference between "if" and "only if".

Of course, any child who can come up with this sort of logical argument may not need to do maths homework.

A double arrow is conventionally used to denote the implication P implies Q, "If P is true, then Q is true."

$$P \Rightarrow Q$$

The converse "If Q is true, then P is true" is denoted by

$$Q \Rightarrow P$$

which can also be verbalised as "P only if Q"

Combining the two symbols give

$$P \Leftrightarrow Q$$

which says that P is true *if and only if* Q is true.

Mathematicians abbreviate "if and only if" to "iff". It's a word that has made its way into the *Concise Oxford Dictionary.*

Existence proofs

Sometimes we can prove that there is a solution to a problem, but have absolutely no way of finding the solution. Here's a simple example.

One morning a monk sets out from his monastery at 6 am and walks to a secluded chapel at the top of a mountain, where he spends the day in fasting and prayer. The next morning the monk sets out at 6 am to return to the monastery by the same route.

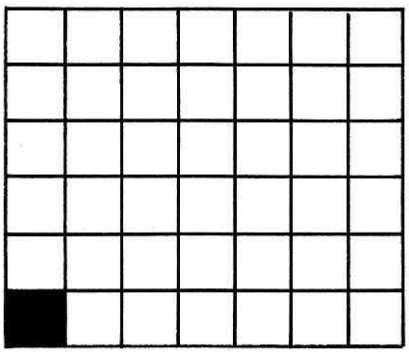
Prove that there is a point on the path at which the monk will be at exactly the same time on the two days.

Note that nothing has been said about the speed at which the monk walks. It could vary, and he could stop for rests. All that we are told is that he starts off each day at the same time, and follows the same path.

To prove that there is a point on the path at which the monk will be at the same time each day, consider an imaginary monk setting off up the mountain on day 2, mimicking the pace on day one of the real monk exactly. Meanwhile, the real monk is coming down. They must meet.

Chomp

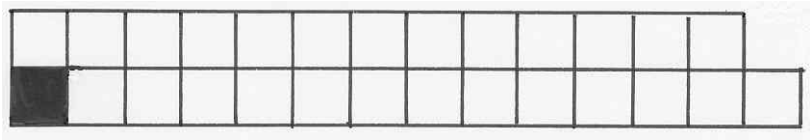
Another example of an "existence proof" is given by a simple game of stragey, called *Chomp.* You have a rectangular chocolate bar made up of smaller square blocks. The square in the bottom left corner is poisoned.



Two players take it in turns to chomp off a rectangular block of squares, from the top right. The loser is the player who is required to eat the poisoned square.

Since the game ends after a finite number of moves, and cannot be drawn, there has to be a winning strategy for one of the players.

In the simple case of 2-Chomp (with a $2 \times n$ bar), the first player can always win by chomping off the top right hand square.



Then, whatever chomp the second player makes, the first player can chomp off a piece to leave a shorter bar, with just the top right hand square chomped out.

The second player will eventually be faced with

which is a losing position.

This strategy does not work when you have a chocolate bar of more than two rows.

However, it turns out for any rectangular starting position bigger than the 1×1 case, it can be proved that there is always a winning strategy for the first player. This can be shown using a "strategy-stealing" argument: assume that the second player has a winning strategy against any initial move by the first player. Suppose then, that the 1st player takes only the top right hand square. By our assumption, the second player has a response to this move that will force victory. But if such a winning response exists, the first player could have played it as his first move, thus forcing victory. The second player therefore cannot have a winning strategy.

So we have proved that the first player always has a winning strategy. However, the proof gives no indication of what that strategy is. Computers can be programmed to find winning strategies for small bars of chocolate, but no general winning strategy is known.

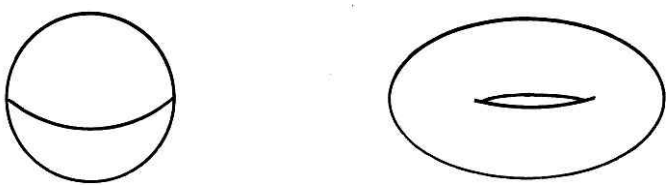
Chomp can also be played using three-dimensional blocks of chocolate, and once again there is always a winning strategy for the first player.

The Hairy Ball Theorem

Another example is the Hairy Ball Theorem. Suppose that you have a sphere, and from each point a hair is growing. Then it is not possible to comb the hairy ball flat in a smooth fashion, so that nearby hairs are combed in nearly the same direction. At least one hair will always be left standing up. This is, in fact, a difficult result to prove.

If, for example, you try to comb the hairs along lines of latitude, the hairs at the north and south poles will be left standing.

A practical consequence of the Hairy Ball Theorem is meteorological: at any time, there must be a point on the Earth's surface at which the wind is not blowing.



Now consider a hairy doughnut. There are at least two ways of combing it flat. That's a fundamental difference between the sphere and the doughnut. It's an important result in topology: the surfaces are significantly different. Another difference is that if you draw a circle on a sphere you can shrink it to a point. You can't do that to all circles on a doughnut.

Ben Trovato managed to bring this important mathematical distinction into one of his *Sunday Times* columns.

> A Japanese study showed that taking 5g of chorella before drinking can prevent hangovers 96% of the time. From what I can make out, chorella seems to be some sort of algae capable of multiplying faster than that Russian maths freak who turned down a medal and a million-dollar prize after proving the Poincaré conjecture which states that in three dimensions you cannot transform a doughnut shape into a sphere without ripping it, although any shape without a hole can be stretched or shrunk into a sphere. How would you like to go up in front of a crowd and explain your thinking on that one? No wonder he still lives with his mother.

The Russian maths freak was Gregorii Perelman, but that's another story that would take up a whole Summer School course.

WORDSWORTH ON INEQUALITIES

$\cdots$ High Heaven rejects the lore
Of nicely calculated less or more.

(Sonnet: *Within King's College Chapel, Cambridge*) $\square$

VARSITY READINESS TEST

How ready are you for university maths? This test focuses on some of the skills your lecturers at varsity will expect you to have. And that's not just the maths profs: courses in physics, chemistry and engineering all expect a high level of algebraic skills, arithmetical fluency and geometrical insight.

Give yourself two or more hours to tackle the problems below, without a calculator or formula sheet. Then turn to page 17 to check your answers. There is no pass/fail mark. Every wrong answer, however, spotlights a maths weakness hat needs to be fixed before you take on Maths I at your chosen university.

1. Simplify: $\dfrac{6^{10}+2^{11}+3^{11}+6}{3^{10}+2}$.

   (A) 999 (B) 1027 (C) 780 (D) 1297 (E) 2189

2. Which is the largest of the following five numbers?

   (A) $2\sqrt{33}$ (B) $3\sqrt{13}$ (C) $4\sqrt{15}$ (D) $5\sqrt{5}$ (E) $8\sqrt{2}$

3. The angles of a triangle are in the ratio $2:6:7$. What is the difference between the largest and the smallest angle?

   (A) $45°$ (B) $52°$ (C) $58°$ (D) $60°$ (E) $64°$

4. The area of a circle is decreased by 64%. By what amount is the radius decreased?

   (A) 32% (B) 36% (C) 40% (D) 50% (E) 20%

5. Jacob runs for 10 km at 6 km/h and then runs back along the same route at 10 km/h. What was his average speed, in km/h, for the whole route?

   (A) 7 (B) 7.5 (C) 7.75 (D) 8 (E) 8.25

6. The current VAT rate is 14%. If the VAT rate is increased to 21%, by what percentage will the cost of taxable items rise? (Round your answer to two decimal places.)

   (A) 6.00% (B) 6.14% (C) 6.52% (D) 6.83% (E) 7.00%

7. $(\sqrt{12}-\sqrt{3}-\sqrt{2})(\sqrt{12}-\sqrt{3}+\sqrt{2})$ is equal to

   (A) $\sqrt{2}$ (B) $\sqrt{3}$ (C) 1 (D) $\sqrt{7}/\sqrt{11}$ (E) $\sqrt{3}+\sqrt{2}$

8. $\dfrac{2.5^{x+1}+3.5^{x+2}}{4.5^{x+1}-3.5^{x}}$ is equal to

   (A) 5^x (B) 5^{x-1} (C) 5^{x+1} (D) 5^{1-x} (E) 5

9. If $\log 2 = a$ and $\log 6 = b$, then $\log(0.45)$ is equal to

(A) $2a + b - 2$ (B) $a + b + \frac{1}{2}$ (C) $2a + 3b - 2$ (D) $2b - 3a - 1$
(E) $ab - a - b$

10. The graph of the function $y = a.2^{-x^2} + b$ has a horizontal asymptote $y = 2$ and passes through $(-1, 4)$. Determine $a + b$.

(A) 2 (B) 4 (C) 6 (D) 8 (E) 10

11. The sum of n terms of an arithmetic series is $\frac{1}{2}n(3n + 11)$. Find the 9th term.

(A) 30 (B) 31 (C) 33 (D) 34 (E) 36

12. The 8th term of an arithmetic series is 38 and the 14th term is 68. Find the 11th term.

(A) 56 (B) 55 (C) 54 (D) 53 (E) 52

13. Solve for x: $\log x = \dfrac{\log 2. \log 64}{\log 4}$

(A) 8 (B) 32 (C) 16 (D) $\frac{1}{2}$ (E) 5

14. Which is the smallest of the following five numbers?

(A) $4\sqrt{5}$ (B) $3\sqrt{6}$ (C) $4\sqrt{3}$ (D) $5\sqrt{3}$ (E) $6\sqrt{2}$

15. Water from Tap 1 will fill a bath in 4 minutes, and water from Tap B will fill the same bath in 6 minutes. How long (in seconds) will it take to fill the bath if both taps are running together?

(A) 120 (B) 125 (C) 128 (D) 132 (E) 144

16. A wooden cube is painted red on five of its six sides, and white on the sixth side. It is then cut into 27 identical small cubes. How many small cubes have exactly one red side?

(A) 8 (B) 9 (C) 10 (D) 11 (E) 12

17. Solve the inequality: $\dfrac{x^2 + 3x - 4}{x^2 - x - 2} < 1$.

(A) $x < -1$ or $\frac{1}{2} < x < 2$ (B) $-4 < x < -1$ or $1 < x < 3$
(C) $x < -2$ or $1 < x < 4$ (D) $-1 < x < -\frac{1}{2}$ or $2 < x < 4$
(E) $x < \frac{1}{2}$

18. Find the circumference of the circle passing through $A(1, 2)$, $B(4, 0)$ and $C(-1, -1)$

(A) $2\sqrt{5}\pi$ (B) $2\sqrt{6}\pi$ (C) $\sqrt{26}\pi$ (D) $3\sqrt{3}\pi$ (E) $4\sqrt{2}\pi$

19. In triangle ABC $AB = 4$, $AC = 8$ and $\angle BAC = 90°$. If D is the midpoint of BC, determine the length of AD.

(A) $2\sqrt{5}$ (B) $3\sqrt{3}$ (C) $4\sqrt{3}$ (D) $5\sqrt{3}$ (E) $6\sqrt{2}$

20. Find the centre of the circle that passes through $A(3,1)$, $B(1,1)$ and $C(1,-3)$.

(A) $(-1,2)$ (B) $(2,-1)$ (C) $(2,1)$ (D) $(-2,-2)$ (E) $(1,-3)$

21. The graph of the equation $xy + 3x - 2y - 7 = 0$ is a rectangular hyperbola with asymptotes

(A) $x = 2$ and $y = 3$
(B) $x = 3$ and $y = -2$
(C) $x = -\frac{1}{2}$ and $y = \frac{1}{3}$
(D) $x = 2$ and $y = -3$
(E) $x = \frac{1}{3}$ and $y = -\frac{1}{2}$

22. Simplify: $\dfrac{6^n + 2^{n+1} + 3^{n+1} + 6}{3^n + 2}$.

(A) $2^{n-1} + 5$ (B) $2^n + 3$ (C) $3^n + 1$ (D) $6^n + 1$ (E) $3^{n-1} + 6$

23. The region inside the triangle with vertices $A(0,0)$, $B(4,1)$ and $C(1,3)$ is described by the inequalities

(A) $y \le 3x$, $x \le 4y$ and $2x + 3y \le 11$
(B) $3x \le y \le 4x$ and $2x - 3y \le 11$
(C) $x \le 3y$, $y \le 4x$ and $3x - 2y \le 11$
(D) $4y \le x \le 5y$ and $3x - 4y \le 11$
(E) $3x \le 4y$, $4y \le x$ and $4x - 2y \le 11$

24. $\log_5 500$ is equal to

(A) $3 + 2\log_5 2$ (B) $5 + \log_5 10$ (C) $10 + \log_5 50$
(D) $\log_5 200 + \log_5 300$ (E) $2\log_5 250$

25. The hypotenuse of a right-angled triangle is 15 and the perimeter of the triangle is 36. Find the area of the triangle.

(A) 60 (B) 64 (C) 72 (D) 54 (E) 48

Answers

BEDCB BCEDB BDACE BACAB DBAAD

VEDIC MATHEMATICS REVIVAL

After yoga and curry, India's next gift to the world could be the secret to lightning quick mental arithmetic, according to its Hindu nationalist government.

Vedic mathematics, a set of supposedly ancient techniques said to help even the most numerically challenged to conquer difficult sums, is surging in popularity as ministers claim they could hold the key to better education.

From next month, three Indian universities will offer courses in the techniques while home learners can watch a digital television channel devoted to the subject. Several thousand teachers have been recruited for private college courses.

Its supporters believe Vedic math could become a major export. Narendra Modi, the nationalist prime minister has attempted to claim the foundations of swaths of knowledge for India. It is argued that algebra, trigonometry, Pythagoras's theorem, the concept of zero and the decimal system all originated in India.

Vedic math is a series of shortcuts for complicated calculations, based on 16 verses discovered in the early 20th century.

One speeds multiplying large numbers by breaking them down to their common bases: To multiply 48 by 52, the numbers are broken into $(50 - 2)$ and $(50 + 2)$ and the square of the smaller number (4) is subtracted from the square of the larger (2500) to reach the answer of 2496.

Similarly, division is simplified by multiplying the denominator into a base 10 number: $44/25 = 176/100 = 1.76$.

(*Daily Telegraph,* 8 January 2015)

Editor's comment:

There is less to "Vedic maths" than meets the eye. Its rules were devised in the early 20th Century by Sri Bharati Krishna Tirtha Maharaj, and published posthumously in 1965 in *Vedic Mathematics*. Though it is said to be derived from 16 previously unknown and ancient Vedic *sutras*, the general agreement is that these sutras are not genuine, and (rather like the verses of Nostradamus) are sufficiently vague to allow all sorts of imaginative interpretation.

Vedic maths enthusiasts have nevertheless distilled from the sutras a collection of arithmetic tricks, applicable mainly in special cases. The examples given in the article above are typical.

Using the "difference of squares" method for multiplying 48 by 52 is fine, but it's not a sensible method for multiplying 50 by 46. The method says correctly that $50 \times 46 = 48^2 - 4$, which is only useful if you happen to know that $48^2 = 2304$. In this case, the calculation is much more easily done by noting that $50 \times 46 = \frac{1}{2}(100 \times 46) = 2300$. And the method crashes when the difference between the two numbers is odd.

As for division, the Vedic rule described in the article (convert the denominator to a power of ten by suitable multiplication) only works when the denominator contains no prime factors except 2 and 5. □

IMO2014 AND SAMF

In *Mathematical Digest* **177** (October 2014) the article *The IMO2014 Story* failed to mention the role of the South African Mathematics Foundation (SAMF).

The SAMF is the organizer of the annual South African Mathematics Olympiad, and since 2007 has been responsible for South African's participation in the Pan African and International and Mathematical Olympiads.

When the idea of bringing the IMO to South Africa was first mooted, it was natural that it should be under the aegis of the SAMF. After the appointment of the University of Cape Town as the IMO2014 host, the main functions of the SAMF office in Pretoria were fundraising, publicity and international relations (invitations and visas). Its role in the success of of the IMO was therefore central, and failing to recognize their contribution in the article is regretted. □

THE UCT MATHS COMPETITION

The annual University of Cape Town Competition will be held on the evening of Thursday 16 April on the UCT campus. It is open to all high schools in the Western Cape.

Online registration is open at `www.mth.uct.ac.za/competition`, closing on 25 March. □

EXAM CHEATING

Letter to the Editor, *Cape Times*, 6 January 2015:

The discovery of widespread cheating in the Senior Certificate Mathematics examination is yet another sad symptom of the state of mathematics teaching in South Africa.

Not only are the maths teachers at these guilty schools dishonest (and criminally so), but they are so incompetent that, even with access to the question papers, they are unable to supply their hapless pupils with correct answers to the problems.

Cheating is not confined to Senior Certificate exams. I have been involved in running mathematics competitions and Olympiads for many years, and cheating is not unknown in these events.

For a number of years I ran a nationwide Mathematics Talent Search, designed to identify and develop the skills of promising high school youngsters in preparation for maths Olympiads. I set up the Talent Search as a correspondence programme in problem-solving, available to all, and free. Sets of problems were sent to all participants, who submitted their solutions. Their marked papers were returned to them, with model answers and the next round of problems.

Because the Talent Search was self-paced, it was quite possible for students at the same school to be at different stages in the programme, with access to the model solutions sent to their friends who were further ahead. They could therefore copy the model solutions and send them in as their own.

This form of cheating was easily detected. The model solutions were often copied imperfectly, and helpful comments like "Hint: think of Pythagoras" were slavishly transcribed.

At a higher level, when national teams for the Pan African or International Mathematical Olympiad were being selected from among the front-runners in the Talent Search, the problem sets were quite difficult, and solutions were not published. Here we sometimes encountered a different form of cheating, sometimes even by pupils from some of our top schools. If they were stumped by a problem, they would post it on a problem-solving website as a challenge. When a whizz-kid somewhere else in the world posted a solution, it was copied by the cheater and blandly submitted as his own.

The cheating was then quickly picked up by the Talent Search marking team, who were regular readers of those same websites.

All these cases involved cheating by individuals, and were relatively infrequent. What is particularly worrying is that cheating is now widespread and institutionalized, with the connivance of maths teachers, school principals and examination invigilators.

Professor John Webb
Department of Mathematics and Applied Mathematics, UCT □

X+Y = IMO

X+Y is a 2014 BBC film directed by Morgan Matthews starring Asa Butterfield, Rafe Spall, and Sally Hawkins.

The film, inspired by the television documentary *Beautiful Young Minds*, focuses on a teenage English mathematics prodigy named Nathan (Asa Butterfield) who has difficulty understanding people, but finds comfort in numbers.

When he is chosen to represent Great Britain at the International Mathematical Olympiad, Nathan embarks on a journey in which he faces unexpected challenges, such as understanding the nature of love.

The film premiered at the Toronto International Film Festival on 5 September 2014. The European premiere was at the BFI London Film Festival on 13 October 2014, and the general release will be on 13 March 2015. □

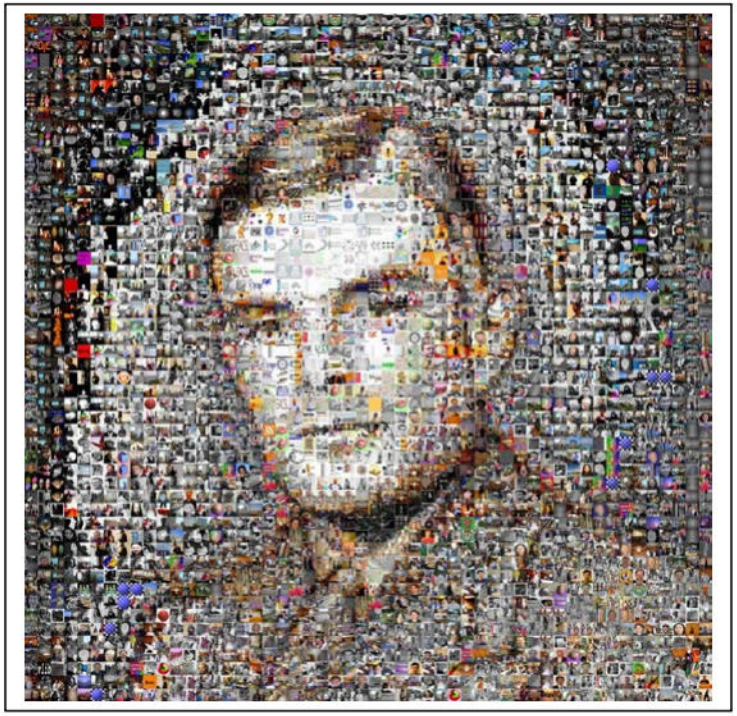
FAS AND MATHS

Foetal alcohol syndrome (FAS) is brain damage caused to the unborn child when a mother-to-be drinks during pregnancy. It is permanent and irreversible. The child's IQ is reduced significantly and mathematical ability is seriously impaired.

> "Brain scans of children with foetal alcohol syndrome showed that virtually their whole brain lit up to do very simple mathematical tasks. ... Children without alcohol damage use only the specific regions normally associated with solving simple mathematical problems."
>
> (*Sunday Times*, 10 January 2015) □

mathematical digest

No. 179 April 2015



Alan Turing

mathematical digest

Number 179 **April 2015**

In the common opinion certainty is to be found in two places and only two—religion and mathematics. In religion you believe by faith, in mathematics you prove what you want to prove. Outside these cosy domains, life is full of doubt and uncertainty and that is what people prefer because it is what they are used to. But it is a comfort to feel that, if the doubt and uncertainty become excessive, one can fall back on the eternal verities of religion or mathematics.

(J. L. Synge, in Kandelman's Krim)

Department of Mathematics and Applied Mathematics
University of Cape Town

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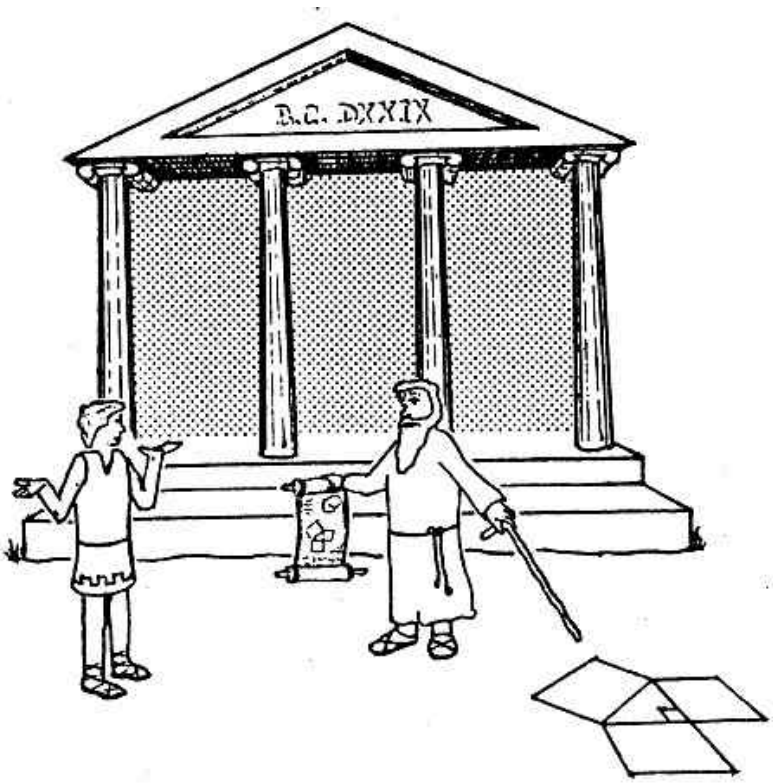


THAT'S ALL VERY INTERESTING, PYTHAGORAS — BUT WILL IT HELP ME GET A JOB ?

Cover: Based on an original portrait by D J Rogers, Arnold Beckman of the Swansea University Department of Computer Science, created a picture mosaic of Alan Turing.

FERMAT'S LAST THEOREM AT LAST

In a famous marginal note, the French lawyer and mathematician Pierre de Fermat announced that he had obtained a "marvellous" proof of the result:

> There are no positive integers $x,\ y,\ z$ and n, with $n > 2$, such that $x^n + y^n = z^n$.

Fermat scribbled this remark in the margin of the book he was reading (*The Arithmetic*, by Diophantus), on the page discussing the solutions of the Pythagorean equation $x^2 + y^2 = z^2$, which has, as is well known, infinitely many integer solutions. Tantalisingly, Fermat added that "the margin is too small to contain my proof."

For three hundred and fifty years mathematicians worked furiously to find a proof of "Fermat's Last Theorem", to no avail, until the Princeton mathematician Andrew Wiles, using very complicated methods and theory which Fermat certainly would not have known, eventually cracked it and published it in 1995, just 20 years ago.

Sensationally, Fermat's original proof has now been found, and it is indeed marvellously simple. Somehow it got mixed up with Fermat's legal papers and litigation records for the period 1660-1662, which were preserved and eventually fell into the hands of the famous nineteenth century American judge Oliver Wendell Holmes. The Oliver Wendell Holmes papers, housed in the Seeley G. Mudd Manuscript Library of Princeton University, had been mulled over by many legal scholars, but none of them had noticed the single sheet with Fermat's long-lost "marginal" proof.

A discussion of Fermat's Marvellous Marginal Proof was published in Princeton at the beginning of this month.

Reference

`http://www.princeton.edu/ aloo/fermat.` □

MATHEMATICAL SURPRISES

John Webb
UCT Summer School
January 2012

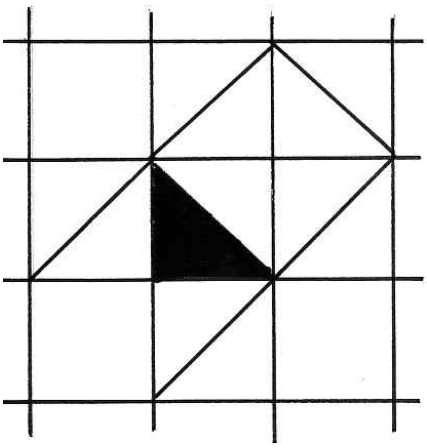
The Paradox of the Irrational

Mathematical ideas were developed in several ancient civilizations (Egypt, Mesopotamia, India and China), but it was the Greeks who gave us real mathematics, with proper logic and proofs. As soon as they got started, paradoxes reared their heads.

It all began with the Pythagoreans. They believed that everything could be described by numbers, by which they meant whole numbers

$$1, 2, 3, \ldots.$$

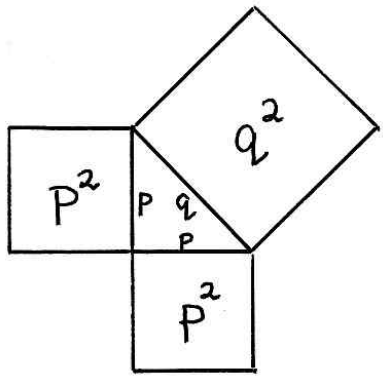
Fractions like 1/2, 3/7 etc were acceptable, because they were ratios of whole numbers; they were *rational*. But it was a real shock for the brotherhood to discover that a consequence of Pythagoras' famous theorem about the square on the hypotenuse led to the realisation that not all numbers were rational.



Pythagoras may well have first thought of his famous result while contemplating a tiled floor. The diagonal of a square tile of unit side has length $\sqrt{2}$.

One of the brotherhood then discovered that this number could not be expressed as a ratio, precipitating a crisis in their faith in numbers. Legend has it that the discoverer was drowned: a classic cover-up.

There are a number of ways of demonstrating this important anomaly. Here is a quick and slick argument.



Suppose, on the contrary, that the sides of the square are p units each, and the length of the hypotenuse q units. Then

$$\sqrt{2} = \frac{p}{q}$$

where p and q are whole numbers.

Multiplying up and squaring gives

$$p^2 = 2q^2.$$

Now imagine that the whole numbers p and q are broken down into their prime factors. Then p^2 has an even number of prime factors, while $2q^2$ has an odd number of prime factors. But there is only one way of expressing a whole number in terms of prime factors (apart from changing the order of the factors), so we have a contradiction. So p^2 cannot equal $2q^2$. That means that $\sqrt{2}$ is irrational.

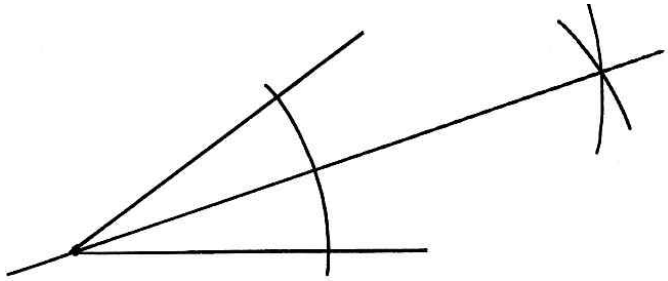
So the Pythagoreans were forced to admit that you can't do everything in mathematics with just whole numbers.

The Three Famous Problems

Turning in frustration to geometry, the Greeks, led by Plato and Aristotle, regarded the straight line and circle as perfect and fundamental, and decided that all their geometry had to be based on these two philosophically ideal figures. In practical terms, all geometrical construction had to be accomplished using only ruler and compasses.

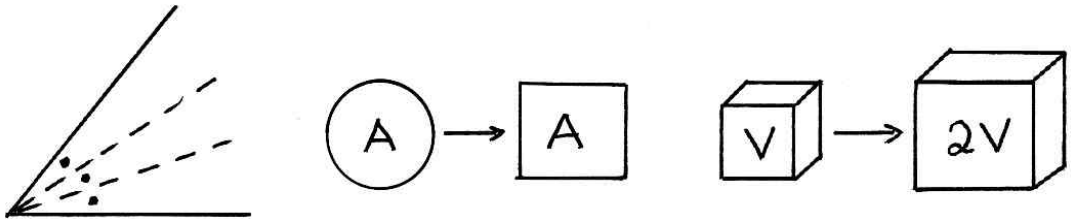
With these tools, they could perform wonderful constructions, dropping and erecting perpendiculars at will and drawing circles inside and outside triangles. Euclid's geometry (13 books of it) ends with the triumphant construction of a regular dodecahedron, with 12 regular pentagonal faces.

One of the easier constructions is the bisection of an angle: three twirls of the compasses and a quick straight line and you're home.



But trisection (dividing an angle into three equal parts) was elusive. The Greeks could produce quite good (but not 100% accurate) trisections with their methods of maths construction, and they devised clever mechanical instruments for doing the job exactly. But they were continually surprised that an exact ruler-and-compasses trisection evaded them.

There were two other impossibilities that baffled the Greeks: squaring the circle (constructing a square equal in area to a given circle) and doubling the cube (constructing a cube double the volume of a given cube. It took mathematicians two thousand years to work out why these constructions are impossible with ruler and compasses.



The Imaginary Square Root of Minus One

Fast forward through the Dark Ages to the Renaissance Italy of Michelangelo and Leonard da Vinci. There was a sudden surge of interest in algebra, and the Italian mathematicians made wonderful progress in developing the algebraic notation we use today in solving equations. The solution of the general quadratic equation was well known (in fact, the Greeks and even the Babylonian mathematicians before them knew how to solve quadratics (their methods were cumbersome by modern standards).

The formula for solving the general quadratic equation

$$ax^2 + bx + c = 0$$

is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Is there a similar sort of formula for solving the general cubic (third degree) equation

$$ax^3 + bx^2 + cx + d = 0?$$

It proved to be a difficult problem, but eventually a rather complicated solution formula was found. A strange feature of the formula is that it sometimes throws up a mysterious concept: the square root of minus one. Such an object makes no sense in the real world, for square numbers are never negative. But if you just grit your teeth, denote this mysterious thing by i and carry on with your algebra as usual, everything miraculously works. It was only in the 19th century that a logical foundation for i was found, by creating a two-dimensional system of "complex" numbers, which soon became an indispensable tool of mathematics. Today's electrical engineers, in particular, can't do without them. Nor can physicists, in their development of quantum mechanics.

String theorists (mathematicians who try to reconcile the mathematics of the small – quantum theory – with the mathematics of the large – relativity) are not happy with just the two-dimensionality of complex numbers. They employ four-dimensional quaternions and eight-dimensional octonions – exotic number systems that were formulated in the 19th century as a generalization of complex numbers and are now at last finding a use.

The Calculus Crisis

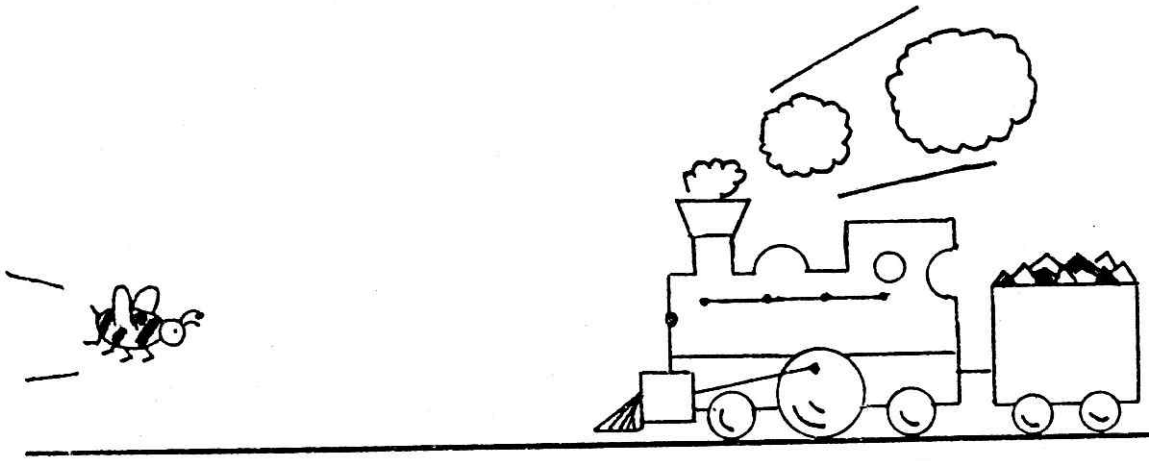
In the 17th century Isaac Newton in England and Gottfried Leibniz in Germany came up with the idea of calculus, which comes in two forms: differential (finding the slope of a curve) and integral (finding areas). The basic ideas of calculus rely upon the concept of a limit. It's a plausible concept, until you think hard about it. And then you have to think VERY hard to convince yourself that it is indeed plausible.

In the 18th Century mathematicians, physicists and astronomers used the new calculus with its subtle limits freely, but generally correctly, and with enormous success, while not really understanding what they were doing. Only after a hundred years of sustained effort was the concept of a limit finally spelt out carefully and correctly. It's no wonder that our students today find the theory of limits rather difficult to grasp.

Let me use a little story to illustrate the problem.

Imagine a locomotive steaming down a railway track at 100 km per hour. Further down the track is a bee, flying at 1 km per hour towards the

locomotive. They meet, and in an instant the bee is transformed to a smear on the front of the locomotive.



What is the velocity of the bee at the moment of impact? Since it was negative before impact (relative to the locomotive) and positive afterwards, it must be zero at the moment of impact. But at the moment of impact the bee is firmly glued to the front of the locomotive, which is travelling at 100 km/h and is scarcely slowed down by the impact. How can the bee be both stationary and moving at the moment of impact?

Infinite Sets and the Continuum Hypothesis

At the end of the 19th century two different crises arose: one in the consideration of infinite sets, and another in mathematical logic.

The Greeks had carefully avoided the concept of infinity, both in their philosophy and in their mathematics. The German mathematician Georg Cantor nailed infinity down, and in the process showed that there are different levels of infinity: some sets could be more infinite than others. A number of sceptical mathematicians found that difficult to swallow.

The first level of infinity is given by the counting numbers 1, 2, 3, ... Cantor called such a set *countably infinite*. He further showed that the set of all real numbers formed a bigger set, in a sense that he made quite precise. He labelled the real numbers "uncountable".

Was his genius a spark of divinity?
When Cantor thought hard for a minute, he
    Said "Not all things, I bet
    Form a countable set —
There is more than one kind of infinity."

Cantor then posed the question of whether there is a set of real numbers that is bigger than the set of whole numbers, but smaller than the set of all real numbers. Cantor tried to find such a set, and failed. He then tried to prove that no such set exists, and failed. Other mathematicians tried and failed in the same way, and the problem became known as the *Continuum Hypothesis*. Only in the 1960s was the issue resolved: the proposition is undecidable. It can be proved that the Continuum Hypothesis cannot be proved true, nor can it be proved false. To prove that something cannot be proved is certainly paradoxical.

The Logical Basis of Mathematics

Meanwhile, Bertrand Russell and others had been trying to build mathematics from a basic system of logic, in the course of which the natural and intuitive concept of a *set* arose. A set, after all, is "just a collection of objects", as school textbooks naively put it.

Russell devised a paradox which showed that the set concept is not at all natural or intuitive.

Consider the set of adjectives in the English language. There are some that have the curious property of describing themselves. "English" is an English word, "Afrikaans" is an Afrikaans word but "French" is not a French word, "red" is not a red word, and "long" is certainly not a long word. It's an amusing little word game to find adjectives that describe themselves. Here are some:

polysyllabic **unhyphenated** **misspellt** **twelve-letter**

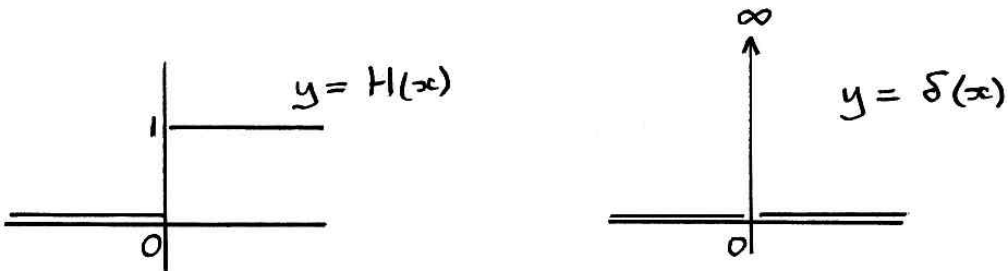
Now suppose we classify all adjectives in English into these two sets, for which we invent new labels: *homolytic* for adjectives that describe themselves, and *heterolytic* for those that don't. Most adjectives turn out to be heterolytic.

Here comes the paradox. Into which category do we put the adjective we have just coined, "heterolytic". If heterolytic is a heterolytic word, it describes itself and should be put in our first category of homolytic words, so must be homolytic. But if heterolytic is a homolytic word, it does not describe itself, so must be in the heterolytic category. Whichever way you turn, you meet a contradiction.

So the concept of a set is not really as simple as it may seem.

The Dirac Delta Function

In the early 20th century physicists tackling partial differential equations invented two simple but very useful tools, which they then proceeded to use in a most unprofessional way. The Heaviside function $y = H(x)$ represents the abrupt change when you switch on an electrical circuit. Its derivative, the Dirac delta function $y = \delta(x)$, is a curious beast that is zero everywhere except at 0, where it takes on a value so infinitely large that the area under the "curve" is 1.



These devices were used with great effect by physicists and engineers, with the pure mathematicians tut-tutting over their cavalier use of a piece of mathematical nonsense. But the practical-minded scientists carried on regardless, for the simple reason that their methods worked. Only in the 1940s was the true nature of the delta function understood and explained in the context of the rigorous theory of distributions.

All these paradoxes presented serious problems in the proper understanding of mathematics, and their resolution led to serious advances in that understanding.

Let's lighten up.

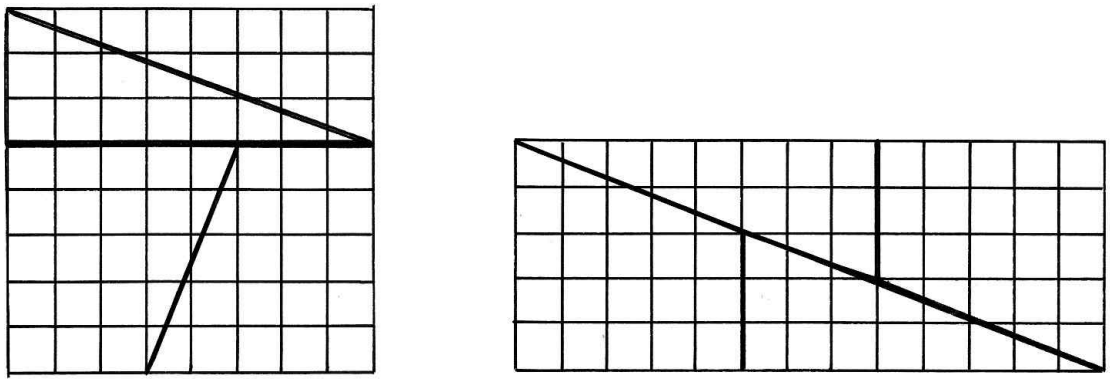
Unlike paradoxes, which require deep thought to unravel, fallacies are light-hearted examples of mathematical reasoning that deceptively lead to absurd results. They are the mathematician's jokes. You follow an apparently correct line of reasoning until suddenly you are confronted with a surprise: something ridiculous.

Fallacies can also be regarded as a conjuring tricks. The conjuror displays his hat, and shows you that it is empty. After waving his magic wand over the hat, he pulls a rabbit out. The game is to uncover the deception.

Good fallacies are excellent teaching tools, because they have the element of deception and fun. But they also carry a health warning: even the slightest misuse of maths can lead to a disaster. Careless thinking can damage your mathematical health.

64 = 65

Look at the 8×8 square. It has been cut into four pieces that have been rearranged to form a 5×13 rectangle. It follows that $64 = 65$.



The resolution of this little puzzle lies in noticing that the pieces don't quite fit together. The diagonal of the rectangle is not a straight line, and there is a long thin lozenge-shaped hole in the rectangle along the diagonal, of area exactly 1.

Warning: Beware of geometrical arguments that rely on cutting up and rearranging figures.

The Pizza Problem

This little puzzle never fails to baffle.

Three office girls agree to share a pizza for lunch, and send a messenger out to buy a R60 pizza, each giving him R20. At the pizza place the messenger finds that the R60 pizza is that day on a special, at R55. He buys it, and takes it back to the three girls, but gives them each only R1 change, pocketing R2 for himself.

So each girl paid R19. Now three pizzas at R19 is R57. Add the R2 kept by the dishonest messenger, and you get R59. But the original price was R60. What happened to the other rand?

The issue here is not mathematical, but a matter of simple bookkeeping. The statement

$$3 \times 19 + 2 = 59$$

is mathematically correct, but does not represent the transaction. The R2 should be subtracted, not added:

$$3 \times 29 - 2 = 3 \times 20 - 5 = 55$$

is a correct representation of the transaction.

Salary Surprise

Suppose you are offered two jobs.

Job A offers a salary of R100 000 per annum, with increases of R4 000 per annum.

Job B offers the same initial salary of R100 000 per annum, but with increases of R1 000 every six months.

Which should you take? At first sight, Job A looks better, since a single increment of R4 000 looks better than two increments of R1 000. However, it is worthwhile to look at the numbers in detail.

In the first year, Job A will pay you a total of R100 000. But Job B will pay R101 000: R50 000 in the first six months of the year, and R51 000 in the second half of the year. So at the end of the first year Job B has paid a total of R101 000, and is therefore R1 000 ahead of Job A.

In the second year, Job A will pay you a total of R104 000. Job B will pay R52 000 in the first half-year, and R53 000 in the second half-year, for a total of R105 000 for the year. So at the end of the second year Job B is once again ahead by R1 000.

In the third year, Job A pays R108 000, but Job B pays R54 000 + R55 000 = R109 000, and is again R1 000 ahead of Job A.

The pattern continues. Every year Job B will be ahead of Job A by R1 000 per annum.

Bowling Averages

The village cricket team was playing an important match, and a keen supporter offered a prize for the best bowling performance by a member of the team. The two star bowlers, Trundle and Bouncer, were keen contestants for the prize. In the first innings Trundle took 3 wickets for 17 runs, averaging 5.67 runs per wicket, and Bouncer took 7 wickets for 40 runs, averaging 5.71 runs per wicket. So Trundle was the better bowler in the first innings by a small margin.

In the second innings Trundle took 7 wickets for 110 runs, averaging 15.71 runs per wicket, and Bouncer took 3 wickets for 48 runs, averaging 16 runs per wicket. It was again very close, but Trundle was again judged the better bowler in the second innings.

The prize for the best bowler was, however, to be awarded on overall match figures. When the numbers were added, Trundle had taken 10 wickets for

127 runs, at a rate of 12.7 runs per wicket. Bouncer had taken 10 wicket for 88 runs (an average of 8.8 runs per wicket).

So even though Trundle's figures were better in both innings, Bouncer had the best overall bowling average, so won the prize.

This story carries the warning: be careful with averages. They can be deceptive. An average is a single number extracted from a large set of data to give an overall impression of the data. Calculating an average means losing information, so it should not be surprising if averages sometimes behave in unexpected ways.

Football League Tables

When football leagues were first started, a team was awarded two points for winning a game, and one point for a draw. Because a draw is a rather boring result, it was decided to encourage teams to play for a win rather than the safety of a draw by increasing the points for a win to three.

The change in awarding league points had a surprising result. It turns out that it is possible for a team to come bottom of the league in the old system, and top in the new system.

Suppose there are 13 teams in the league, each playing every other team just once. One of the teams, Team A, wins five of its matches and loses the other 7. All other matches in the league are drawn.

With 2 points for a win, Team A scores 10 points. The teams that beat A score 13 points and the teams that lose to A score 11 points. On the old league table, Team A is bottom.

What happens under the second system? With 3 points for a win, Team A scores 15 points. The teams that beat A score 14 points and the teams that lose to A score 11 points. On the new league table, A is top.

Chess League Surprise

Imagine a Chess League of three teams, each consisting of three players. The players are numbered from 1 to 9, according to their ranking.

Team A consists of Players 1, 6, 8
Team B consists of Players 3, 5, 7
Team C consists of Players 2, 4, 9

By the simple-minded method of adding up their rankings, the three teams seem to be evenly matched.

In a tournament, the team members play Top Board, Second Board and Third Board, according to their rankings.

In the matches, Team B beats Team A, Team C beats Team B and Team A beats Team C. That doesn't seem logical.

Now suppose that in a different tournament, each player plays all the players in the other team.

When Team A meets Team B, Team A wins four games (1 beats 3, 5, and 7 and 6 beats 7) and loses the other five. So Team B beats Team A.

When Team B meets Team C, Team C wins four games (3 beats 4 and 9, 5 beats 9 and 7 beats 9) and loses the other 5. So Team C beats Team B.

When Team C meets Team A, Team C wins four games (2 and 4 beat 6 and 8) and loses the other five. So Team A beats Team C.

The moral here is that you should not expect a ranking system to be transitive, i.e. it may not follow the rule that "If A is better than B and B is better than C, the A is better than C".

There is a serious form of this paradox.

Arrow's Paradox

In an election with three or more candidates, all voting systems have some sort of built-in unfairness.

Arrows paradox, states that, when voters have three or more distinct options, no voting system can convert the ranked preferences of individuals into a community-wide (complete and transitive) ranking while also meeting a specific set of criteria. In short, the theorem proves that no voting system can be designed that satisfies these three "fairness" criteria:

- If every voter prefers alternative X over alternative Y, then the group prefers X over Y.
- If every voter's preference between X and Y remains unchanged, then the group's preference between X and Y will also remain unchanged (even if voters' preferences between other pairs like X and Z, Y and Z, or Z and W change).
- There is no "dictator": no single voter possesses the power to always determine the group's preference.

The theorem is named after economist Kenneth Arrow, who demonstrated the theorem in his Ph.D. thesis and popularized it in his 1951 book *Social*

Choice and Individual Values. The original paper was titled "A Difficulty in the Concept of Social Welfare". Arrow was a co-recipient of the 1972 Nobel Memorial Prize in Economics.

Gödel's Theorem

At the beginning of the 20th Century the renowned German mathematician David Hilbert set out an agenda of major problems for the mathematical world to tackle. Hilbert had a wider aim: to establish a compete system of axioms on which all mathematics could be based. He had already achieved this aim for Euclid's geometry. The task was now to do the same for the whole body of mathematics.

In 1931 a young Austrian mathematician Kurt Gödel shattered the dream. He proved that in any system of mathematics rich enough to include ordinary arithmetic, there would always be statements that made sense, but were undecidable: they could be proved neither true nor false. Such a statement could then be added as an axiom, but the enriched system would once again contain an undecidable proposition.

Some thirty years later it was established that the first problem on Hilbert's famous list, the Continuum Hypotheses, was one. The issue of whether there is a set of numbers that is bigger than the integers, but not as big as the set of all real numbers, cannot be proved either true or false. It is undecidable.

Gödel's Theorem came out at about the same time as the creation of quantum mechanics, in which the Heisenberg Uncertainty Principle states that if you cannot observe both the position and the momentum of a particle to a high degree of accuracy at the same time.

When Heisenberg went on vacation
His colleagues were in consternation.
    For while studies had shown
    That his speed was well known,
His position was pure speculation.

Those colleagues then carried out testing
And found a result interesting.
    They found his position
    With perfect precision
But couldn't be sure he was resting.

On a lighter note,

Q: How many Heisenbergs does it take to change a light bulb?

A: If we knew the answer to that question, we would not be able to find the light bulb.

Quite recently, we had heard news from the new super-collider in CERN that faster-than-light neutrinos have been observed, suggesting the possibility of time travel.

The barman says: "We don't allow faster than light neutrinos in here."

A neutrino walks into a bar.

There have been better jokes, but not many that manage to combine, in two lines, quantum mechanics, special relativity and time travel.

The weird features of the world of the quantum have had deep philosophical repercussions, and there has even been speculation about whether there is some deep connection between Heisenberg's Uncertainty Principle and Gödel's Undecidability Theorem.

Hilbert died in 1942, never accepting the implications of Gödel's achievement. The rest of us are happy, for Gödel has proved that mathematics will never be finished. Let's end on that very positive note. □

TURING'S MILLION-DOLLAR NOTEBOOK

A notebook of Alan Turing's that hadn't been seen in public until recently will be auctioned in April. The notes, which foreshadow some of Turing's later foundational work on computing and logic, were made while Turing was working in Bletchley Park on breaking the Enigma Code.

Andrew Hodges, author of *Alan Turing: The Enigma*, says: "This notebook shines light on how, even when he was enmeshed in great world events, he remained committed to freethinking work on pure mathematics."

Turing left the notebook to his close friend Robin Gandy, who added his own personal notes to the notebook and kept it until his death in 1995. Because of interest in Turing generated by *The Imitation Game*, the notebook is expected to sell for over US$1 million.

Financial Times, 19 January 2015 □

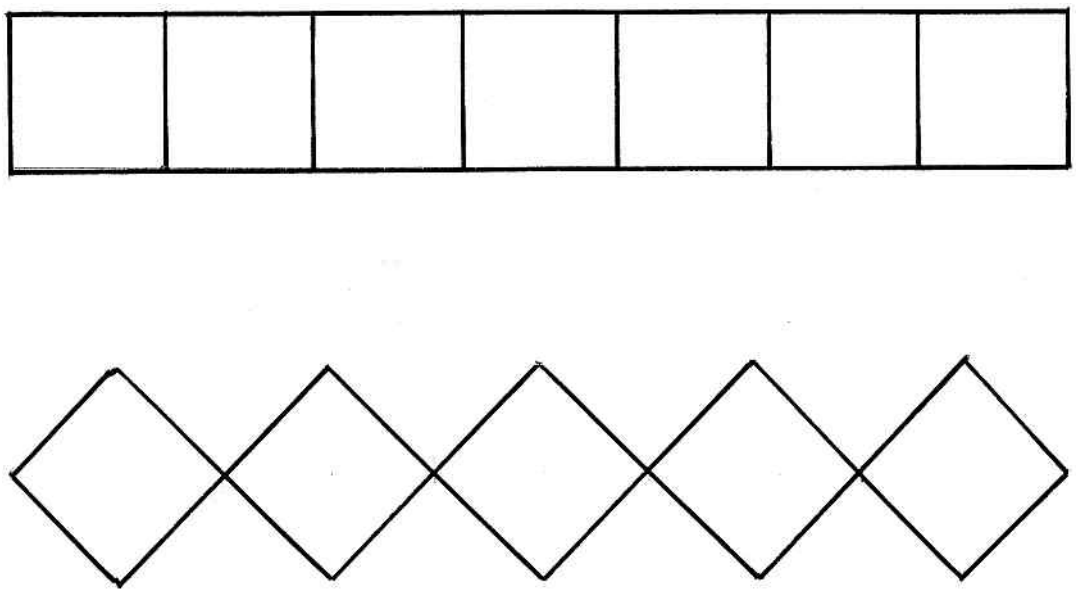
GEMS OF GEOMETRY

John Webb
UCT Summer School
January 2012

A geometrical proof that $\sqrt{2}$ is irrational

In my first lecture I mentioned the Pythagorean scandal: the irrationality of the square root of 2.

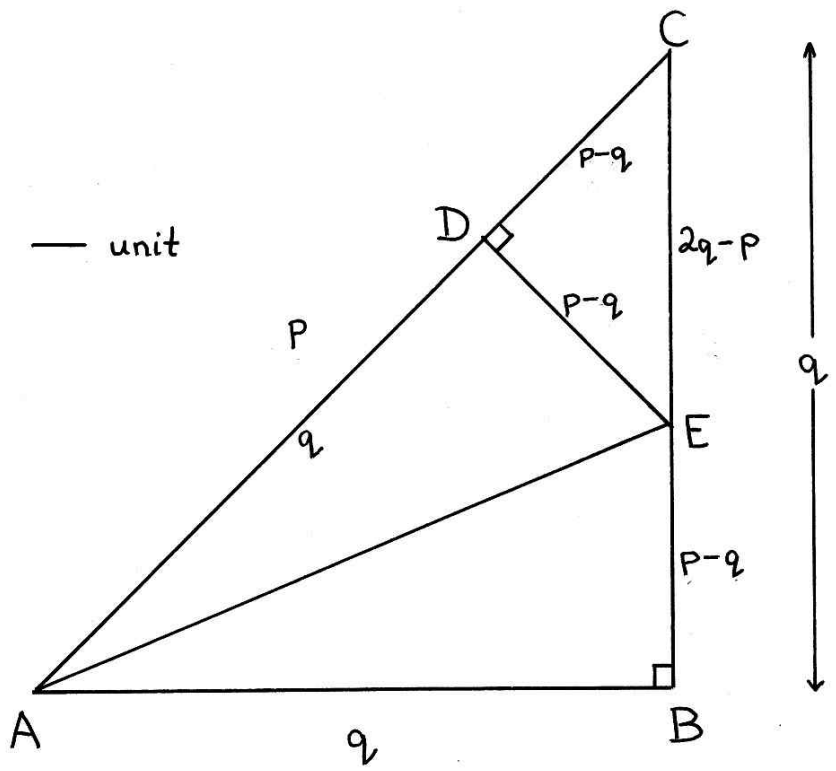
A practical description of this result can be given in terms of paving stones. With square stones, you can pave a path in two ways:



The path of seven square paving stones is almost (but not quite) the same length as a path of five diamonds. A path of 17 squares is even closer in length to a path of 12 diamonds, but again not exactly. However many stones you use, you cannot get a path of squares and a path of diamonds of exactly the same length. That is equivalent to saying that $\sqrt{2}$ is irrational.

I gave an argument that the square root of 2 is irrational by appealing to the prime factorization theorem. Here is a geometrical proof.

Suppose that there is a unit of length so that the hypotenuse AC of the right-angled triangle shown is p, and the sides AB and BC are of length q, relative to an agreed unit.



Mark off D on AC such that $AD = p$, and let the perpendicular from D meet BC in E. Then $CD = DE = p - q$.

Joining AE, we see that triangles ADE and ABE are congruent, so that $BE = p - q$. It follows that $CE = q - (p - q) = 2q - p$.

Now look at triangle CDE. Its sides are of integer length, measured in the same units. And it's a *smaller* right-angled isosceles triangle.

Repeating the construction, we can find a yet smaller triangle with sides of integer length, and yet another smaller triangle, and yet another, all with sides of integer length. But eventually we will get a triangle with sides of integer length less than the agreed unit of length, which is a contradiction.

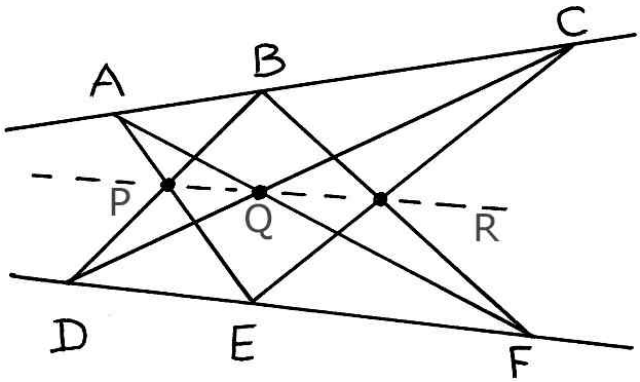
This, by the way is an example of a method of proof known as *infinite descent*, which is equivalent to mathematical induction.

The Theorem of Pappus

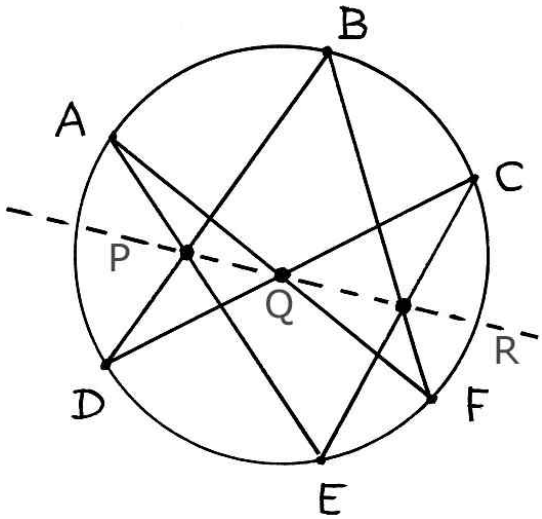
Some results in geometry are easy to believe. The Pons Asinorum, mentioned in the previous lecture, is one. Here's one that is really surprising.

Draw any two lines, and mark off three points A, B, C and D, E, F on each, as shown. Denote the intersections of
AE and DB by P
AF and CD by Q, and
BF and CE by R.

Then P, Q and R always lie on a straight line.



Now do the same with six points on a circle. The intersection points are again collinear.



This result was discovered by the French philosopher and mathematician Blaise Pascal, who gave his name to the unit of pressure, in newtons per square metre.

A pascal of pressure on top of your head
Is the same force that butter exerts on sliced bread.

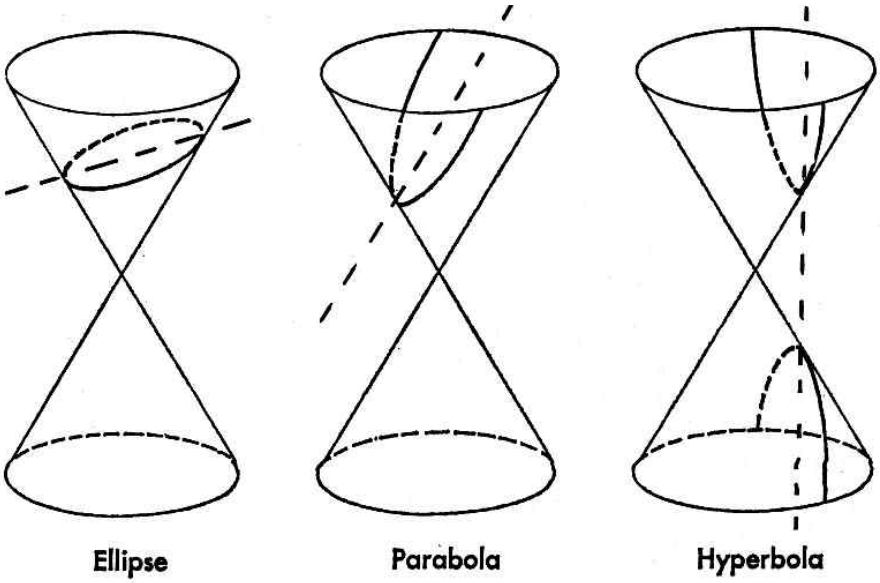
The pascal is a very small unit. So when we have our tyres pumped, we give the pressure in kilopascals.

The unit of force used to define the pascal is the newton, and most people would have no idea of its magnitude. Here's a tip. It's quite interesting that one newton is the weight of an average apple.

Back to Pappus and Pascal. The same theorem is true for six points on an ellipse, or on a parabola, or an a hyperbola.

The overhead projector can be used to see why Pascal's Theorem for a circle can be morphed into a theorem for the ellipse. Under projection, the circle becomes an ellipse, but straight lines remain straight and intersections are unaltered.

The ellipse, parabola and hyperbola are all curves in the family known as *conic sections*: they can be realised as the intersection of a plane with a cone.



The conic sections

When the plane is perpendicular to the axis of the cone, you get a circle.

When the slope of the plane is less than the slope of the side of the cone, the intersection is an ellipse.

When the slope of the plane is equal to the slope of the side of the cone, the intersection is a parabola.

When the slope of the plane is greater than the slope of the side of the cone, the intersection is in two parts: a hyperbola.

There is a quite interesting analogy here between the names of these figures, and figures of speech.

Ellipsis: understatement

Parable: analogy

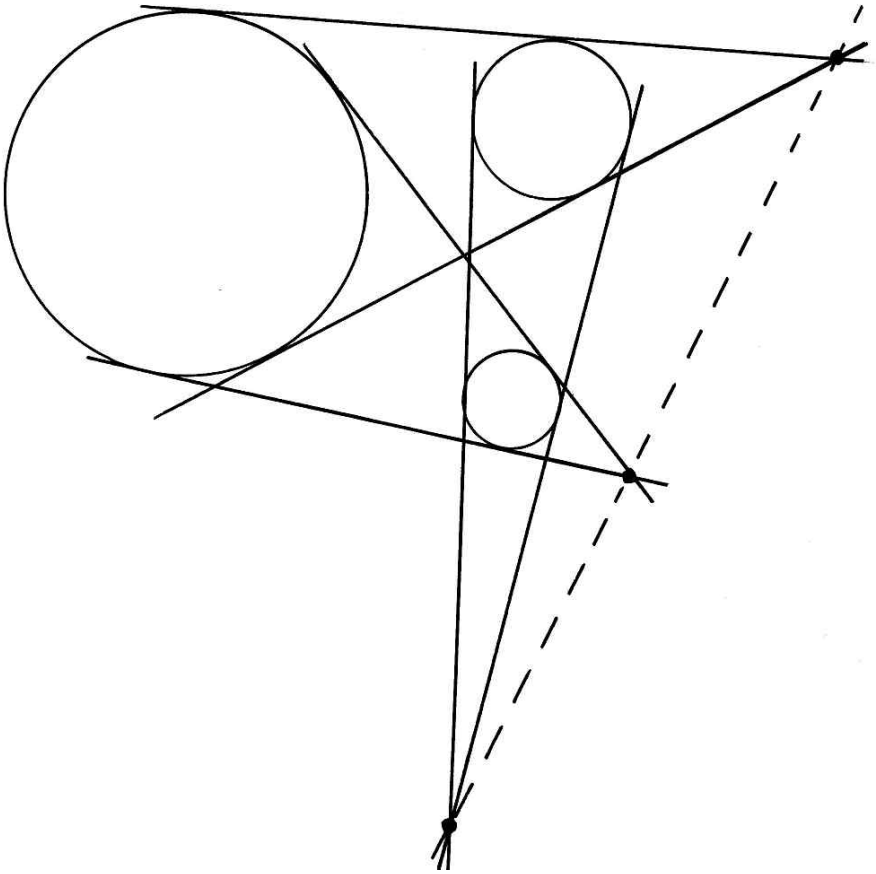
Hyperbole: exaggeration

The proof of this family of theorems about six points on a conic section can be done in one go, which I will not attempt because it takes a little time to develop the background tools. But let me give you the idea of how

it might work by demonstrating a related result, which I mentioned in my second lecture.

Monge's Theorem

This result is due to Gaspard Monge (1746-1818). Draw three disjoint circles in a plane, and draw the common tangents to each pair of circles. Then the three points of intersection are always collinear.



I will try to persuade you without writing anything on the diagram that this result is obvious.

Imagine instead three spheres lying on the plane of the paper, and instead of two lines tangent to a pair of circles, think of a cone enclosing two spheres. The vertex of the cone lies in the plane.

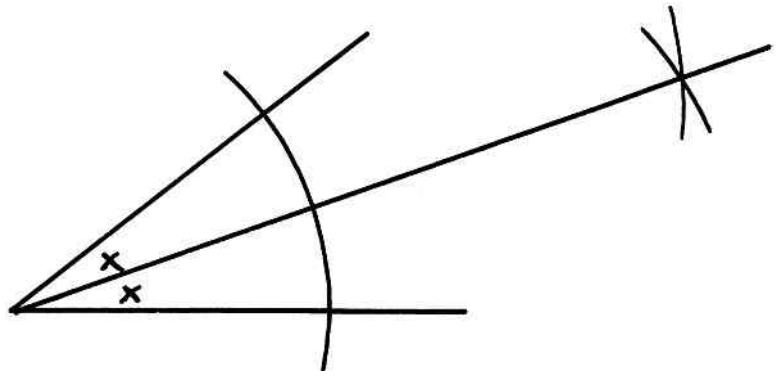
So we have three cones lying on the plane, enclosing the three spheres in pairs. The vertices of the three cones lie on the plane.

Now think of a second plane lying on top of these three spheres. The three cones are tangent to this plane also, and the three vertices lie on the second plane also.

Now consider the two planes. How do the planes intersect? In a straight line, which contains the vertices of the three cones.

Bisecting angles

Let's get back to bisecting an angle. There is a very simple ruler-and-compasses construction of the bisector of an angle.



If you bisect the three angles of a triangle, the bisectors all meet in the same point: they are *concurrent*. That's one of those mathematical words derived from Latin or Greek. It means "running together". Other technical words like concurrent, isosceles, hypotenuse, collinear, equilateral do not tell you what they mean unless you have a classical eduction. It is quite interesting that the corresponding Afrikaans words are quite clear as soon as you meet them:

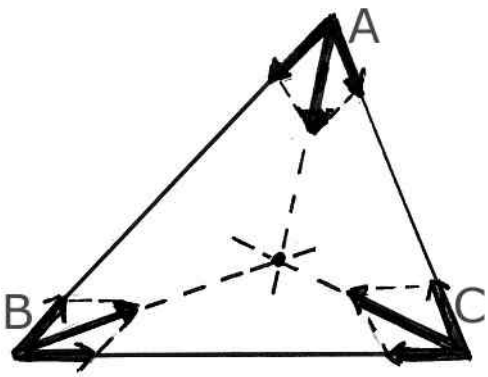
samelopend, gelykbenig, skuinssy, samelynig, gelyksydig

These words in Afrikaans come from the Dutch, and go back to Simon Stevin (1548-1620), a Flemish engineer and mathematician whose major claim to fame is the creation of decimal fractions. Stevin decided to write in his native Dutch, not Latin, and coined many words still in use today.

In other languages, mathematics is mathematiques, matematika, mathematik. Stevin coined the Dutch word *wiskunde*, and that's also the Afrikaans word for mathematics.

Back to the bisectors. Their concurrency can be proved reasonably simply by standard geometry techniques. Here is a proof by means of physics.

Put a force at A, directed towards B, and put an equal and opposite force at B. These forces are in equilibrium. Do the same at B and C, with forces of the same magnitude. The six forces are in equilibrium.

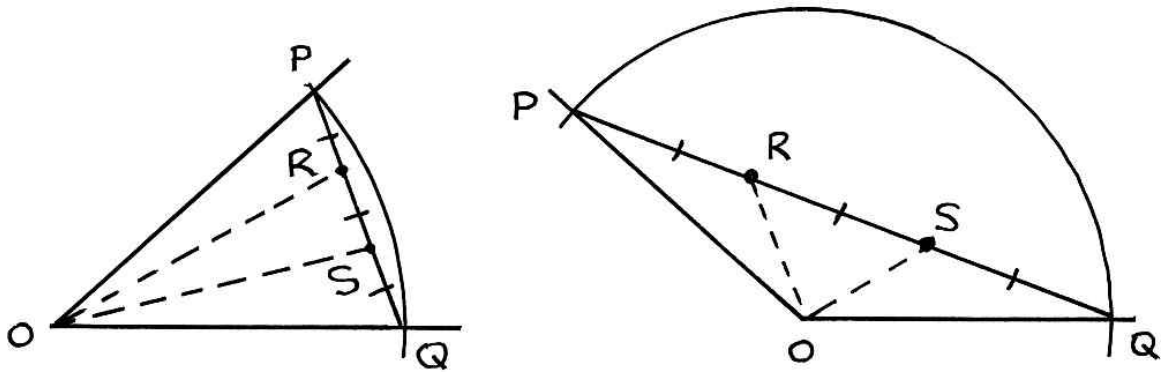


Now combine the two forces at A into a single force. Since they are equal in magnitude, their resultant is directed along the bisector of the angle at A. Do this for the other pairs of forces at B and C. Now look at the three resultants. These three forces are in equilibrium. So they have to meet in a point, since otherwise there would be a rotational force, and the triangle would not be in equilibrium.

After angle bisection, you would think that it would be a small step to finding a way of *trisecting* an angle.

If you set this as an exercise in a school geometry lesson, it won't take them long to offer this solution:

> Use the compasses to mark off $OP = OQ$ on the arms of the angle, join PQ and trisect it at R and S. Then OR and OS are the required trisectors.



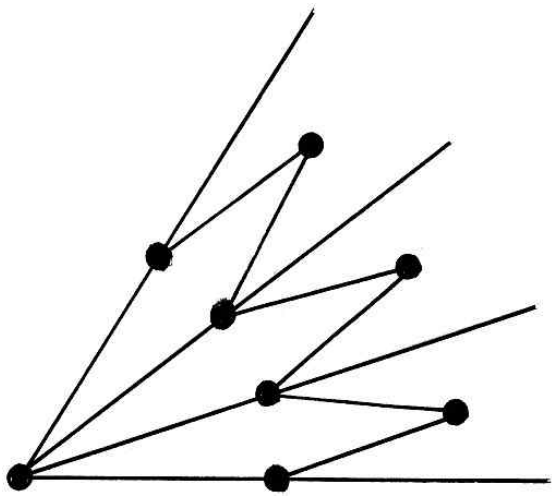
The trisection looks plausible for a small angle, but is not at all accurate for all angles, as is shown by repeating the construction for an obtuse angle, where it clearly fails.

It turns out that an accurate ruler-and-compass trisection is impossible. The impossibility was only proved in the 19th century. The key is to translate the basic Euclidean constructions using straight lines and circles

lines into algebra: solving linear and quadratic equations. These techniques will allow construction of square roots, or 8th roots, or 6th roots, etc. But it can be shown that a trisection amounts to finding a cube root, which cannot be done via quadratic equations.

What happens if you *trisect* the angles of a triangle? Although a ruler-and-compasses trisection is not possible, it is easy to devise gadgets that trisect angles.

It's not difficult to construct a gadget to trisect angles.



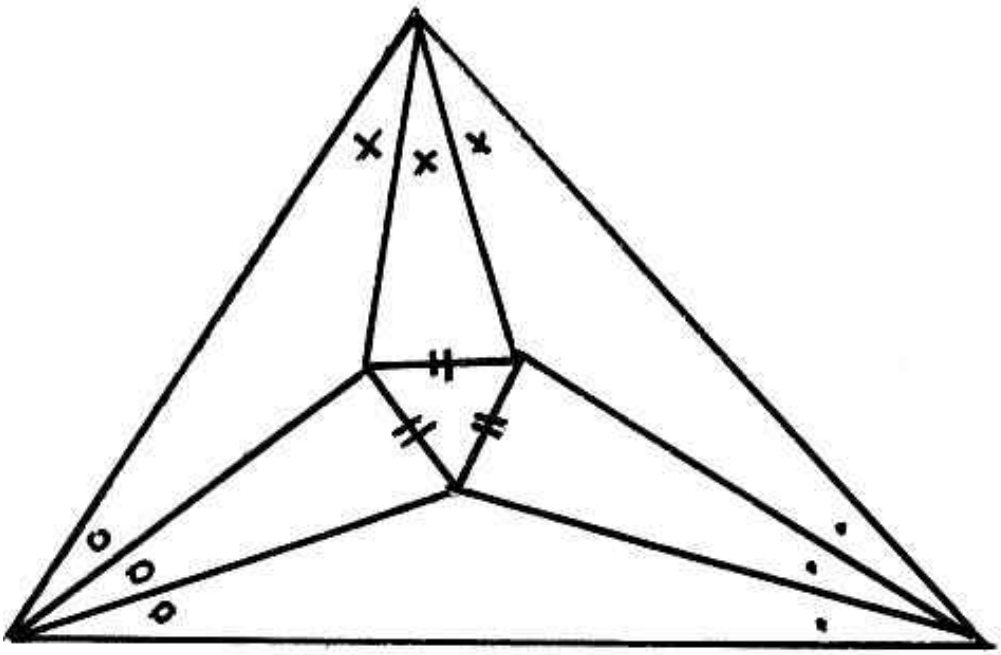
The mechanism of hinged rods shown will trisect angles accurately,

You can also trisect an angle with a clock.

With the hands set at 12.00, position the angle with one arm at 12. Start the clock. When the minute hand has rotated by the angle θ, the hour hand will have moved $\theta/12$. Multiply that angle by 4 and you have your trisection.

(This is something that a digital clock can't do.)

The Greeks never considered trisecting the angles of a triangle, because they didn't know how to do it with their ruler-and-compasses restriction. So they did not discover a remarkable result, first noted by Frank Morley in 1899: the trisectors of the angles of a triangle meet in three points which form an equilateral triangle.



The best proof that I know of this result was published by a Cambridge undergraduate, now a famous cosmologist, Roger Penrose. The proof is too long for now, but is totally within the reach of high school geometry.

Combinatorial geometry

So far my geometry gems have been in the classical Euclidean setting (although the Theorems of Pappus and Pascal are really part of what is known as *projective geometry*). Let me move to a completely different area, known as combinatorial geometry, which I will illustrate by discussing two problems.

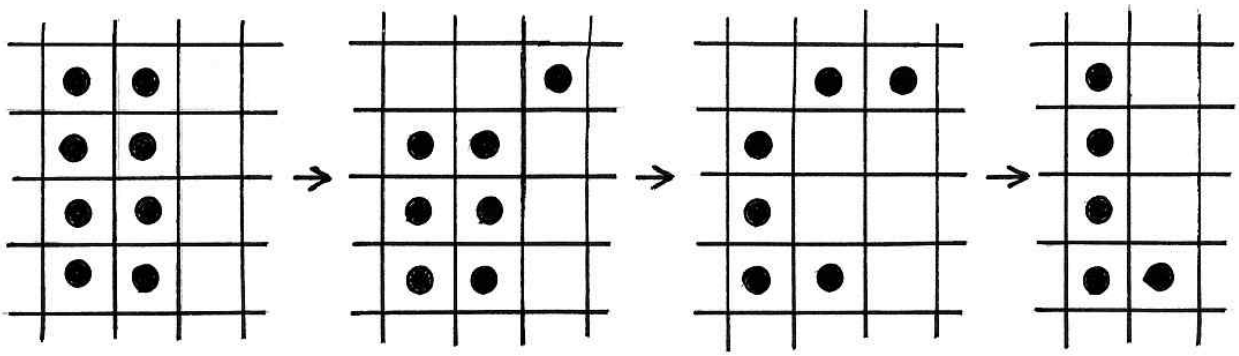
The first is a problem set in the International Mathematical Olympiad in 1993.

On an infinite chessboard, a game is played as follows.

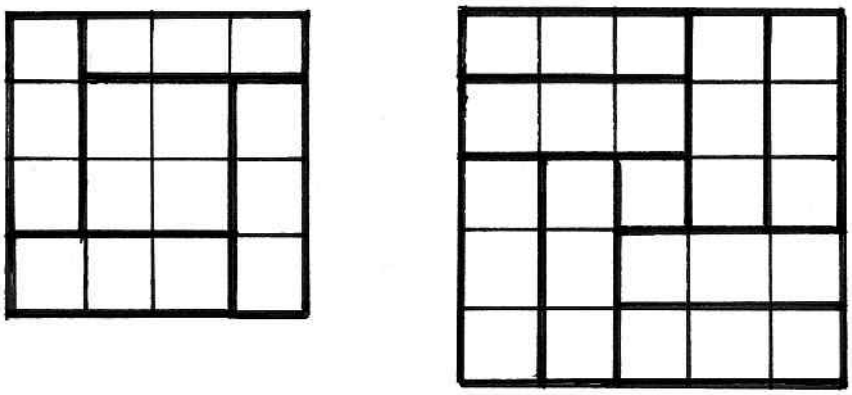
At the start, n^2 pieces are arranged on the chessboard in an $n \times n$ block of adjoining squares, one piece in each square. A move in the game is a jump in a horizontal or vertical direction over an adjacent occupied square to an unoccupied square immediately beyond. The piece which has been jumped over is then removed.

Find those values of n for which the game can end with only one piece remaining on the board.

The problem can easily be solved in the 2×2 case, but 3×3 gives trouble. Then 4×4 and 5×5 are fine, but not 6×6. Playing around like this leads to a key observation, a sequence of moves, which we shall call a triple jump:



In this way, we can peel off the layers of the square, and it can quickly be seen that any square with n not a multiple of 3 can be solved.



It remains to show that when n is a multiple of three, then the problem cannot be solved.

Colour the squares red, white and green as shown. When a jump is made, two occupied squares become unoccupied, and one unoccupied square becomes occupied. The number of occupied squares in the three colours goes down by 1 in two cases, and up by 1 in the third.

R	W	G	R	
W	G	R	W	
G	R	W	G	
R	W	G	R	

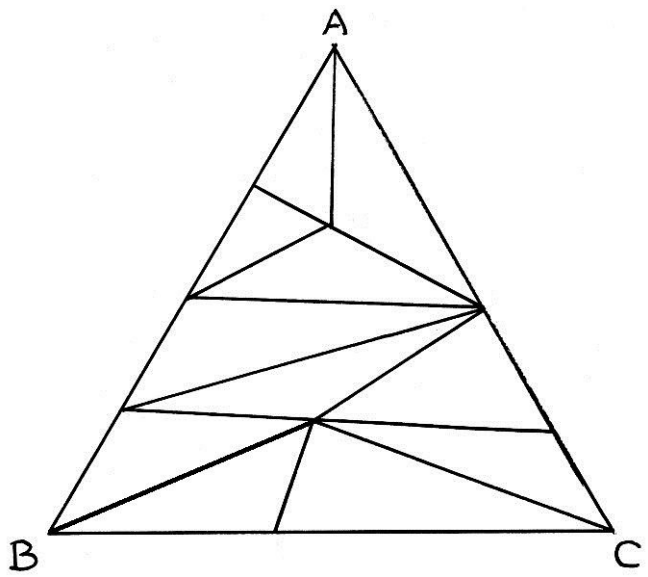
Now suppose we start off with a 6×6 square, and draw up a table. The colour count starts (12, 12, 12), and any jump will change it to (11, 11, 13).

The next stage could be (10, 10, 14), or perhaps (12, 10, 14). Continuing in this way, we make a crucial observation: at every stage, the triple of numbers are either all even, or all odd. But in the end we are aiming for the triple (1, 0, 0) (odd-even-even). That goal cannot be attained.

Sperner's Lemma

Here is a problem that would make a lovely IMO problem, except that it is a rather well-known result.

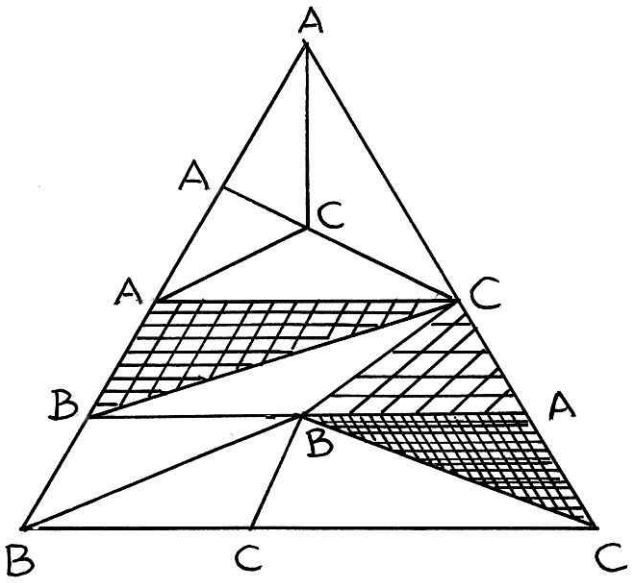
Start with a triangle ABC, and subdivide it into smaller triangles in any way you like.



The new points are then labelled A, B or C, subject to the constraints:

points on the edge AB are labelled either A or B
points on the edge BC are labelled either B or C
points on the edge CA are labelled either C or A
points inside the triangle may be labelled wither A, B or C.

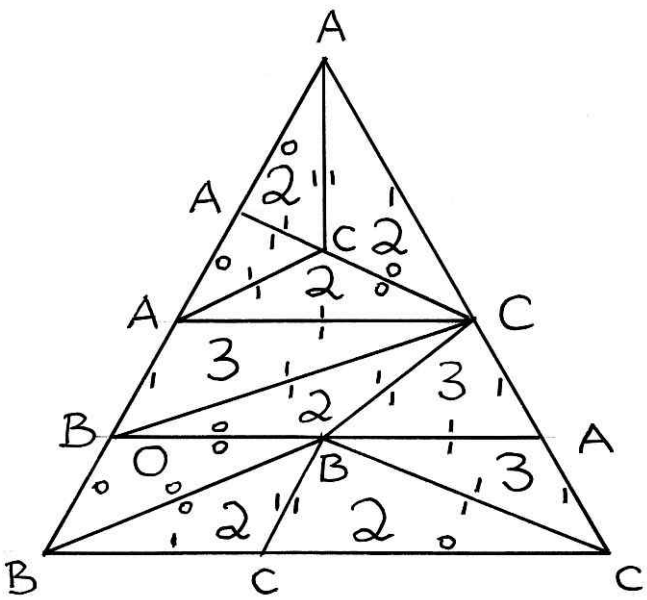
Now look at all the little triangles, and shade those that have vertices A, B, C.



No matter how the triangle is divided, or how the vertices are labelled, the number of shaded triangles will always be odd. This result is known as Sperner's Lemma, and is fundamental in algebraic topology.

The proof is by a clever little exercise of double counting.

Label each little triangle in the figure on the inside of each of its edges with 1, if the edge has two different labels, and with 0 if the edge has two labels the same.



Note that each interior edge occurs in two triangles, so is labelled twice, with the same label on either side.

Now add up all the numbers. Call this sum N. Along each edge of the big triangle, the sum will be odd, since the edge starts and ends with different

labels. So the total of all the outside edge numbers is odd. Inside the triangle, every number appears twice, so the sum of the inside numbers is even.

So N is the sum of an odd number (from the outside edges) and an even number (from the inside edges). Thus N is odd.

In each triangle, write the sum of the labels on its edges. That sum will be

0 if the three vertices are the same
2 if two edges are the same, and one different
3 if all three vertices are different.

Suppose now that there are m triangles labelled ABC, n triangles with just two vertices the same and p triangles with three vertices the same. Then the sum of all the triangle numbers is $3m + 2n + 0p = 3m + 2n$.

Now this sum must be equal to N: $3m + 2n = N$, an odd number.

It follows that m must be odd.

This activity can be set as a classroom game. Two players take turns to label the vertices according to the rules stated, and when they form a triangle ABC they shade it and score a point.

Our theorem says that this game can never end in a draw.

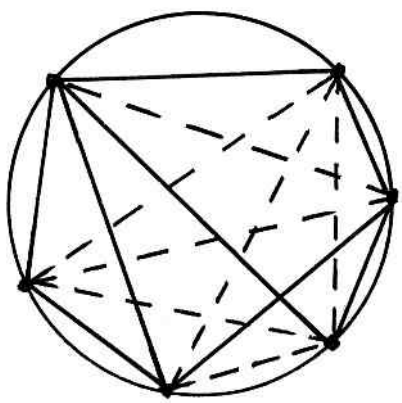
Here is another problem in combinatorial geometry

Ramsey Theory

Frank Ramsey was a mathematician who died young. His brother was Michael Ramsey, 100th Archbishop of Canterbury from 1961 to 1974.

In general terms, Ramsey theory shows that in many chaotic situations there are always elements of order: total randomness is impossible.

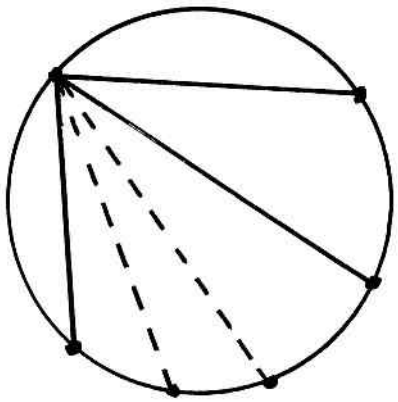
Here's a simple example.



Six points on a circle are joined in all possible ways, giving a figure with 15 edges. Each of these edges is coloured either red or blue.

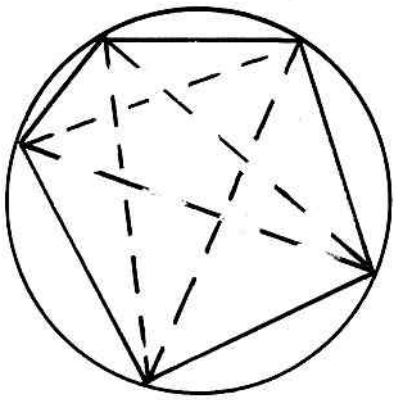
However the colouring is done, there will always be a triangle in the figure with all its edges of the same colour.

The proof starts by picking any vertex. It will be joined to five other vertices. Of these lines, at least three must be of the same colour (since we have only two colours); suppose they are red.



Now look at the three vertices at the end of the three lines. They form a triangle. If any of this triangle's edges is red, then it completes a red triangle. If they are all blue, we have a blue triangle.

Note that this result is false if we have only five points joined to each other.



We can crank the problem up to a three-colour version.

Seventeen points on a circle are joined in all possible ways, and every edge is coloured in one of three colours. Prove that there is a monochrome triangle in the figure.

This latter problem has featured in both the South African and International Mathematical Olympiads. □

THE VARSITY MATHS PROBLEM

Why do so many first-year students fail varsity maths?

Thousands of students across South Africa have started their university careers, and many of them have enrolled for a course in Mathematics. Some will be aiming at a maths major, in particular those who hope to teach mathematics at school level. But far more will be doing maths as a requirement for their degrees in a whole range of areas. Computer science, statistics, physics, actuarial science, engineering, economics and many other fields of study all need a solid grounding in mathematics.

Within a couple of weeks of starting varsity many of these students will be floundering in their maths lectures, and after a disastrous performance in their first class test will drop out or move to a bridging or slow-stream course with extra tutorial support. This early failure will inevitably mean taking an extra year to get their degree.

Many of these students will have arrived at university with an A for matric maths, and often they will have six or more subject A's. Why are these A students failing university maths?

Grade inflation

In the last few years the matric maths syllabus has been severely cut. The trickier parts of the algebra, geometry and trigonometry syllabus have been left out, and the final exams are now much easier.

Achieving a good maths mark is much easier than before. There has been huge grade inflation in matric results. In the past a good school could expect about half a dozen A's for Senior Certificate Mathematics. Now they are boasting 50 or more A's, most of whom would have earned C's and D's in the past.

Maths is not the only subject to suffer from grade inflation. Under the old pre-OBE curriculum, six As would get you into the top ten in the province. Today, a number of schools now boast 30 to 40 students with six or more As.

The problem is, as George Orwell might have put it, that all A's are equal, but some are less equal than others.

What can be done?

The answer is to offer high school students who are planning to continue with maths at university an opportunity, while still at school, to sharpen their maths skills by tackling problems that are more challenging than those of the matric exam.

Varsity Maths Prep

This book of problems, by Professor John Webb of the UCT Department of Mathematics, is aimed to identify weaknesses in a student's preparedness for university mathematics. Some exercises focus on popular algebraic errors. Others originated from questions that students would ask during his lectures about a simple piece of school geometry or algebra. Full solutions are provided.

Failure to solve any of the problems will indicate a weakness in the student's background that must be fixed before entering the Maths 101 lecture hall.

Will Varsity Maths Prep help students prepare for the National Benchmark Test?

The exercises in Varsity Maths Prep will certainly be good preparation for the National Benchmark Test (NBT) in Mathematics, but that is not its primary aim. The NBT looks back at what has been learnt at school. Varsity Maths Prep looks ahead to the sort of mathematical skills and thinking that varsity maths lecturers will expect of students in their classes. And it's not just the maths profs: lecturers in Physics, Computer Science, Engineering and Economics all expect their students to have fluent algebraic skills, numerical ability and good geometrical and spatial perception in two and three dimensions

Dumbed-down maths

A good example of how school maths is dumbed down can be found in trigonometry. There are six basic trig functions, but when Higher Grade Maths was abolished, only three were left in the curriculum. The array of trig formulas, which have wide application in physics and engineering, was cut down to just a couple, and students are no longer expected to prove, or even know them, since a formula sheet is now part of the final exam.

The same can be said about algebra and the important formula for solving a quadratic equation. Since it is on the formula sheet, students don't know it, and don't know how to derive it.

Students are also very weak on basic arithmetic, reaching for a calculator even when confronted with the task of adding or multiplying single-digit numbers.

As university students start their studies, those in mathematics courses will be shocked to realize that their matric Maths A symbol does not mean that they can handle varsity maths.

It's as if a cricket coach showed his novice batsmen only how to play gentle full tosses and longhops, without ever bowling them a bouncer, yorker or googly. With such instruction, ducks in a real cricket match would be inevitable. It's the same with maths. If you only meet a limited range of simple problems at school, at varsity you will be stumped, caught out or bowled over.

The book is not for high school students who are battling to pass maths. It is aimed at those who are good at maths and will collect a comfortable A in Senior Certificate mathematics. They may get into Science, Engineering, and Commerce degree programmes with ease, but will unexpectedly find varsity maths a serious challenge.

The book costs R95 and can be ordered from

The Answer
50 Imam Haron Road (formerly Lansdowne Road) Claremont 7708
Tel: 021 671 0837 Fax 021 671 2546
Email: `info@theanswerseries.co.za`
Website: `http://www.theanswer.co.za/shop/varsity-maths-prep/` □

HERON'S FORMULA

Heron (sometimes referred to as Hero) was a Greek mathematician who published a book called *Metrica* in 60 A.D. In it Heron gives the formula for the area of triangle, given the lengths of its three sides.

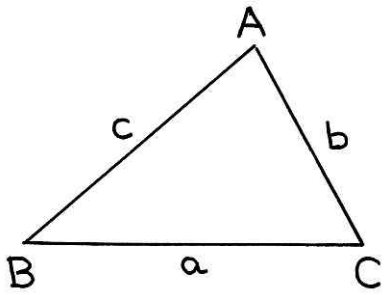
If the sides of a triangle are a, b and c,

then the area of the triangle is

$\sqrt{s(s-a)(s-b)(s-c)}$

where $s = \frac{1}{2}(a+b+c)$,

the *semiperimeter* of the triangle.



Let the area of the triangle be A. Then the Sine Formula for area gives $A = \frac{1}{2}ab\sin C$. We must find a way of eliminating the $\sin C$ term from this formula. The key idea is to note that $\sin^2 C = 1 - \cos^2 C$, and that the Cosine Formula $c^2 = a^2 + b^2 - 2ab\cos C$ suggests how to proceed.

It is convenient to work with A^2 rather than A.

We have $A^2 = (\frac{1}{2}ab\sin C)^2 = \frac{1}{4}a^2b^2\sin^2 C = \frac{1}{4}a^2b^2(1-\cos^2 C)$.

From the Cosine Formula we get $\cos C = \dfrac{a^2+b^2-c^2}{2ab}$, and making this substitution gives

$$A^2 = \tfrac{1}{4}a^2b^2\Big(1 - \Big(\frac{a^2+b^2-c^2}{2ab}\Big)^2\Big)$$

$$= \tfrac{1}{4}a^2b^2\Big(1 - \frac{a^2+b^2-c^2}{2ab}\Big)\Big(1 + \frac{a^2+b^2-c^2}{2ab}\Big) \text{ (difference of squares)}$$

$$= \tfrac{1}{4}a^2b^2\Big(\frac{2ab-(a^2+b^2-c^2)}{2ab}\Big)\Big(\frac{2ab+(a^2+b^2-c^2)}{2ab}\Big)$$

$$= \tfrac{1}{16}(2ab - a^2 - b^2 + c^2)(2ab + a^2 + b^2 - c^2)$$

$$= \tfrac{1}{16}(c^2 - (a-b)^2)((a+b)^2 - c^2)$$

$$= \tfrac{1}{16}(c-a+b)(c+a-b)(a+b-c)(a+b+c)$$

$$= \tfrac{1}{2}(c+b-a)\cdot\tfrac{1}{2}(c+a-b)\cdot\tfrac{1}{2}(a+b-c)\cdot\tfrac{1}{2}(a+b+c)$$

$$= (s-a)(s-b)(s-c)s, \text{ setting } s = \tfrac{1}{2}(a+b+c).$$

Taking square roots and rearranging the terms gives Heron's Formula

$$A = \sqrt{s(s-a)(s-b)(s-c)}.$$

$\square$

THE COMPOUND ANGLES FORMULAS

The formulas

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

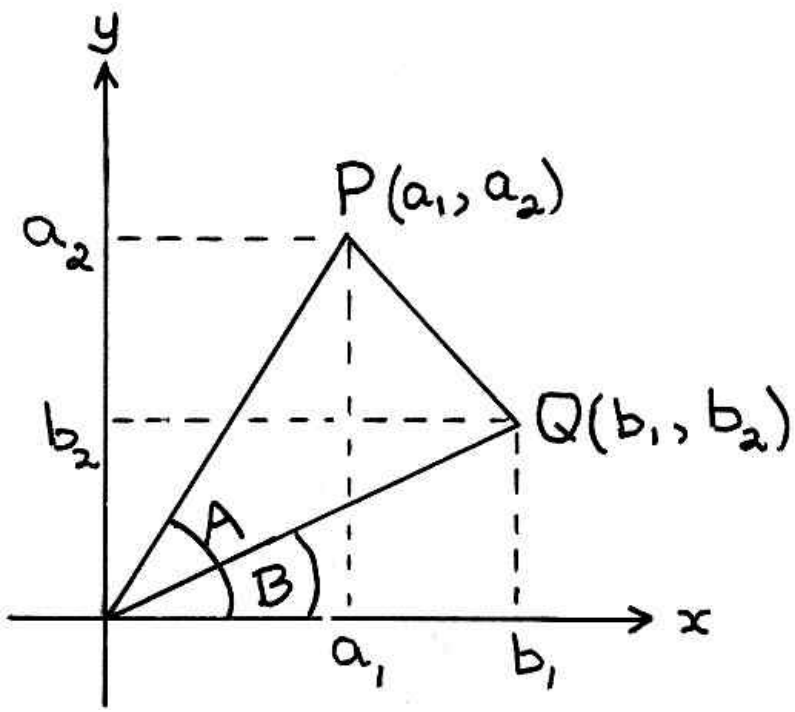
are fundamental in trigonometry.

They are all closely related, and its is necessary to prove just one of them, namely

$$\cos(A - B) = \cos A \cos B + \sin A \sin B$$

from which the other three follow quickly.

The proof, via coordinate geometry, relies on a neat combination of the Cosine Rule and the distance formula.



In the diagram, $OP = OQ = 1$.

The Cosine Rule in triangle OPQ gives

$PQ^2 = OP^2 + OQ^2 - 2OP.OQ.\cos(A - B)$

$$= 2 - 2\cos(A - B) \tag{1}$$

since $OP = OQ = 1$.

Since P and Q have coordinates (a_1, a_2) and (b_1, b_2), respectively, the distance formula (which is just a coordinate version of Pythagoras) gives

$PQ^2 = (a_1 - a_2)^2 + (b_1 - b_2)^2$
$= a_1^2 + a_2^2 - 2a_1a_2 + b_1^2 + b_2^2 - 2b_1b_2$
$= 2 - 2(a_1a_2 + b_1b_2)$ (2)
since $a_1^2 + a_2^2 = b_1^2 + b_2^2 = OP^2 + OQ^2 = 2$.
From equation (1) and (2) it follows that

$$\cos(A - B) = a_1a_2 + b_1b_2 = \cos A \cos B + \sin A \sin B.$$

The other three formulas follow quickly from the first.
Replacing B by $-B$ gives

$$\cos(A + B) = \cos A \cos(-B) - \sin A \sin(-B) = \cos A \cos B + \sin A \sin B,$$

since $\cos(-B) = \cos B$ and $\sin(-B) = -\sin B$.
To get the formula for $\sin(A+B)$, just remember that the sine of an angle is equal to the cosine of its complement and replace A by $90° - A$ in

$$\cos(A - B) = \cos A \cos B + \sin A \sin B.$$

That gives

$$\cos(90° - A - B) = \cos(90° - A) \cos B + \sin(90° - A) \sin B$$

and so

$$\sin(A + B) = \sin A \cos B + \cos A \sin B.$$

Now switch B to $-B$ to get the last of the four formulas

$$\sin(A - B) = \sin A \cos B - \cos A \sin B.$$

Putting $A = B$ in the formulas gives $\cos 2A = \cos^2 A - \sin^2 A$ and $\sin 2A = 2 \sin A \cos A$.
We can now also get the formulas for $\tan(A \pm B)$ very quickly.
$\tan(A \pm B) = \dfrac{\sin(A \pm B)}{\cos A \pm B} = \dfrac{\sin A \cos B \pm \cos A \sin B}{\cos A \cos B \mp \sin A \sin B} = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$
(dividing top and bottom by $\cos A \cos B$).
Putting $A = B$ gives $\tan 2A = \dfrac{2 \tan A}{1 - \tan^2 A}$.
We needn't stop there: it is now not too difficult to get formulas for $\sin 3A$, $\cos 3A$ and $\tan 3A$. Try it yourself. $\square$

THE DOUBLE ANGLE FORMULAS

The formulas

$$\sin 2x = 2\sin x\cos x$$

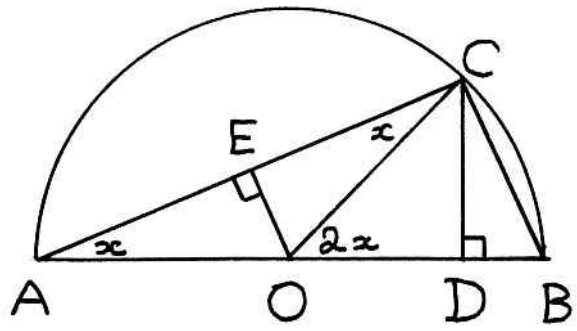
and

$$\cos 2x = 2\cos^2 x - 1 = 1 - 2\sin^2 x = \cos^2 x - \sin^2 x$$

are usually derived as special cases of the formulas for $\cos(x+y)$ and $\sin(x+y)$.

Here is a neat way of getting them directly.

In the figure, AOB is the diameter of a circle of radius 1 and CD and OE are perpendiculars dropped onto AB and AC, respectively.



With $\angle A = x$, $\angle COB = 2x$.

Then $\sin 2x = CD = AC\sin x = (2AE)\sin x = 2\cos x.\sin x$.

and $\quad \cos 2x = OD = 1 - BD = 1 - BC\sin x = 1 - 2\sin^2 x$.

These arguments are neat and easy, but rely on the diagram, in which $0° < x < 45°$.

For angles between 45° and 90° the diagram is different, but very similar arguments apply. $\square$

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IMO in Chiang Mai

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Learn from yesterday, live for today, hope for tomorrow.
The important thing is not to stop questioning.

(Albert Einstein)

Department of Mathematics and Applied Mathematics
University of Cape Town

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Cover: The 56th International Mathematical Olympiad took place in July in Chaing Mai, Thailand. See page 10 for the story.

ASTRONOMY IN SOUTH AFRICA

Professor René Kraan-Korteweg
Department of Astronomy
University of Cape Town

An address delivered at the Prize Giving
of the UCT Mathematics Competition, 21 May 2014

First of all, my heartfelt congratulations to all of winners in the audience tonight – and also to all the people who were influential in getting you where you stand today: your undoubtedly very proud parents, and the dedicated teachers without whom none of us would be here.

It is a real honour for me to stand here in front of you today, sharing this moment of celebration of your fantastic accomplishments in this year's UCT mathematics competition. It is a great feat. And one should not forget that success does not just happen. On the contrary, it usually needs hard work and dedication. So why is it **you** are sitting here tonight in this audience?

I personally think it is a drive that grows out of curiosity for knowledge, possibly ignited by a challenge like this math competition, which can then grow into passion – a passion for knowledge, a passion for excellence. All of you prize winners here in the audience have already started to develop that passion – and it is actually this – more than the prizes – that makes you the greatest winners overall. Why? Because it is passion that inspires, passion and energy that leads to achievement and pave the pathway to success. It is success that creates a springboard of opportunity. A passion for a subject – whatever subject – leads to excellence, and excellence will lead the way to the freedom to choose nearly any career path you like. Because excellence is regarded highly everywhere in the working world. Let me tell you, in this address, about my own passion which is closely linked to mathematics, or rather a kind of applied mathematics, and that is astronomy. But I will only tell you a little about **my** passion, because I rather want to spend most of the time to highlight the phenomenal opportunities that have opened up in South Africa through the investment of the government in astronomy.

What sparked my interest? When I was about your age my original interest lay in mathematics. But it moved more and more towards astronomy. The latter was probably influenced by my father who was a navigator on airplanes. At that point of time, navigation was not completely automated

and computerised, but flight paths were calculated taking measurements of the location of the Sun, bright stars and planets with the help of a sextant. His closeness to the sky led him to have a passion for astronomy. He even studied it in his spare time. And this made me aware of the existence of astronomy as a career path. And that was how my career started off, and kept me hooked throughout my life. My own research passion is the study of galaxies, agglomerations of billions of stars like our own Milky Way, and their distribution in space. For instance, why are they distributed in a foam- or sponge-like topology. But many other facets add to the fascination as well. For instance the variety the job entails and the various travel opportunities: I have used telescopes all over the world, in the most remote locations like the Andes in Chile, Sutherland in South Africa – even before I moved here – and Australia. The international astronomy community is closely connected, and I collaborate with people over most of the continents. I actually myself worked in six different countries on three different continents – and dragged my family along with me – with the additional benefit that my children are all fluent in six different languages.

But let me divert to the main point of this address now. I want to tell you more about the exciting developments that are currently happening in South Africa with regard to astronomy – and which ultimately is the reason that has brought me here to South Africa a bit over ten years ago.

In recent years South Africa has invested heavily in astronomy, both in infrastructure and in human capacity development programs, like bursary schemes for students and artisans. About 10 years ago we built the Southern African Large Telescope, SALT. This 10m telescope is the largest single optical telescope in the southern hemisphere. We are furthermore currently building a really cool radio telescope right here in the northern Karoo. It is called the MeerKAT and will consist of 64 individual radio telescopes of 13.5 m each that will all work in tandem to make it one the most powerful radio telescopes in the southern hemisphere and in the world.

So why is that happening? Why is South Africa investing in such world-class flagship projects? The South African government is seeking to make the transition from a resource-based to a "knowledge-based" economy. What does a "knowledge-based"' economy mean? It means that South Africa wants to build up a large contingency of people who are skilled in areas like Science, Technology, Engineering and Mathematics, a new generation of South Africans who are able to drive innovative research in science and technology, and be competitive, leaders, at an international

level. Hence, they are looking for young people just like you.

What then is the role of astronomy in this bigger scheme? In 2002, the Department of Science and Technology decided to promote certain areas of research in which South Africa has a distinct geographic advantage. Astronomy was identified as one of them with its long tradition in optical astronomy, and access to the southern skies. This was what led to the building of these large world-class facilities like SALT and MeerKAT.

But the ultimate goal of the government was not the construction of MeerKAT. MeerKAT was really just a springboard: The ultimate goal was to attract one of the largest and coolest international Mega Science projects to South Africa, the so-called Square Kilometre Array, or SKA for short, a major radio telescope of about 3000 telescopes linked together across distances of up to 3000 km. The total collecting area of the telescope dishes combined will be about one square kilometer, giving 50 times the sensitivity, and 10 000 times the survey speed of the best current-day radio telescopes.

Dr Rob Adams, who in that period (1999 - 2006) was the Director General of the Department of Science and Technology said: "What I wanted to do was make Southern Africa the world's number one astronomy hub. We already had optical (SALT) and gamma (HESS). What was left was radio, so bidding to host the SKA was a no-brainer. When I heard about it I made the decision to go for it almost instantly."

By building MeerKAT we showed the world that we not only have a perfect site for radio astronomy, but we also have highly competent engineers who can locally design, build and operate radio telescopes, and that we have and are training people with skills to lead cutting-edge science projects with such instruments.

And the gamble paid off. After nearly a decade of intensive activity to prepare the hosting bid – in severe competition first with four other countries – the international SKA Office announced in 2012 that South Africa was the big winner: the major part of the SKA would be built in South Africa with outlier stations in the eight African partner countries.

So why is this so exciting? There are two reasons for it. The first one is the science: the SKA will revolutionise our understanding of the Universe. It can address questions that have occupied us for decades: like what is dark matter and dark energy, and how and when the very first stars born. The latter is a fascinating question because everything you and I are made of,

actually every piece of furniture in this room, and every element on Earth and in the whole Universe has been formed through nuclear fusion in the hot interiors of stars. We would not be here if the stars would not exist and produce their radiation.

Another aspect that fascinates many people: The SKA may even detect signs of extra-terrestrial intelligent life: the SKA will be so sensitive that it will be able to detect an airport radar on a planet 50 light years away. 50 light years!! 50 light years means that it takes the emitted radiation 50 years to reach us if it travels at the speed of light; and I remind you that the speed of light is 300 000 km/s.

While the science excites the researchers, it is not the science that excites and drives the politicians. They care about the benefits and spin-offs to South Africa. And the biggest spin-off will be Big Data. Astrophysics and the SKA is at the forefront of the Big Data revolution. Why? Because the SKA will produce more data much faster than the entire internet today, in fact it will produce 50 times as much data traffic as the entire worldwide web. At the moment we could not even manage the data incoming data flow, store, reduce and analyse the data the SKA will produce. We do not have the know-how yet. As such the SKA is a real technology driver. What is happening with the preparation for the SKA is therefore of great interest to the economy of South Africa and the rest of the

In South Africa we are not sitting still. We are developing the Big Data Africa programme for the MeerKAT and SKA. This will position South Africa and Africa to be leading players in the big data industry by 2020. The UK and China have already announced that they want to investment in this, other countries are doing the same.

It is for all these reasons together that the government rated astronomy as a priority research area and aimed for the big price, the SKA. Through it, doors and numerous new and exciting career options are opening up. Not just for budding astronomers – the skills needed in this Mega Project are huge, ranging from computing, to modelling, to innovative instrumentation design, signal processing, education, communication! You name it. Is this career path for you? That is hard for me to say. But the future perspective for youngsters skilled in mathematics, like you, have never been better.

Let me close with quoting our Minister of Science and Technology, Dr Naledi Pandor, a real ambassador for the SKA, who summarized this much more eloquently than I can, when we won the major fraction of the SKA bid:

"This project is giving effect to our dream that Africa must become a global science and technology destination, and that cutting-edge science will be done in Africa by African scientists. Our SKA success is also reversing the brain drain into brain gain by bringing top researchers to the continent to do cutting edge work on African soil. This could be a game-changer for Africa, bringing about a science Renaissance across the continent. Let's use it to make South Africans proud and to inspire young people about a future in science and technology." □

EINSTEIN LIMERICK

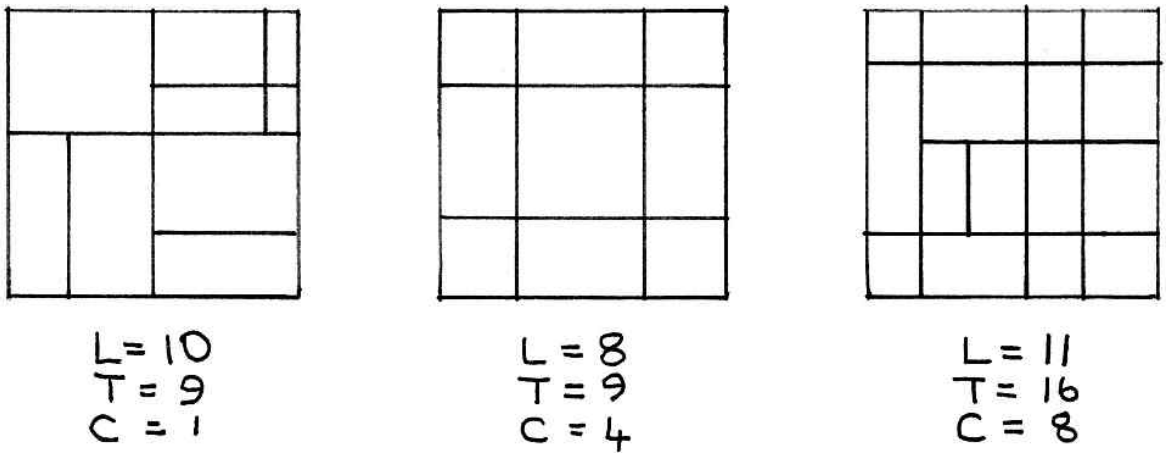
The Fitzwilliam Museum in Cambridge contains a fine bronze of Albert Einstein, by the sculptor Jacob Epstein, reminding one of a limerick dating from the 1920s.

Have you heard of the family Stein?
Theres Gert and theres Ep and theres Ein.
    Gert's poems are bunk,
    Ep's statues are junk
And no one can understand Ein.

Gertrude Stein (1874–1946) was an American literary innovator and pioneer of Modernist literature. □

A TILING FORMULA

The figures below show rectangles divided in different ways into smaller rectangles (called *tiles*) by horizontal and vertical lines. In each example the number of *lines* L, the number of *tiles* T and the number of *crossings* C is given.

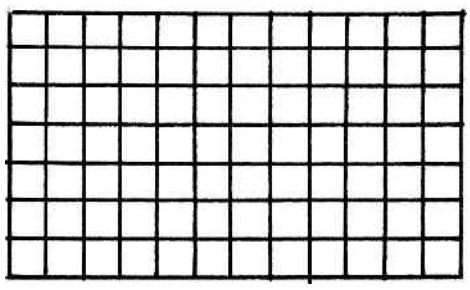


After drawing a number of such figures, one comes to the conclusion that there is a simple formula connecting L, T and C, namely

$$L - T + C = 3.$$

Can we be sure that the formula holds for every tiled rectangle, no matter how complicated? A proof is needed.

The way to prove that the formula is generally true is to start with a special case: when the rectangle is a simple $m \times n$ chessboard.



In this case there are $m + 1$ horizontal lines and $n + 1$ vertical lines, so

$$L = m + n + 2.$$

Next, there are clearly $m \times n$ tiles, so

$$T = mn.$$

Finally, we count the crossings, which form an $(m-1) \times (n-1)$ rectangle inside the figure. So

$$C = (m-1)(n-1).$$

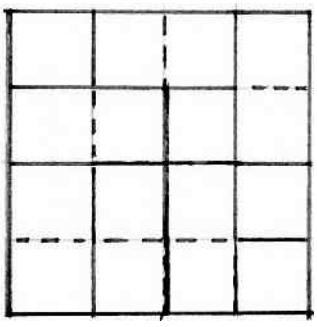
We now have

$$\begin{aligned}
&L - T + C \\
&= m + n + 2 - mn + (m-1)(n-1) \\
&= (m + n + 2) - mn + (mn - m - n + 1) \\
&= 3,
\end{aligned}$$

as required.

So the formula works for this specific type of tiling. Now we need to show that the formula holds for every possible tiling.

The key idea is to notice that any tiled rectangle can be modified by adding in extra line segments until it becomes a full chessboard-style grid.



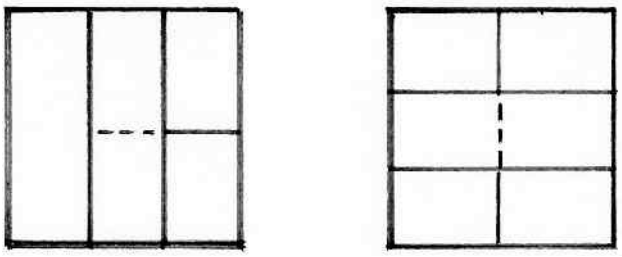
This can be systematically done.

Start with a tiling, and look for any T-junctions inside. Draw in an extra line segment from that T-junction across the tile

There are two possibilities:

On the first case (see the figure on the left below), the extra line segment simply meets the opposite side. There are the same number of lines in the figure, but the number of tiles has gone up by 1 and the number of crossings has gone up by one. In this case, $L - T + C$ remains unchanged after the extra line segment has been inserted.

In the second case (see the figure on the right below), the extra line segment starts and ends at a T-junction. In this case, the number of lines has been reduced by 1, the number of tiles has been increased by 1 and the number of crossings has been increased by 2. Once again, $L - T + C$ has not been changed.



Thus starting with an arbitrary tiling, we add line segments one by one until it becomes a complete chessboard-style grid, for which we know that $L - T + C = 3$.

Since at each step the value of $L - T + C$ has not changed, we can deduce that in the original figure we must have had $L - T + C = 3$. $\square$

A CURIOUS PROPERTY OF 82000

Our Hindu-Arabic system of writing numbers uses 10 as a base. Before writing numbers was invented, people counted on their fingers, and counting in batches of ten was natural.

Computers count in base 2, because electric currents are easiest to represent as either ON or OFF. Using just the digits 0 and 1, any natural number can be written as sums of powers of 2: the base 10 natural numbers

$$1, 2, 3, 4, 5, 6, 7, \cdots$$

become, in base 2

$$1, 10, 11, 100, 101, 110, 111 \cdots$$

In the same way, natural numbers can be written in base 3, base 4, etc.

Suppose we just look at the numbers that can be written in other bases, using just the digits 0 and 1. This is where 82000 pops up. Its representations in all the bases 2, 3, 4 and 5 use only 0's and 1's.

Since $82000 = 2^{16} + 2^{14} + 2^6 + 2^4$, its base 2 representation is

$$10100000001010000.$$

Since $82000 = 3^{10} + 3^9 + 3^7 + 3^6 + 3^5 + 3^4 + 3^3 + 3^0$, its base 3 representation is

$$11011111001.$$

Since $82000 = 4^8 + 4^7 + 4^3 + 4^2$, its base 4 representation is

$$110001100.$$

Since $82000 = 5^7 + 5^5 + 5^4 + 5^3$, its base representation is

$$10111000.$$

All these representations of 82000 use only 0's and 1's. That, however, is where the pattern stops. The base 6 representation of 82000 is 1431344.

Is there a number (necessarily bigger than 82000), for which its base representations in all the bases 2, 3, 4, 5 **and 6** consist only of 0's and 1's?

Nobody knows!

What is known is that the number would have to be huge. Written in base 10, a computer search has shown that it would have to consist of more than 2184 digits $\square$

IMO IN THAILAND

John Webb
Department of Mathematics and Applied Mathematics
University of Cape Town

The 56th International Mathematical Olympiad took place in Chiang Mai, Thailand, from 4th to 16th July. I am a member of the IMO Advisory Board as organizer of the 2014 IMO in Cape Town, and attended the event as a guest of the Thai IMO organization. My travel expenses were covered by the South African Mathematics Foundation.

I left Cape Town on 3rd July, on flights to Chiang Mai routed through Dubai and Bangkok, where I had an overnight stop. An early morning flight of 55 minutes to Chiang Mai next day got me to the IMO venue: the Chiang Mai Holiday Inn.

The IMO Advisory Board

My first duty was to attend a meeting of the IMO Advisory Board. I presented my report on the 2014 IMO and passed out copies of the Final Report. The report was well received.

One issue raised at the AB meeting was how the IMO could get some official structure. At present the IMO Foundation is a registered entity, set up in 2011 in the Netherlands to manage the Google donation of a one million Euros. The Foundation has supported the host countries of the last five years. The Google grant will not be renewed, and discussion took place on finding other sponsors.

But there has always been resistance to the IMO itself being given official status by embedding it in a body such as the International Mathematical Union. On the one hand many IMO Team Leaders do not wish to be subservient, as they see it, to some outside organization. On the other hand, some prominent members of the IMU are hostile towards the IMO, believing that "theory-building" and not "problem-solving" is what mathematics is all about. The lack of an official umbrella body, however, means that the IMO has no protection when, for example, an upstart commercial organization in India describes itself as the "International Mathematics Olympiad", and uses the acronym IMO, causing confusion, especially in India. The discussion was indecisive.

The primary function of the IMO Advisory Board is to ensure that suitable

host countries are in place every year. The hosts for the next four years are established: Hong Kong (2016), Brazil (2017), Romania (2018) and the United Kingdom (2019). There is a possibility that Russia will offer to host the IMO in 2020 or 2021.

An interesting observation came from the Korean member of the Advisory Board. In the past a high achievement in the Korean Maths Olympiad counted officially in determining university entrance. This was abolished a couple of years ago, and registrations for the Korean Maths Olympiad plummeted. The meeting continued in the afternoon and ended at 4.30 pm.

The IMO Jury

Jury meetings started the next morning. It was announced that a total of 104 teams will take took part in the 2015 IMO, with 577 contestants, of whom 52 are girls. After other routine matters, the Jury set to work selecting six problems from a Short List prepared by an international Problems Selection Committee.

The first decision of importance was agreeing on a protocol for setting the papers. There are four IMO topic areas: Algebra, Combinatorics, Geometry and Number Theory. Each year, the Problems Selection Committee presents the Jury with a Short List of seven or eight problems in each topic area, ranging from easy to difficult. With the two contest papers each containing three questions (one easy, one medium, one difficult) the proposal is to first select four problems, one from each topic, for the easy and medium problems. The difficult problems are then chosen from any two of the four topic areas. This protocol was first used in Colombia in 2013, and again in South Africa in 2015, and has met with general approval. The Jury agreed to follow the same method this year.

There followed a thorough discussion of the problems. At one stage I was called out to meet with the group of Observers for Hong Kong, the hosts of next year's IMO, for detailed discussions of the ins and outs of hosting an IMO.

After lengthy discussion and long sequences of voting, the final selection of the six problems was made. Then the definitive English version was set. This always takes a long time because the English language committee is large and consists of English speakers, both native and approximate, and all talkative. Then the ball is in the court of the other 55 languages. There may well be a problem with the Arabic versions, which are different for

Algeria, Morocco, Tunisia, Syria and Saudi Arabia. It's not just variations in Arabic dialects. While all Arabic is written from right to left, like Hebrew, some of them write their maths from left to right (like Hebrew) but others write everything including the maths from right to left. There are also two versions of Chinese: traditional (Taiwan) and modernized (China), and North and South Korea insist on setting separate versions, as do Serbia and Bosnia-Herzegovina.

In due course the translations (55 in total) were pasted up on a long wall for all to see and point out corrections. I was pleased to be the first to find an error. The dates of the papers were Friday 10 June and Saturday 11 June, but the Danish paper said "Sontag" on the second paper, Danish for Saturday. Indeed it isn't — I discovered that the Danish for Saturday is "Lordag", which means "Washing Day". It's almost the same in the other Scandinavian languages

The team from Saudi Arabia arrived at the last moment in time to do their translation. They had experienced visa problems, almost certainly the spin-off of a bitter diplomatic row between Thailand and Saudi Arabia in what is known as the "Blue Diamond Affair". The sensational details of this 25-year-old diplomatic row, with a massive jewel theft, fraud and several murders, does not concern us here, but may be found in Wikipedia.

Meanwhile, Thailand had issued visas to the Kosovo team. It was a great source of embarrassment in 2014 when the South African Government refused to give visas to Kosovo, despite representations at the highest level.

Africa and the IMO

When I took the first South African team to the IMO in 1992, only three other African countries were participants: Algeria, Morocco and Tunisia. Through South African IMO involvement and support of the Pan African Mathematics Olympiad, there are now more African participants, peaking at 12 last year. This year the number was down to nine: Algeria, Botswana (their first participation, after being an Observer in 2014), Ghana, Morocco, Nigeria, South Africa, Tanzania, Tunisia and Uganda. African participants in 2014 which did not take part in 2015 were Benin, Burkina Faso, Gambia, Ivory Coast and Zimbabwe. It was disappointing that Madagascar, an Observer in 2014, did not send a team. Kenya and Egypt sent Observers to Thailand.

Several discussions about the Pan African Mathematics Olympiad (PAMO) took place during the IMO. Two IMO Leaders are PAMO stalwarts: Paul

Vaderlind (Sweden) and Charles Leytem (Luxemburg) and will be going to this event, scheduled to take place in Abuja, Nigeria in the last two weeks of August. The plans for PAMO2015, announced by Nigeria rather suddenly and without any previous discussion, include integrating it with their idea of a new Pan African Mathematics Olympiad for Girls, PAMOG. The double event will be preceded by a week-long training programme for all the teams. It was reported that Nigeria would be entering 8 teams.

Subsequent discussion with the African Team Leaders revealed that they were highly unlikely to find funds to go to Nigeria just one month after the IMO. Some teams do not have government or corporate support, and the students' parents have paid for their air tickets to Thailand. The same parents will be unwilling to pay for another event next month, especially after poor results.

There was general agreement that PAMO should be held in April, not August, and should be promoted as a stepping stone to the IMO. The sponsors of a student who gets a PAMO medal in April could well be persuaded to support an IMO ticket for the following July.

There are few African countries able to host future PAMOS. Tunisia, which has hosted several PAMOs and was a possible host for 2014, can no longer be considered a safe venue. There was a suggestion by the Observer from Kenya of the possibility that Kenya may consider hosting PAMO2016. The message is that South Africa should be prepared to host PAMO at frequent intervals.

The Opening Ceremony

The IMO was under the patronage of Princess Maha Chakri Sirindhorn, and formed part of her 60th birthday celebrations. She was Guest of Honour at the Opening Ceremony. Teams had to be assembled more than an hour in advance to given strict instructions on how to behave in her presence. Teams had been told to dress neatly and behave with decorum, and, somewhat to my surprise, they complied. The Parade of Teams, usually an informal and spontaneous affair, with students fooling about on the stage and throwing gifts to the audience, had to be stiff and formal. Students had to be seated more than an hour beforehand, and were shown how to advance with due respect towards the Princess and bow (or curtsey), before leaving with appropriate decorum. Tourist videos were shown to while away the time. When the Princess arrived there were some brief formal speeches but no entertainment and the parade took about 20 minutes. It ranks as the shortest - and dullest - Opening Ceremony ever.

The Contest

There was an unexpected hitch. It is normal practice to give the Deputy Leaders copies of the first paper while the students are writing it. Unfortunately, some Deputies were given copies of the next day's paper in error. So a new paper had to be set by a small committee in record time. This was perhaps the main reason why the second paper was rather more difficult than usual, and overall the scores this year were down.

Coordination Although there were a couple of disputes over marking by Leaders who insisted on bringing their arguments to the full Jury, they were quickly settled, and the whole coordination process went smoothly.

The Contestants' Programme

The students were housed in a separate hotel some distance from the Leaders. In past years the Leaders have moved to be with the students after the contest, but this did not happen this year and the Leaders stayed in the same hotel throughout. There was no opportunity to meet the students until the Farewell Banquet. They enjoyed two full-day excursions and a programme of celebrity lectures.

Excursions

The Leaders enjoyed two excursions. The first was a trip to the Maetana Elephant Camp, including an elephant ride through the jungle, followed by an oxcart transfer to an embarkation point for a gentle raft ride down a river. The second was a visit to two of the finest temples in Chiang Mai. There were also opportunities for participants to arrange their own trips to the Chiang Mai Zoo (famous for its pandas), some specialist silk shops and the famous Sunday night street market.

Meeting of the Advisory Board with the Jury

The final meeting of the Jury began with a discussion of financial and other matters with the Advisory Board. This was the last of the five IMOs which have enjoyed the support of the million-Euro Google donation, and no new donors have been found. The Jury stood in silence to remember the passing of IMO stalwarts.

There will be an election next year, for the position of Secretary and one elected member. Gregor Dolinar, who took over from me as Secretary, has been nominated unopposed for a second term. There are two candidates (one from Norway, the other from Venezuela) for an upcoming vacancy on the Board. The election will take place next year.

I have been on the Advisory Board for 15 years, with three four-year terms as Secretary and the last three years as a co-opted member, representing South Africa as 2014 Host Country. The Chairman announced that I would be leaving the Advisory Board, and gave me a nice tribute, including reference to my efforts to increase African participation. A long round of applause was heart-warming.

The Jury then moved to consider the IMO results. The first task was to finalise the scores, and then the crucial decision was made on how many medals in each category were to be awarded. The rule is that no more than half the students should be awarded medals. In earlier years a simple list of scores, without names, has been shown, and the half-way cut decided on that basis. Since the numbers never work out exactly right, when a cut had to be at either 13 or 14 points, say, Leaders with students on 13 would vote for the lower cut, even when that meant that the rule about the number of medals was violated. Last year the Jury irresponsibly voted for far too many medals, and we had to produce ten extra Silver Medals at short notice.

So a new system was devised. To prevent Leaders knowing where the cut should be placed in the list of scores, the Jury was shown histograms, without scores, with options for the proportions of Gold/Silver/Bronze medals. They were then required to select the best-fitting histogram. Any deviation from the "no more than half" rule will in future require a two-thirds majority vote. This system produced a sensible result, and the scores were immediately posted on the website.

The top ten countries are always of great interest, and for the first time since 1994 the USA headed the list with 185 points. China followed with 181, then South Korea (161), North Korea (156), Vietnam (151), Australia (148: their best ranking ever, with two Golds and four Silvers), Iran (145), Russia (141), Canada (140) and Singapore (139) in tenth place.

South Africa was 55th out of 104 countries, with one Bronze (Yaseen Mowzer, of Fairbairn College) and two Honourable Mentions (Andrew McGregor, of Rondebosch Boys' High School) and Bronson Rudner (South African College High School). South Africa was the best of the African countries, and ahead of several European countries. But there is room for improvement.

The other African scores were dismal - a matter of great concern. There were no easy questions for beginners, and the marking schemes were very strict: the first point out of the allotted 7 always requires significant

progress. The remedy for Africa lies in training programmes and PAMO experience. South Africa should be more proactive in this regard.

Closing Ceremony

The Closing Ceremony consisted of the usual speeches and presentation of medals, with no style-cramping royalty. There was a long video featuring the teams and all their student activities, showing that they certainly have enjoyed a very good programme of events. There was only one musical item. As a member of the IMO Advisory Board I took part in the presentation of the Silver Medals, and it happened that one of the Australian boys ended up in front of me, so I was able to congratulate him and his team for their best result ever. The Golds were handed out, and then the top student Alex Song (Canada) was called back for special recognition: one Bronze and five Golds in a row (the fifth with a perfect 42) puts him at the top of the IMO Hall of Fame.

On arrival at the Farewell Banquet venue we were confronted with a vibrant "street scene" of Thai vendors offering a huge variety of traditional snacks on sticks or wrapped in leaves. It was an excellent idea, establishing a happy informal atmosphere. Then a large buffet featuring a huge array of Thai dishes was opened. There were tables of ten, so we claimed one for SA and I met the team for the first time. They were in good spirits. Having given out all the printed IMO2014 Final Report books, I sought out suitable recipients for the report in CD format. I singled out two of the triumphant USA team who were in SA last year, the Australian Alexander Gunning (another Gold for him this year, but he missed out on a second perfect score by a couple of points), the other Chinese perfect scorer of 2014, Jiyang Gao, and Alex Song.

Farewell

There were lots of farewells, with many old friends asking me whether I will be back next year, and expressing genuine sorrow when I said that after 25 years, IMO 2015 was my last IMO. □

THE 2015 ABEL PRIZE

The Abel Prize, established in 2002, is awarded annually for outstanding work in mathematics. It is named in honour of Niels Hendrik Abel, who was born in 1802 and died at the age of only 26. He is ranked as the greatest mathematician Norway has produced. The Abel Prize is worth 6 million Norwegian kroner (about $1 million).

The Norwegian Academy of Science awarded the Abel Prize for 2015 to John Nash (Princeton University) and Louis Nirenberg (Courant Institute, New York University) "for striking and seminal contributions to the theory of nonlinear partial differential equations and its applications to geometric analysis."

Nirenberg (right) was born in Canada in 1925 and Nash (left) was born in 1928. Although Nirenberg published many papers in collaboration with other mathematicians, none were with John Nash, though their work was closely related,

John Nash became famous when he was awarded the Nobel Prize in Economics in 1994 for his work in Game Theory. The story of his life and long battle with schizophrenia was the subject of Sylvia Nasar's biography *A Beautiful Mind,* made into a movie of the same name in 2001 with Russell Crowe playing the role of Nash. The movie story differed from the real one in many ways, ignoring Nash's mathematically more significant research into geometry and partial differential equations. It was geometrty, not Game Theory, that won John Nash the Abel Prize jointly with Nirenberg.

The award of the 2015 Abel Prize was announced in March this year, but attracted little attention in the press. Nor did the actual award ceremony in May, when Nirenberg and Nash received the prize from the Norwegian King, Harald V. The headlines came, tragically, just a few days later, when Nash and his wife Alicia were on their way home from Newark airport, having just returned from Norway. Their taxi crashed and they were both killed. □

UCT MATHEMATICS COMPETITION RESULTS

The annual University of Cape Town Mathematics Competition took place on the UCT campus on 16 April this year, attracting 7694 participants from Western Cape high schools. Each school could enter up to five individuals and five pairs, in each grade (8 to 12).

Grade 8: Individuals

1	Adri Wessels	Curro Durbanville
2	Taariq Mowzer	Fairbairn College
3	Jiyoon Jeon	Rustenburg Girls' H S
3	Sean van Wyk	Home School
5	Peter Stead	El Shaddai Christian School
6	Declan Heathcote-Marks	Constantia Waldorf School
7	Katinka Wilkinson	Herschel Girls' H S
8	Liam Foxcroft	Bishops
9	Thomas Warner	Bishops
10	Sangah Lee	St. George's Grammar School

Grade 8: Pairs

1	Gary Allen / Matthew Blows	Rondebosch Boys' H S
2	Kelly-Anne Hockly / Ornella Nusca	Springfield Convent
3	Tomas Slaven and Zac Young	Bishops
3	Tallulah Graziani / Molly Ryan	Westerford H S

Grade 9: Individuals

1	Razeen Parker	Rondebosch Boys' H S
2	Timothy Schlesinger	Rondebosch Boys' H S
2	Du Toit Spies	Parel Vallei H S
2	Ralph McDougall	Curro Durbanville
5	Emile Tredoux	Parklands College
6	Yong Jun Park	Rondebosch Boys' H S
7	Jane Park	El Shaddai Christian School
8	Harry Kim	El Shaddai Christian School
9	Coleridge Faraday	Wynberg Boys' H S
10	Alaric McGregor	Bishops

Grade 9: Pairs

1	Dillon Cilliers / Liam Schreiber	Parel Vallei H S
2	Conor Colquhoun / Lloyd Watermeyer	Rondebosch Boys' H S
3	Juan Barnard / Niel Theron	Durbanville Hoërskool

Grade 10: Individuals

1	Stephan le Roux	Parel Vallei H S
2	Phillip Swart	Durbanville H S
3	Charl du Toit	Parel Vallei H S
4	Daniël Hugo	D F Malan H S
4	Abdullah Karbanee	Rondebosch Boys' H S
4	Sumin Park	El Shaddai Christian School
7	Nicholas Lambrecht	Westerford H S
7	Yunjae Na	Groote Schuur H S
9	Jialong Weng	Rhodes H S
10	Onur Ozkaya	Star International

Grade 10: Pairs

1	Eugène Fouché / Michael Mariano	Bellville H S
2	Christiaan Marais / Lourens Strydom	Paarl Gimnasium
2	David Mills / Daniel (Joon-Soo) Park	Wynberg Boys' H S

Grade 11: Individuals

1	SangEun Lee	St George's Grammar School
2	Andrew McGregor	Rondebosch Boys' H S
3	Yaseen Mowzer	Fairbairn College
4	David Broodryk	Westerford H S
5	Bronson Rudner	S A College H S
6	Matthew Booth	Rondebosch Boys' H S
7	Torsten Babl	Paarl Boys' H S
7	Sanjiv Ranchod	Westerford H S
9	Hannah Clayton	Rustenburg Girls' H S
10	Jun Lu	CBC St John's Parklands

Grade 11: Pairs

1	Sean Cheney / Liam Didcott	Wynberg Boys' H S
2	Robert Lancefield / Francois Stassen	Bishops
3	Sandra Snyman / Carla van der Westhuizen	Parel Vallei H S

Grade 12: Individuals

1	Tae Jun Park	Rondebosch Boys' H S
2	Soo-Min Lee	Bishops
3	Christoff van Zyl	Paarl Gimnasium
4	Jeremy Wilkinson	Bishops
5	Amy Chung	Herschel Girls' H S
6	Christian Cotchobos	Bishops
7	Sam Jeffery	Bishops
8	Scott Hallauer	Reddam House Constantia
9	Andrew Harrison-Migochi	Parel Vallei H S
9	James Robertson	Westerford H S

Grade 12: Pairs

1	Joshua Knipe / Matthew Kuttel	Bishops
2	Georg Janowski / Art van der Lingen	Constantia Waldorf
3	Desmond Fairall / Rayhaan Survé	Bishops

School ranking

A school score is obtained by adding the top 30 scores (individual or pairs) in each school. By this measure the top 20 schools were:

1	Rondebosch Boys' High School
2	Bishops
3	Westerford High School
4	Parel Vallei High School
5	Wynberg Boys' High School
6	Reddam College Constantia
7	Herschel High School
8	Somerset College
9	Stellenberg High School
10	El Shaddai Christian School
11	Rustenburg Girls' High School
12	Paarl Boys' High School
13	Herzlia High School
14	South African College High School
15	Paul Roos Gymnasium
16	Paarl Gimnasium
17	Curro Durbanville

18 Elkanah House
19 Wynberg Girls' High School
20 Strand High School

A full report on the 2015 UCT Mathematics Competition has been published and is available at R65. The book contains all the question papers, solutions, statistics, awards and lists of participants. Use the order form on the website:

`www.uct.mth.ac.za/outreach/competition`

to obtain your copy.

The 2016 UCT Mathematics Competition will take place on Thursday 7 April 2016.

mathematical digest

No. 181 October 2015

The Webb-Ellis Trophy

mathematical digest

Number 181 **October 2015**

Ever since I was engaged on *Principia Mathematica*, I have had a certain method of which at first I was scarcely conscious, but which has gradually become more explicit in my thinking. The method consists in an attempt to build a bridge between the world of sense and the world of science. I accept both as, in broad outline, not to be questioned. As in making a tunnel through an Alpine mountain, work must proceed from both ends in the hope that at last the labour will be crowned by a meeting in the middle.

Bertrand Russell, in *My philosophical development*

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Cover: The Webb-Ellis trophy (see page 2).

EDITORIAL

This edition of *MATHEMATICAL DIGEST* has two main themes: the UCT Mathematics Circle, and Olympiads.

The UCT Mathematics Circle is a programme of mathematics enrichment offered to the top students in the UCT Mathematics Competition. Classes meet on Monday afternoons in school term, and the second half of 2015 had two themes: Fibonacci Numbers and Graph Theory. There are three articles on each of these themes in this edition.

The results of the South African Mathematics Olympiad, the Computer Programming Olympiad and the Pan African Mathematics Olympiad cover the second theme.

In between you can read about the Webb-Ellis Trophy (nothing to do with rugby), news of a major breakthrough in solving a famous 83-year-old problem by Terence Tao (who has featured many times in this magazine), a very unlikely claim to have cracked the Riemann Hypothesis (by a little-known Nigerian mathematician) and some whacky statistics culled from a local newspaper. Enjoy the read! □

THE WEBB-ELLIS TROPHY

October 2015 saw the Rugby World Cup taking place in England, with the teams competing for the Webb Ellis Cup, named after William Webb Ellis, a schoolboy at Rugby School in the English Midlands who is credited with introducing the game at his school.

It was eight years ago, when the 2007 Rugby World Cup was in progress, that a light-hearted suggestion by Professor John Webb to Professor George Ellis took place in the tea room of the UCT Mathematics Department. "George" he said "We should create a trophy for achievement in Mathematics, and call it the Webb-Ellis Mathematics Trophy."

The joke was generally appreciated by the maths tea-drinkers, and might well have been forgotten, had not George Ellis proposed taking the trophy idea seriously. And so it was that Professors Webb and Ellis purchased a handsome silver cup which was named the Webb-Ellis Mathematics Trophy. They declared that their trophy should be awarded every year to the best first-year UCT student in Mathematics and Applied Mathematics, considered together, on the joint recommendation of the Course Conveners of the two first-year major courses in Mathematics and Applied Mathematics.

The Webb-Ellis Mathematics Trophy is a floating trophy, and each year the names of the winners are engraved on the base. Each year's winner receives a miniature version to keep.

First awarded in 2008, the winners have been:
2008: Neil Goldberg
2009: Timothy Trewartha
2010: Alastair Grant-Stewart
2011: Gregory Jackson
2012: Robert Moerman
2013: Dylan Nelson

The award of the Trophy for 2014 was made during the Medals Ceremony in the UCT Department of Mathematics and Applied Mathematics, when all Class Medallists of 2014 were presented with their medals. The date was coincidentally appropriate: 17th September, the eve of the start of the 2015 Rugby World Cup in England.

The winner of the 2015 Webb-Ellis Mathematics Trophy, featured on the cover with Professors Webb and Ellis, was Munbodh Pankaj, a B Sc student.

John Webb and George Ellis were undergraduate students at UCT some fifty years ago, and both went on to achieve PhDs at the University of Cambridge. John's research work was in Functional Analysis, an area of Pure Mathematics with many applications in Physics and Engineering. George's research in Applied Mathematics and Cosmology led to a fruitful collaboration with Stephen Hawking, and the publication in 1973 of their book *The Large Scale Structure of Space-Time*, now considered a classic.

Webb and Ellis returned to UCT and have been members of the Department of Mathematics and Applied Mathematics ever since. Both of them have taken an active interest in school and undergraduate mathematics education. □

THE DISCREPANCY PROBLEM

In 1932 the Hungarian mathematician Paul Erdös conjectured that for any sequence of numbers, which are all either $+1$ or -1, and for any integer C, there exist integers k and d such that

$$\Big|\sum_{i=1}^{k} x_{i\cdot d}\Big| > C.$$

Putting that into words: in any ± 1 sequence, and for any number C, no matter how large, it is always possible to find a finite subsequence of the numbers, with the terms in regular jumps, adding up to more than C.

For 80 years the "Erdös Discrepancy Problem" remained unsolved. Then, in September this year, Terence Tao announced that he had proved that the conjecture of Erdös is true.

Just in case you were beginning to think that such esoteric problems are for ivory towers only, it should be pointed out that such ± 1 sequences are important to the electrical engineer in the understanding of pulse compression radar systems.

The mathematician Terence Tao was born in Australia in 1975 and was a child prodigy, winning a Bronze Medal at the International Mathematical Olympiad at the age of age 10, followed by Silver and Gold Medals in the next two IMOs in 1987 and 1988 (the 1988 IMO was, appropriately, in Australia).

He entered Flinders University at the age of 14, graduating with a Masters degree in two years. A Fulbright Scholarship took him to the USA, where he obtained a PhD at the age of 21. He has since won awards for his mathematical work, including a Fields Medal (the top prize in mathematics) in 2006. In 2014 he was awarded the first Breakthrough Prize in Mathematics, funded by Yuri Milner and Mark Zuckerberg and worth $3m. He is currently at the University of California, Los Angeles.

Other major mathematical advances made by Terence Tao include a proof (with Ben Green) that there are arbitrarily long sequences of primes in arithmetic progression (see *Mathematical Digest* **136**, July 2004) and major progress in solving a 270-year-old problem, the Goldbach Conjecture (see *Mathematical Digest* **136**, July 2004).

For more information on the Discrepancy Problem, see *Quanta Magazine* (1 October 2015):`https://www.quantamagazine.org/20151001`.

The picture above shows Paul Erdös in conversation with the 10-year-old Terry Tao. Were they discussing the Discrepancy Problem? □

FIBONACCI AND MATHEMATICAL INDUCTION

The Fibonacci numbers F_n are defined by:

$$F_1 = F_2 = 1$$

$$F_{n+2} = F_{n+1} + F_n \text{ for } n \geq 1.$$

The numbers are 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, ...

The Fibonacci numbers have many interesting properties, and the proofs of these properties provide excellent examples of *Proof by Mathematical Induction.* Here are two examples. The first is quite easy, while the second is more challenging.

Theorem

Every fifth Fibonacci number is divisible by 5.

Proof

We note first that $F_5 = 5$ is certainly divisible by 5, as are $F_{10} = 55$ and $F_{15} = 610$. How can we be sure that the pattern continues?

We shall show that
IF the statement "F_n is divisible by 5" is true for some natural number m, THEN the statement is also true for the natural number $m + 5$.

Now $F_{m+5} = F_{m+4} + F_{m+3}$
$= (F_{m+3} + F_{m+2}) + F_{m+3}$
$= 2F_{m+3} + F_{m+2}$
$= 2(F_{m+2} + F_{m+1}) + (F_{m+1} + F_m)$
$= 2F_{m+2} + 3F_{m+1} + F_m$
$= 2(F_{m+1} + F_m)3F_{m+1} + F_m$
$= 5F_{m+1} + 3F_m$.

Since we know that F_m is divisible by 5, it is now clear that F_{m+5} is also divisible by 5.

But F_5 is divisible by 5, therefore so is F_{10}, and also F_{15}, and so on.

By the Principle of Mathematical Induction, every fifth Fibonacci number is divisible by 5.

Theorem

$$F_{n+1}^2 - F_n F_{n+2} = (-1)^n \text{ for all } n$$

Proof

We note first that the statement is true when $n = 1$, since

$$F_{1+1}^2 - F_1 F_{1+2} = F_2^2 - F_1 F_3 = 1^2 - 1 \times 2 = 1 - 2 = -1 = (-1)^1.$$

We now show that

IF the statement true for some natural number m,

THEN it is also true for the next number, namely $m + 1$.

So let us assume that $F_{m+1}^2 - F_m F_{m+2} = (-1)^m$. (*)

Then

$F_{m+2}^2 - F_{m+1} F_{m+3}$

$= F_{m+2}^2 - F_{m+1}(F_{m+2} + F_{m+1})$

$= F_{m+2}^2 - F_{m+1} F_{m+2} - F_{m+1}^2$

$= F_{m+2}(F_{m+2} - F_{m+1}) - F_{m+1}^2$

$= F_{m+2} F_m - F_{m+1}^2$

$= -(F_{m+1}^2 - F_{m+2} F_m)$

$= -(-1)^m$, using (*)

$= (-1)^{m+1}$

and we have shown that the result is true for the next number, $m + 1$.

But we noted at the beginning that the result is true for $m = 1$.

It is therefore true for the next number, 2, and therefore true for the next number, 3, and for 4, 5, 6, ... and so on.

By the Principle of Mathematical Induction, the result is true for all natural numbers. □

FIBONACCI AND BINET

The Fibonacci numbers

$$1,\ 1,\ 2,\ 3,\ 5,\ 8, 13,\ 21,\ 34,\ 55,\ 89,\ 144,\ \cdots$$

are defined by a recurrence relation

$$F_1 = F_2 = 1,\ F_{n+2} = F_{n+1} + F_n \ \text{ for } n \geq 1.$$

This definition allows one to find any Fibonacci number, but for large values of n it would be time-consuming to compute F_n this way. Worse, any error of calculating a term would be repeated and magnified in every subsequent term. So a natural question to ask is whether there is a simple formula, in terms of n, which enables one to calculate F_n without having to find all the earlier Fibonacci numbers.

In 1843 the French mathematician Jacques Philippe Marie Binet announced that he had found a Fibonacci Formula. Although it was later revealed that other mathematicians, including Leonhard Euler and Abraham de Moivre, had worked out the same formula a hundred years earlier, the formula is today still labelled *Binet's Formula*.

Deriving the Binet's formula starts with a simple little quadratic equation

$$x^2 = x + 1.$$

Multiplying the equation by successive powers of x, and simplifying at each step, gives

$x^3 = x(x^2) = x(x+1) = x^2 + x = (x+1) + x = 2x + 1$

$x^4 = x(x^3) = x(2x+1) = 2x^2 + x = 2(x+1) + x = 3x + 2$

$x^5 = x(x^4) = x(3x+2) = 3x^2 + 2x = 3(x+1) + 2x = 5x + 3$

$x^6 = x(x^4) = x(5x+3) = 5x^2 + 3x = 5(x+1) + 3x = 8x + 5$

and so on.

A pattern is beginning to emerge. It appears that

$$x^n = F_n x + F_{n-1} \text{ for all } n \geq 2.$$

The pattern can easily be established to be true for all natural numbers n by Mathematical Induction. It is clearly true when $n = 1$, and if we assume that it is true for some positive integer m, i.e. that

$$x^m = F_m x + F_{m-1}$$

then

$x^{m+1} = F_m x^2 + F_{m-1}x = F_m(x+1) + F_{m-1}x = (F_m + F_{m-1})x + F_m$

$= F_{m+1}x + F_m,$

showing that the formula is valid for the next number, $m+1$.

Now go back to the original quadratic equation $x^2 = x + 1$, which the standard quadratic formula shows has solutions $x = \dfrac{1 \pm \sqrt{5}}{2}$.

Substituting these two values into the equation

$$x^n = F_n x + F_{n-1}$$

gives

$$\Big(\frac{1+\sqrt{5}}{2}\Big)^n = F_n\Big(\frac{1+\sqrt{5}}{2}\Big) + F_{n-1}$$

and

$$\Big(\frac{1-\sqrt{5}}{2}\Big)^n = F_n\Big(\frac{1-\sqrt{5}}{2}\Big) + F_{n-1}\ .$$

Since we want to find an expression for F_n, the obvious thing to do now is subtract the two equations:

$$\Big(\frac{1+\sqrt{5}}{2}\Big)^n - \Big(\frac{1-\sqrt{5}}{2}\Big)^n = F_n\Big(\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}\Big) = \sqrt{5}F_n$$

from which Binet's Formula follows:

$$F_n = \frac{1}{\sqrt{5}}\Big(\Big(\frac{1+\sqrt{5}}{2}\Big)^n - \Big(\frac{1-\sqrt{5}}{2}\Big)^n\Big)\ .$$

□

AVOCADO LOGIC

The Professor of Logic was sent to buy a litre of milk by his wife.

"Get one litre of milk," she said, "and if they have avocados, get six."

He dutifully went to the corner shop and came back with six litres of milk.

"Why did you buy six litres?" said his wife.

He replied: "They had avocados." □

FIBONACCI AND THE GOLDEN RATIO

The Fibonacci numbers

$$1,\ 1,\ 2,\ 3,\ 5,\ 8, 13,\ 21,\ 34,\ 55,\ 89,\ 144,\ \cdots$$

are defined by the recurrence relation

$$F_1 = F_2 = 1,\ F_{n+2} = F_{n+1} + F_n \ \text{ for } n \geq 1.$$

Consider now the ratios $\dfrac{F_{n+1}}{F_n}$:

$$\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \frac{21}{13}, \frac{21}{13}, \frac{34}{21}, \frac{55}{34}, \frac{89}{55}, \frac{144}{89}, \ \cdots \ .$$

.

The sequence of fractions appears to rise and fall alternately. This observation can be confirmed by reference to the equation

$$F_{n+1}^2 - F_n F_{n+2} = (-1)^n$$

which, when divided by $F_{n+1}F_n$ gives

$$\frac{F_{n+1}}{F_n} - \frac{F_{n+2}}{F_{n+1}} = \frac{(-1)^n}{F_{n+1}^2} \ ,$$

showing that the difference between successive terms in the sequence of ratios is alternately positive and negative.

Calculating the first few terms suggest that successive ratios converge to 1.618 ... . This may be established by using Binet's formula

$$F_n = \frac{1}{\sqrt{5}}\Big(\Big(\frac{1+\sqrt{5}}{2}\Big)^n - \Big(\frac{1-\sqrt{5}}{2}\Big)^n\Big)$$

from which we have

$$\frac{F_{n+1}}{F_n} = \frac{\Big(\frac{1+\sqrt{5}}{2}\Big)^{n+1} - \Big(\frac{1-\sqrt{5}}{2}\Big)^{n+1}}{\Big(\frac{1+\sqrt{5}}{2}\Big)^n - \Big(\frac{1-\sqrt{5}}{2}\Big)^n}$$

Now look at the second terms in the numerator and denominator.

Since $-1 < \dfrac{1-\sqrt{5}}{2} < 1$, these terms tend to zero as n tends to infinity, so are insignificant in comparison with the first terms. It follows that

$$\frac{F_{n+1}}{F_n} \to \frac{1+\sqrt{5}}{2} \quad - \quad \text{the Golden Ratio.}$$

$\square$

THE 2015 PAN AFRICAN MATHEMATICS OLYMPIAD

The 2016 Pan African Mathematics Olympiad was held in Abuja, Nigeria, in August. A new feature of the event was the introduction of a special section for girls. Instead of sending a team of four, countries were invited to enter a team of three boys and three girls.

Nine countries took part: Benin, Burkina Faso, Ghana, Mali, Niger, Nigeria, South Africa, Tanzania and Tunisia. Not all countries entered full teams of six.

The South African team was Glenda Watt (grade 11, Durban Girls' College), Hannah Clayton (grade 11, Rustenburg Girls' High School), SangEun Lee (grade 11, St George's Grammar School), Jianliang Yu (grade 10, Crawford College, Pretroria), Tim Schlesinger (grade 9, Rondebosch Boys' High School) and Arthur Qampi (grade 11, Star College, Durban).

The Team Leader was Liam Baker (University of Stellenbosch), with Deputy Team Leader Dylan Nelson (University of Cape Town).

The girls and the boys wrote the same two papers. As in the International Mathematical Olympiad, each paper contained just three questions, to be tackled in $4\frac{1}{2}$ hours.

In the final ranking, SangEun was ranked second, winning a Gold Medal. Tim came 4th, winning a Silver Medal, and Jianliang, Arthur, Hannah and Glenda all won Bronze Medals.

Then the girls were ranked separately, and were awarded their own medals. SangEun came first, winning a Gold Medal, Hannah took 3rd place, winning a Silver Medal,and Glenda came 7th, winning a Bronze Medal.

When the scores were added up, Nigeria was ranked top with 156 points, with South Africa second on 129 and Tunisia 3rd, with 84 points. But the results are controversial. The score sheet revealed that two of the Nigerian girls who had low scores had been replaced in the team by two boys with higher scores. Had the Nigerians observed the rules, their team of three boys and three girls would have scored 124 points, 5 points behind South Africa. □

SOLVING PROBLEMS

Ngo Bao Chau was born in Hanoi, Vietnam in 1972 and took part in the 29th and 30th International Mathematical Olympiads. He became the first Vietnamese student to win two IMO gold medals,of which the first one was won with a perfect score of 42.

He studied in France and obtained a PhD in 1997 from the Université Paris-Sud. He is today Professor of Mathematics at the University of Chicago. In 2010 he was awarded the highest honour available to a mathematician: a Fields Medal.

His message below, to all the participants of the 2015 IMO in Thailand, was published in the IMO daily newsletter, *IMO Post 2015*.

> It has been a quarter of a century since I participated in the IMO. During that period, I have been wondering whether the IMO experience helped shape my career as a mathematician. To tell the truth, I did not find it easy to to adjust myself to higher mathematics in the university, as I had been exposed to so little of it in my growing up in an underdeveloped country. I did not find higher mathematics as challenging, as "spicy", as the olympiad problems that I got used to. In higher mathematics the emphasis seems to lie on formulating abstract concepts instead of solving concrete problems.
>
> With more time and experience my perception grows more nuanced. I now believe that doing mathematics, higher or not, is

about formulating and solving beautiful problems. Abstract concepts would be useless if they do not help us formulate and prove beautiful theorems.

I share with you this testimony, for I hope some of you will later continue the fantastic human endeavour of pursuing mathematical truth. There will be excellent universities around the world to offer you a guiding hand, should you want to become a mathematician. The University of Chicago, where I teach is one of them.

I wish you the best success in this IMO organized in Thailand. Enjoy the legendary hospitality of the host country! □

MATHS ON DRUGS

In a front page article on drugs in the Western Cape (*The Times*, 30 September 2015), statistics attributed to the UCT Centre of Criminology are given.

According to the report, over the last ten years

> "The price of tik dropped by 183.7% and the price of heroin by 183.75% in the Western Cape."

This is an extraordinary statement. If the price of drugs has fallen by over 100%, i.e. into negative territory, the drug lords must be paying the people of the Western Cape to take the drugs off their hands.

Further figures in the report give some hint at how the percentage drops of over 100% were arrived at.

> "Ten years ago tik cost R225 per gram, now it is R217.50. Had the inflation rate been taken into account it would have cost R426 today. While heroin costs R119, it was R215 a decade ago. The price should have appreciated to R421."

This paragraph could give one an idea of how the outlandish percentages of 183.7% and 183.75% were obtained.

We start by trying to find out what average annual inflation figure was used.

Using the figures for tik, (R225 ten years ago and an inflation-adjusted price of R426 today), the price has increased by $426/225 = 1.8933$ in ten years, so the increase per year is $1.8933^{\frac{1}{10}} = 1.0659$, which is an annual increase of 6.6%.

Using the figures for heroin (R215 ten years ago and an inflation-adjusted price of R421 today), the price has increased by $421/215 = 1.9581$ in ten years, so the increase per year is $1.9581^{\frac{1}{10}} = 1.0695$, which is an annual increase of 7.0%.

It's not clear why two different figures for average annual inflation over the last ten years were used, but something between 6% and 7% is reasonable.

Now let's compare the original prices of the two drugs with their inflation-adjusted prices.

For tik the price decrease (from R426 to R225) is R201.
Since $201 \div 426 = 0.471$, the percentage drop in ten years is 47%.

For heroin the price decrease (from R421 to R119) is R302.
Since $302 \div 421 = .717$, the percentage drop in ten years is 72%.

Actually, there is something strange about R119, the price of heroin ten years ago. It could be that it is a misprint for R219, which looks more likely.

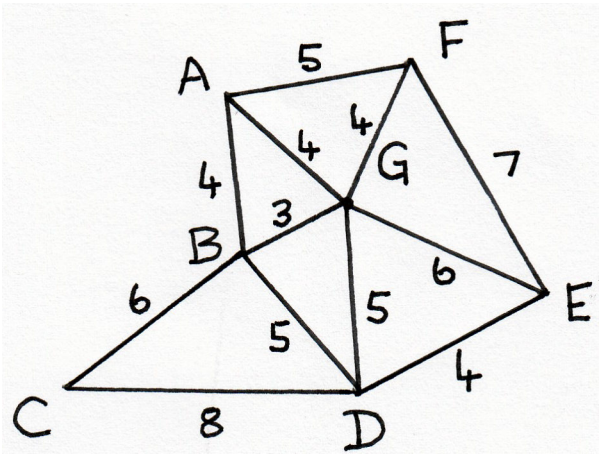
In that case the price decrease (from R421 to R219) would be R202.
Since $202 \div 421 = 0.480$, the percentage drop is 48%, which is more in line with the 46% figure for tik.

But whichever way round one looks at the figures, how those ridiculously huge percentages of 183.7% and 183.75% were conjured up remains a mystery.

$\square$

KRUSKAL'S ALGORITHM

A number of villages in a rural area are connected by roads, with the distances between them shown on the map below.



The roads are all gravel, and after receiving many complaints from local residents the local council agrees to tar some of the roads. There are 7 villages, and 11 roads connecting them, with a total length of 61 km. However, the council does not have enough money to tar all the roads, so they decide to tar just enough roads so that it will be possible to travel between any two villages completely on tar. They ask their roads engineer to work out the most economical way of connecting the villages.

The engineer's first idea is to tar the ring of roads $ABCDEFGA$, for a total length of 37 km. He then realises that tarring the road GA is not necessary, as one can drive from G to A on tar through all the other villages. That reduces the total length to 33 km. Another cost-saving measure would be to tar the 5km BD instead of the 8 km CD, for a further saving of 3 km of tar.

This sort of trial-and-error method is not satisfactory. Is there a systematic way of finding the most economical network?

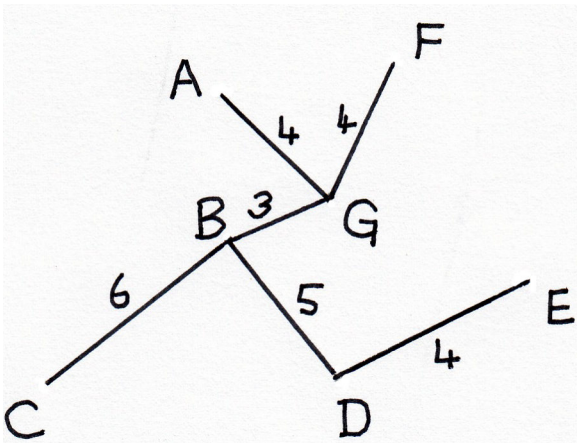
Problems of this sort can be solved by a method known as *Kruskal's Algorithm*, first published by Jacob Kruskal in 1956.

- Tar the shortest road. If there are two shortest roads, tar either.
- Look for the shortest untarred road which does not complete a closed circuit of tarred roads, and tar it.
- Continue in this way, until any further tarring would create a circuit.

Using this algorithm, the roads for tarring are chosen in order:

3 km roads: BG
4 km roads: AG, GF, DE (Not AB: it would create a circuit $ABGA$.)
5 km roads: BD (Not AF or DG: they would create circuits.)
6 km roads: BC

All the villages are now connected by tarred roads, with no circuits. Any further connection would create a circuit. The total length of this network is 26 km.

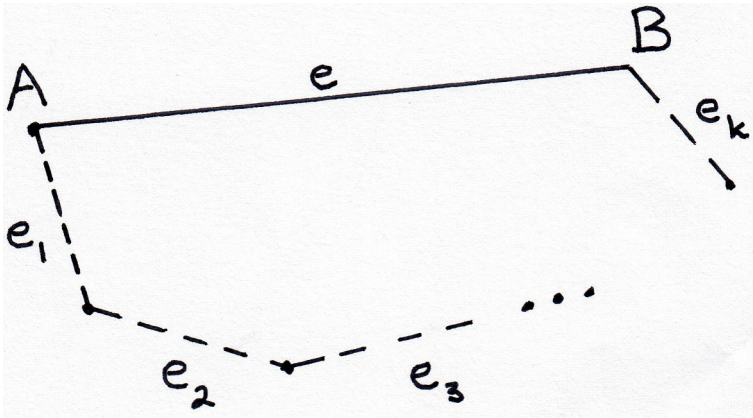


When submitted to the council, they agree that the proposal to tar the six roads is within their budget. But, they ask, is this the most economical network?

How can the engineer convince the council that Kruskal's Algorithm has produced the best solution and that no shorter network of tarred roads is possible?

The first point to notice is that, although there are very many ways of tarring selections of roads, there are still only finitely many, and among these there has to be a minimal solution. Call this minimal network **N**, and denote the network that Kruskal's Algorithm produces by **K**. We now prove that **N** and **K** have the same total length.

Suppose there is a road in **K** of length e, joining two villages A and B, which is not in **N**. Since the road AB is not in **N**, A and B must be connected by a sequence of roads through other villages of lengths $e_1,\ e_2,\ \cdots, e_k$



Now if $e < e_i$ for some i, then the road of length e_i could be replaced by the road of length e, giving a network of total length less than the most economical network **N**, which is a contradiction. So we conclude that $e \geq e_i$ for $i = 1, 2,\ \cdots, k$.

We now show that, for some i, $e = e_i$.

For suppose $e_i < e$ for all i. Then, when AB was selected by Kruskal's algorithm, each of those shorter roads in **N** must have been rejected, and that means that they must have closed a circuit in **K**. But then A and B would be joined by roads in **K**, and the road AB would not have been chosen by Kruskal's Algorithm.

So the minimal network **N** could be modified by replacing the road of length e_i by AB, with the modified network of the same length, and therefore still minimal.

Carrying on in this way, we can successively replace the roads in the minimal network **N** by roads in Kruskal's network **K**, while keeping the total length the same at every step, until eventually the minimal network becomes **K**. This proves that **K** is minimal.

Conclusion: Kruskal's Algorithm always produces a network of minimal total length.

Kruskal's Algorithm is called a "Greedy Algorithm" because each step chooses the cheapest option.

There is another version of Kruskal's Algorithm.

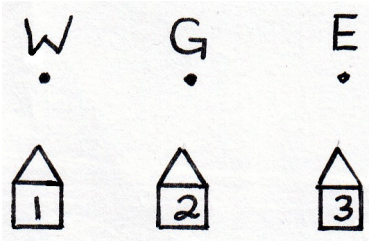
- Start with the whole network of gravel roads which could be tarred, and choose the longest road. Eliminate it, as long as it does not disconnect the network.
- Look for the longest untarred road which does not disconnect the network, and eliminate it.
- Continue in this way, until any road you choose will disconnect the network, then stop.
- Tar this network of remaining roads.

Use this version of Kruskal's Algorithm on the network of roads above. You may not get the same network as before, but it will be the same length.

Can you prove that this version of Kruskal's Algorithm also gives a network of minimal length? □

THE UTILITIES PROBLEM

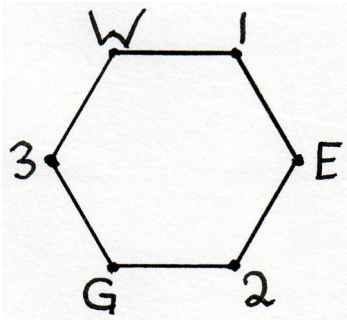
Three houses are to be connected to three utilities: water, gas and electricity. There is an extra requirement: two lines may not cross.



After trying a few connection diagrams, none of them seems to work, and one soon becomes convinced that the problem cannot be solved. Since one can never be sure that all possibilities have been eliminated, a proof of impossibility is needed.

Here are two completely different approaches.

Labelling the utilities W, G and E, and numbering the houses 1, 2,and 3, we can start by putting them at the points of a regular hexagon, with the houses alternating with the utilities.



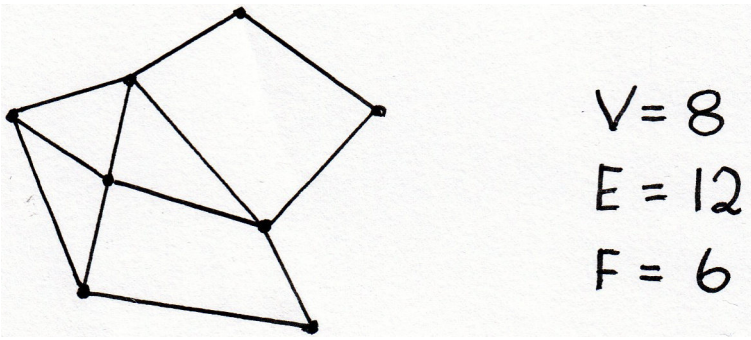
In this diagram, house 1 has been connected to water and electricity, house 2 has been connected to electricity and gas, and house 3 has been linked with gas and water.

Three more connections must be made: house 1 to gas, house 2 to water and house 3 to electricity. Of these three connecting lines, two cannot both be inside the hexagon, for they would have to cross. Therefore two have to be outside the hexagon — and again they would have to cross. So the connections cannot all be made without two of them crossing.

The second argument is a very neat piece of reasoning involving Euler's Formula

$$V - E + F = 2.$$

Euler's formula is usually set in the context of the vertices V, edges E and faces F of a polyhedron, but applies equally well to a polygonal graph dividing the plane into regions (counting the region outside the polygon as a face).



The second proof is by contradiction: we assume that the utilities problem can be solved with a suitable plane graph, and derive a contradiction.

In the utilities graph there are six vertices (three utilities and three houses) and nine edges (each house is linked to three utilities).

From Euler's Formula $V - E + f = 2$ it follows that

$$F = 2 + E - V = 2 + 9 - 6 = 5.$$

Now consider the faces from a different angle. In a utilities graph there can be no triangular faces, any edge joins a house with a utility, and no two utilities are joined by an edge, and no two houses are joined by an edge. So every face has four or more edges.

Since we have established that the utilities graph has exactly five faces, each with four or more edges, the faces together have at least $5 \times 4 = 20$ edges. Of course, every edge is shared by two faces, and so there must be at least $20 \div 2 = 10$ edges. There's the contradiction.

We are forced to conclude that no plane graph can be drawn to solve the utilities problem. $\square$

THE FOUR CUBES PUZZLE

You are given four cubes, with their faces painted in four colours.

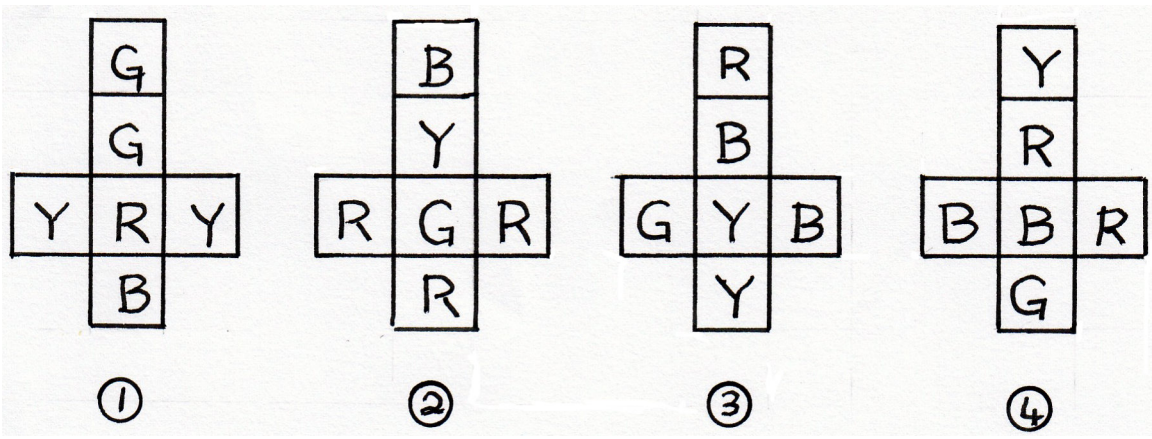
The four cubes must be arranged in a line so that on each of the four long faces all four colours are shown.

The puzzle has been around for over a century, and has been marketed as *The Great Tantalizer*, *The Katzenjammer Puzzle* and *Instant Insanity*.

It is a rather difficult puzzle. There are thousands of ways of putting the cubes in a line, taking into account the 24 different orientations of just one cube, and random fiddling with the cubes in the hope that a solution will just happen can be very frustrating.

This is where maths comes to the rescue. An application of some simple Graph Theory shows how the problem can be solved.

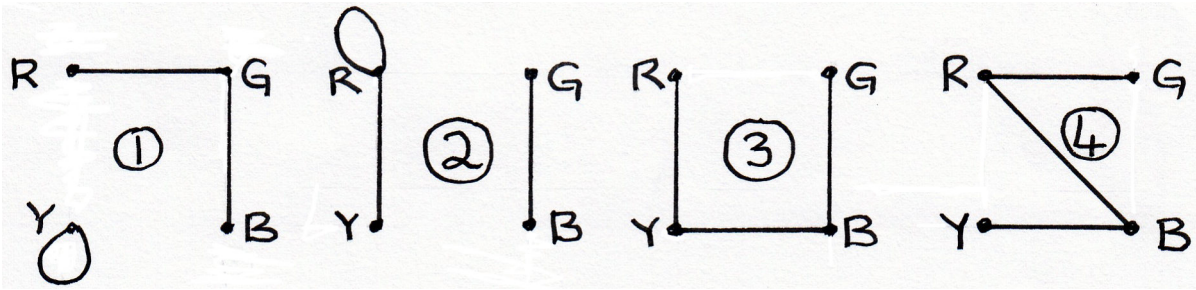
Suppose that the cubes have nets as shown below, with the faces labelled Red, Green, Yellow and Blue.



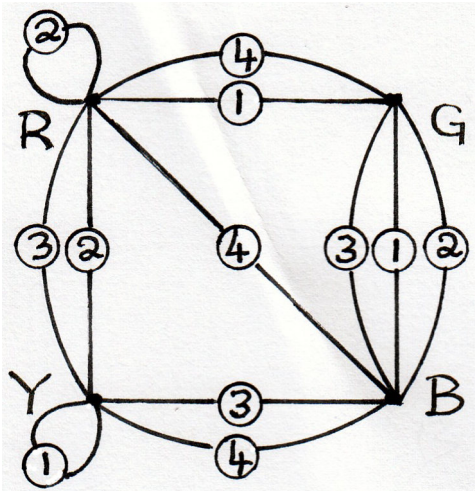
The first step is to draw four graphs describing the cubes.

Each graph has four vertices and three edges. The vertices of the graphs correspond to the four colours, and a line joining two vertices indicates that those two colours are on opposite faces of the cube. If two opposite faces have the same colour, a little loop is drawn at that vertex.

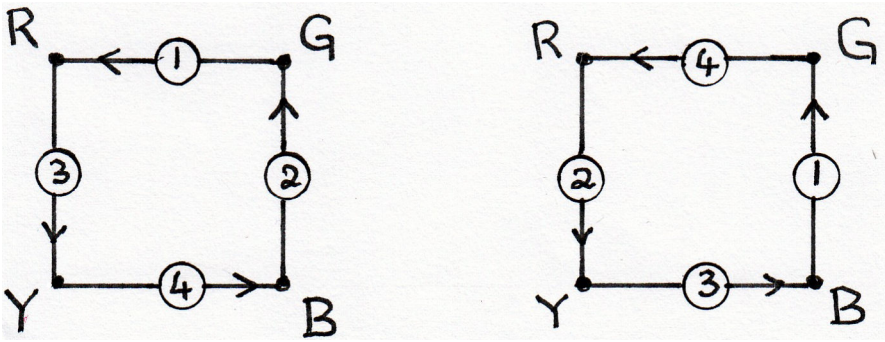
The graphs for the four cubes are shown below.



Now we combine all four graphs into one composite graphs, labelling each edge with the number of its cube.



From the composite graph we now select two circuits, with no common edges.



These two circuits now show how to arrange the cubes.

Start with the first circuit.

The edge GR indicates that Cube 1 must be set with Green in the front and Red at the back.
The edge RY indicates that Cube 2 must be set with Red in the front and Yellow at the back.
The edge YB indicates that Cube 3 must be set with Yellow in the front and Blue at the back.
The edge BR indicates that Cube 4 must be set with Blue in the front and Red at the back.

The cubes have now been arranged so that the front and back faces contain four different colours.

The top and bottom faces can now be adjusted. Use the second circuit to rotating each of the four cubes about an axis through the front and back faces so that four different colours are shown on the top and bottom faces, but the front and back faces have not changed. □

SA MATHEMATICS OLYMPIAD RESULTS

Western Cape pupils dominated the lists of Gold and Silver medallists in the 2015 South African Mathematics Olympiad were announced at an Awards Function in Durban on Saturday 19 September, taking took eight out of the ten Junior medals and six of the ten Senior medals, including the Senior Gold Medal. Rondebosch Boys' High School captured four medals: three Junior and one Senior.

The SA Mathematics Olympiad is a three-round contest for high schools, written at two levels: Junior (grades 8 and 9) and Senior (grades 10, 11 and 12). In each division the first round is a straightforward multiple-choice paper, and those who score 50% or more are invited to take part in the second round, in which problems require only answers. The third round is for the top 100 in round 2, and now the problems require full answers, explanations and proofs.

The Dawie du Toit Gold Medal in the Junior division was won by Aaron Naidu (grade 9, St. Henry's Marist Brothers College).

Silver medals were awarded to
2. Emile Tredoux (grade 9, Parklands College)
3. Ralph McDougall (grade 9, Curro Durbanville)
4. Tim Schlesinger (grade 9, Rondebosch Boys' High School)
5. Harry Kim (grade 9, El Shaddai Christian School)
6. Razeen Parker (grade 9, Rondebosch Boys' High School)
7. Du Toit Spies (grade 9, Parel Vallei High School)
8. Taariq Mowzer (grade 8, Fairbairn College)
9. Tae Beom Kim (grade 9, Rondebosch Boys' High School)
9. Dricus Oerlemans (grade 9, Hoërskool Menlopark).

In the Senior division the Dirk van Rooy Gold Medal was awarded to Bronson Rudner (grade 11, South African College School).

Silver Medals were awarded to
2. Andrew McGregor (grade 11, Rondebosch Boys' High School)
2. Roger Song (grade 12, Reddam House College, Bedfordview)
4. Yaseen Mowzer (grade 11, Fairbairn College)
5. Nicholas Kroon (grade 11, St Andrew's College, Grahamstown)
5. Sanjiv Ranchod (grade 11, Westerford High School)
7. SangEun Lee (grade 11, St George's Grammar School)

8. Jeffrey Russell (grade 12, Pretoria Boys' High School)
9. David Broodryk (grade 11, Westerford High School
9. Jianglong Liao (grade 12 Waterford Kamhlaba School, Swaziland).

Important dates for 2016:

5 February: Closing date for entries
9 March: Round 1
11 May: Round 2
27 July: Final Round

Full details may be obtained from

The South African Mathematics Foundation
Private Bag X173
Pretoria 0001)
Tel +27 (0)12 392 9372
Fax +27 (0)12 392 9312
Email: info@samf.ac.za

Past papers and results are on the website `www.samf.ac.za`. □

LAST TASK DECIDES OLYMPIAD WINNER

Right until the very last question on the last day of the Programming Olympiad the two top participants matched each other question by question and mark by mark. "It was uncanny" said Peter Waker, manager of the Olympiad. "Both Ulrik de Muelenaere and Yaseen Mowzer had 100% for the first task on the first day. That can easily happen. Both again scored 100% for the second task. A happy coincidence. Both got 55% for the last task of the day. It could still be a coincidence. However, when both got 100% for the first two tasks on the second day it looked as if a tie for gold was likely. Only the last task on the last day separated them, with Ulrik beating Yaseen by 36 points."

Both Ulrik de Muelenaere and Yaseen Mowzer are experienced participants at international level. Ulrik won bronze in 2014 and 2015 at the International Olympiad in Informatics. Yaseen has won bronze at both the International Maths Olympiad and the International Olympiad in Informatics. Yaseen is only in Grade 11 at Fairbairn College in Cape Town and will have another shot at winning gold in 2016. Ulrik is in Grade 12 at the Hoërskool Waterkloof in Pretoria and cannot compete again in 2016. Ulrik intends to study Computer Science at the University of Pretoria next year.

Ulrik won the prestigious Standard Bank Trophy, the gold medal, R11 000 for himself and R5 000 for his school.

Yaseen won a silver medal, R7 000 for himself and R3 000 for his school.

The second silver medal went to Bronson Rudner, a Grade 11 learner at SACS.

Bronze medals went to Tae Jun Park of Rondebosch Boys' High, Sanjiv Ranchod from Westerford and Laurens Weyn from Abbotts College Century Gate.

The runners up were:

Torsten Babl from Grade 11 at Paarl Boys' High School
Donal Davies from Grade 10 at Woodridge College, Port Elizabeth
Jarryd Dunn from Grade 12 at the International School of Cape Town
Daniel Jenkins from Grade 12 at York High School, George
Jiaqi Ke from Grade 11 at Roedean School, Johannesburg

Ralph McDougall from Grade 9 at Curro Durbanville
Joseph Rautenbach from Grade 11 at Glenwood House, George
Jeremy Wilkinson from Grade 12 at Bishops, Cape Town.

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□

RIEMANN HYPOTHESIS SOLVED!

A flurry of internet announcements earlier this month trumpeted the news that a Nigerian mathematician, Opeyemi Enoch, of the Federal University in Oye-Ekiti, had "solved the Riemann Hypothesis", and had therefore won the million-dollar Millennium Mathematics Prize, offered by the Clay Mathematics Institute.

Suspicions that the reports may be a hoax were aroused right from the start. The Riemann Hypothesis is the most famous unsolved mathematics problem of today, and false or crank proofs are not unknown. In this case, it was strange that the Nigerian reports all said that the Riemann Hypothesis had been "solved". What everybody would want to know is: Which way was it solved? Is the Riemann hypothesis true? Or is it false? The reports did not say. Further, they all stated that this was the fourth of the seven famous Millennium Mathematical Problems to be cracked, whereas only one (the Poincaré Conjecture) has been solved.

The reports were more or less copy-and-paste versions of each other, though one managed to inflate the million dollars to a billion. Another added the intriguing information that Dr Enoch "had previously designed a Prototype of a silo for peasant farmers and also discovered a scientific technique for detecting and tracking someone on an evil mission."

No news source outside Nigeria ran the story, and it is not mentioned on the Clay Institute's website. Nor, even more significantly, is the story on the website of Opeyemi Enoch's university. □

WESTERN CAPE HAS VINTAGE OLYMPIAD YEAR

For Western Cape schools, 2015 was a bumper year of maths and computer olympiad successes.

Of the six members of the team representing South Africa at the **International Mathematical Olympiad** in Thailand in July, five were from the Western Cape: David Broodryk (Westerford High School), Andrew McGregor (Rondebosch Boys' High School) Yaseen Mowzer (Fairbairn College),Sanjiv Ranchod (Westerford High School) and Bronson Rudner (S A College High School). Yaseen won a Bronze Medal and Andrew and Bronson won Honourable Mentions.

Of the four members of the South African team in the **International Olympiad of Informatics** in Kazakhstan in August, three were from the Western Cape. Yaseen Mowzer (Fairbairn College), Thomas Orton (Bishops) and Robin Visser (St George's Grammar School) all won Bronze Medals.

Of the six members of the South African team in the **Pan African Mathematics Olympiad**, held in Nigeria in August, three were from the Western Cape. SangEun Lee (St George's Grammar School) won a Gold Medal and was ranked 2nd overall (and was top girl), Tim Schlesinger (grade 9, Rondebosch Boys' High School) was ranked 4th and won a Silver Medal. Hannah Clayton (grade 11, Rustenburg Girls' High School) won a Bronze Medal.

Eight Western Province teams of ten were entered in the nationwide **Inter-Provincial Mathematics Olympiad**, held in September. In the Junior division (grades 8 and 9), in which 49 teams took part, the Western Province A team was ranked first, and all four teams ranked in the top ten. In the Senior division (grades 10, 11 and 12) with 50 teams taking part, the Western Province Senior A team took first place.

At the Awards Ceremony of the **South African Mathematics Olympiad**, held in Durban on Saturday 19 September, eight of the nine Silver Medals in the junior division (grades 8 and 9) went to Western Cape schools: Emile Tredoux (grade 9, Parklands College), Ralph McDougall (grade 9, Curro Durbanville), Tim Schlesinger (grade 9, Rondebosch Boys' High School), Harry Kim (grade 9, El Shaddai Christian School), Razeen Parker (grade 9, Rondebosch Boys' High School), Du Toit Spies (grade 9, Parel Vallei

High School), Taariq Mowzer (grade 8, Fairbairn College) and Tae Beom Kim (grade 9, Rondebosch Boys' High School).

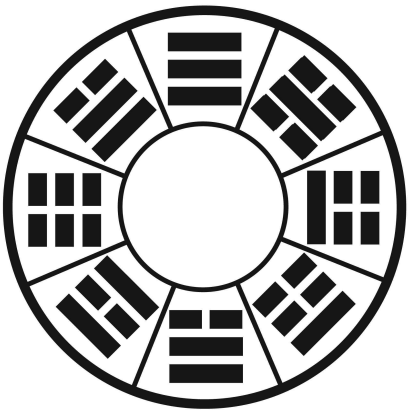
In the senior division the Dirk van Rooy Gold Medal was awarded to Bronson Rudner (grade 11, S A College School), and five of the nine Silver Medals were taken by Western Cape schools: Andrew McGregor (grade 11, Rondebosch Boys' High School), Yaseen Mowzer (grade 11, Fairbairn College), Sanjiv Ranchod (grade 11, Westerford High School), 7. SangEun Lee (grade 11, St George's Grammar School) and David Broodryk (grade 11, Westerford High School).

Of the 14 finalists in the **Computer Programming Olympiad**, 11 were from the Western Cape. The results were announced at an Awards Dinner in Cape Town on Monday 12 October, and the Western Cape took five of the six medals. Tae Jun Park (Rondebosch Boys' High School), Sanjiv Ranchod (Westerford High School) and Lauren Weyn (Abbott's College, Milnerton) won Bronze Medals, while Yaseen Mowzer (Fairbairn College) and Bronson Rudner (SA College high School) won Silver Medals. □

MATHEMATICS AT UCT IN 2016

A full programme of mathematics competitions and other events will be presented by the UCT Department of Mathematics in 2016.

The biggest event of the year, the annual UCT Mathematics Competition, will take place on Thursday 7 April. This year 170 schools registered, and even more are expected next year. Online registration will open after the start of the new school year, and the closing date for registration is 18 March.



A Saturday afternoon programme in the regular *UCT Mathematics Series* will take place on 12 March, and another will be held on 26 August. These popular programmes are for general audiences, and the attendance of around 200 attest to their popularity.

The top students in the UCT Mathematics Competition are invited to take part in the *UCT Mathematics Challenge*, to be written on Saturday 7 May, and the top 50 in this event will be invited back the next Saturday (14 May) to write the *UCT Mathematics Olympiad*.

The results of the UCT Mathematics Competition, Challenge and Olympiad will be announced at a Prize Giving on 19 May.

In the second half of the year the focus will be on selecting and training teams in preparation for the *Interprovincial Mathematics Olympiad* (also known as the SA Mathematics Team Competition). A open invitation will be issued to all schools to send their top students to take part in trials for selecting Western Province teams. Last year over 250 took part in the trials, and eight teams of ten were selected.

The trials are provisionally scheduled for Saturday 3 September, with the IPMO taking place the following Saturday.

Watch the website www.mth.uct.ac.za/competition for full details. □